OPTIMAL SUSTAINABLE AGRICULTURAL DECISIONS:
A MIXED INTEGER APPROACH

by

Santhi Wicks and Richard Howitt

Department of Agricultural and Resource Economics

Selected Paper prepared for presentation at the American Agricultural Economics Association Annual Meeting, Providence, Rhode Island, July 24-27, 2005

I. INTRODUCTION

Soil is a capital asset and a renewable resource that requires continued maintenance to ensure the sustainability of agricultural output in the long-run. Growers are continually adjusting the soil's productivity by adding and extracting soil nutrients, and modifying the soil structure through cultivation. Given the small per unit cost of chemical fertilizers, applications of these fertilizers are cost effective in the short run if externalities are ignored. Farmers can add chemical fertilizers to temporarily increase soil fertility and yields, but generally ignore technologies that maintain or build soil quality. However, if soils are considered a renewable resource and a valuable capital asset over the long term, profit-maximizing farmers should be interested in optimally managing the long-run stock of soil resources.

Winter cover crops are one sustainable agricultural practice that farmers may use to increase soil quality by building soil organic matter, reducing leaching of nutrients, increasing nitrogen availability (using legumes), improving soil surface permeability, and crop establishment. Agronomists at the University of California Cooperative Extension have shown that in spite of the private benefits associated with the use winter cover crops, farmers in the Central Valley of California have not as yet widely adopted this technology (Mayo, G; per comm.).

In this paper we use two theoretical models, a short-run static profit maximizing (SRSP) model and a long-run model is a mixed integer dynamic optimization (LRMD) model. We propose five propositions that characterize the optimal use of cover crops. Static proposition I, if the cost of chemical nitrogen is sufficiently low relative to the cost of cover crop, then profit maximizing farm managers will not plant cover crops in crop rotations in the short-run. Static Proposition II, A manager who observes that soil nitrates levels are low will compensate for the low soil fertility by planting cover crops, unless the cost of nitrogen fertilizer is low. Dynamic Proposition I, there exists a cost of chemical nitrogen sufficiently low that profit maximizing farm managers will not plant cover crops in the long-run. Dynamic Proposition II: while it is unprofitable to plant cover crops in the short-run it may be profitable to plant them in the long-run. Dynamic Proposition III, applying restrictions on the maximum applications of chemical nitrogen increases the value marginal product of soil nitrates, and consequently the extent of cover cropping.

2. BACKGROUND

The general form of the managers profit maximizing problem is identical for both the SPSP and LRMD model: maximize total farm profits by selecting the level control variables namely, chemical nitrogen and the management choice to plant a cover crop or leave the ground fallow prior to planting the cash crop, subject to soil fertility levels.

Profits are generated as the difference between farm revenues from the sales of the single seasonal cash crop and total input costs from the cash crop and/or cover crop. Farm profits depend on the difference in farm revenues and the input costs from the cash and/or cover crop. Total revenue per acre is a product of the market price and the per acreage yields for the cash crop. Total per acre variable costs include the cost of irrigation water, the cost of applying chemical nitrogen, and the costs of planting cover crops. Other per acre costs of planting the cash crop (i.e. machine, labor,….) are not included in total variable costs as they are constant and do not affect the marginal conditions for optimization.

The cash crop general production function is quadratic functional form, which has the usual concavity properties and diminishing marginal rates of substitutability between the two inputs: chemical nitrogen and soil nitrates. The production function takes the following form,

$$Y_t = \alpha_1 N_t + \alpha_2 NO_{3t} - \begin{bmatrix} N_t & NO_{3t} \end{bmatrix} \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} N_t \\ NO_{3t} \end{bmatrix}. \tag{1}$$

Where 't' is an index representing each season, Y_t is the seasonal yield of the cash crop, N_t the seasonal application of chemical nitrogen, and NO_{3t} is the seasonal stock of soil nitrates. Parameters α_1 and α_2 represent the linear effect of chemical nitrogen and soil nitrates on yields which are positive. It is required that the quadratic term $\begin{bmatrix} -z_{11} & -z_{12} \\ -z_{21} & -z_{22} \end{bmatrix}$ is negative definite. For this to occur the matrix $A = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$ is positive definite which implies that the determinant of this matrix is positive $\left(|A| = z_{11}z_{22} - z_{12}z_{21} > 0\right)$. It is assumed that the direct effects z_{11} and z_{22} are positive, which implies that the determinant of A is positive if z_{12} and z_{21} are negative or have relatively small positive values. However, by agronomic reasoning the values of z_{12} and z_{21} are restricted to be positive, as applications of chemical nitrogen and stocks of soil nitrates are substitutes in production. This implies that the marginal impact of a unit increase in the stock of soil nitrates reduces the marginal product of chemical nitrogen $\left(\partial MP_{N_t} / \partial NO_{3t} < 0\right)$, and the marginal impact of an increase in the application of chemical nitrogen reduces the marginal product of soil nitrates $\left(\partial MP_{NO_{3t}} / \partial N_t < 0\right)$. Also by Young's theorem of symmetry it is fair to assume that $z_{12} = z_{21}$ so we rewrite the condition as $z_{11}z_{22} - z_{12}^2 > 0$. In every season (t) at least one cropping activity is conducted, first the manager has the option to plant cover crop or leave the ground fallow and second the cash crop is planted. Chemical nitrogen is a continuous control variable that may take any positive value. Soil fertility is represented by the stock of soil nitrates that varies continuously and depends on the level of the control variables each season. Input costs for the cash and cover crops are defined on a per acreage basis for simplicity of presentation.

In making a profit maximizing decision each season the manager considers the stock of soil nitrates. An equation of motion estimates the seasonal adjustments in the stock of soil nitrates, and is developed using the mass balance concept. Agronomists have traditionally used mass balance equations to define the stock of nitrates in the root zone as a function of the total nitrates entering and leaving the root zone (Powlson, 1993). The general form of a mass balance equation for soil nitrates is,

$$\dot{NO_3} = N_t + CC_t \times NFIX_t - PKRN_t + HUM_t - YLN_t \tag{2}$$

Where the change in soil nitrates between seasons is $NO_3 = NO_{3,t+1} - NO_{3,t}$, $NFIX_t$ is nitrogen fixed by legume cover crops, YLN_t is the nitrate removed in the cash crop, $PKRN_t$ is the nitrate leached into ground water, and HUM_t is the stock of soil nitrates gained in demineralization or lost mineralization. The cover crop management choice (CC_t) is the second control variable. It is a binary variable that takes the value one if cover crops are planted and zero if the ground is left fallow.

The mass balance equation above is simplified for the SPSP and LRMD models. Legume cover crops may have two impacts on soil fertility and soil nitrate levels. They may increase soil nitrate levels through fixing of nitrogen ($CC_t \times NFIX_t$) or demineralization of plant biomass (HUM_t). These terms are aggregated into one term $a_2 CC_t$ that represents the increase in soil nitrates from the adoption of a winter cover crop. Parameter a_2 is positive and defines the proportionate increase in soil nitrates from adoption of a cover crop at the beginning of the current season. Second, variations in soil nitrates that result from applying of chemical nitrogen, leaching into ground water supplies, and reduction in soil nitrates by uptake in yields, are aggregated into $-a_1 NO3_{t-1}$. It is assumed that $-a_1$ is negative, so the carryover of soil nitrates between seasons is depreciated and soil nitrates decreased unless cover crops are planted. The equation of motion or flows in the stock of soil nitrates for the SRSP model is,

$$NO_{3t} = (1 - a_1) NO_{3t-1} + a_2 CC_t \text{, and} \tag{3}$$

for the long-run or mixed integer dynamic model equation (3) is transformed by subtracting the current level of soil nitrates ($NO3_{,t}$) from both sides of equation (3) to yield,

$$\dot{NO}_{3t} = -a_1 NO_{3t} + a_2 CC_t \,. \tag{4}$$

3. SHORT-RUN STATIC ANALYSIS

In the SRSP optimization model is a one period optimization problem. The manager maximizes farm profits in the first season by choosing the levels of chemical nitrogen to apply on the cash crops and deciding whether to plant cover crops or leave the ground fallow prior to planting the cash crop. The SRSP problem facing each manager is as follows,

$$\underset{\{Nt, CCt\}}{\text{Max}} \quad \Pi(.) = pY_t - c_N N_t - c_c CC_t$$

$$\text{subject to} \quad NO_{3t} = (1 - a_1) NO_{3t-1} + a_2 CC_t$$

$$\text{where} \qquad Y_t = \alpha_1 N_t + \alpha_2 NO_{3t} - \begin{bmatrix} N_t & NO_{3t} \end{bmatrix} \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} N_t \\ NO_{3t} \end{bmatrix}, \text{ and}$$

$$N_t \geq 0, NO_{3t} > 0, \text{and } CC_t = 1 \text{ or } 0.$$

The price of the cash crop is represented by the parameter p, c_N is the constant per unit variable costs of chemical nitrogen, and c_c is the constant per acre cost of planting cover crops.

Substituting the equation of motion for soil nitrates and yield function into the profit maximizing equation results in the following modified problem,

$$\underset{\{N_t, CC_t\}}{\text{Max}} \quad \Pi(.) = p\left\{\alpha_1 N_t + \alpha_2 NO_{3t} - \begin{bmatrix} N_t & NO_{3t} \end{bmatrix} \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} N_t \\ NO_{3t} \end{bmatrix}\right\},$$

$$-c_N N_t - c_c CC_t$$

$$\text{subject to } NO_{3t} = (1 - a_1) NO_{3t-1} + a_2 CC_t .$$

Differentiating the above expression with respects to N_t and CC_t to obtain the first order conditions for the SRSP problem which results in the following,

$$\frac{\partial \Pi(.)}{\partial N_t} = p\left(\alpha_1 - 2z_{11} N_t - 2z_{12} NO_{3t}\right) = c_N \tag{5}$$

$$\frac{\partial \Pi(.)}{\partial CC_t} = p\left(\alpha_2 a_2 - 2z_{12} a_2 N_t - 2z_{22} a_2 NO_{3t}\right) \begin{matrix} > \\ = \\ < \end{matrix} c_c \tag{6a}$$

$$NO_{3t} = (1 - a_1) NO_{3t-1} + a_2 CC_t . \tag{7}$$

Recall that $z_{12} = z_{21}$ and so the first order condition for equation (5) and (6a) have been simplified by this assumption.

Equation (5) is the marginal condition on the application of chemical nitrogen, which requires that the value marginal product of chemical nitrogen VMO_{N_t} is equal to the marginal cost of chemical nitrogen. Re-arranging equation (5) to derive an expression for the optimal allocation of chemical nitrogen (N_t) as a function of soil nitrates and other predetermined parameters.

$$N_t = \frac{1}{2z_{11}}\left(\alpha_1 - 2z_{12}NO_{3_t} - \frac{c_N}{p}\right) \tag{8}$$

The switch condition for the static problem (equation (6a)) defines the point where it is profitable or unprofitable to plant cover crops. This equation states that it is profitable to plant cover crops if the value marginal product of soil nitrates ($VMO_{NO_{3_t}}$) is greater than the marginal cost of cover crops (c_c/a_2). Alternatively, if the value marginal product of soil nitrates is less than the marginal cost of cover crops, then it is unprofitable to plant cover crops.

In summary the solution to the SRSP optimization problem is characterized by,

$$p\left(\alpha_2 - 2z_{12}N_t - 2z_{12}NO_{3t}\right) \begin{array}{c} > \\ = \\ < \end{array} \frac{c_c}{a_2} \tag{6a}$$

where $NO_{3t} = (1 - a_1)NO_{3t-1} + a_2CC_t$, and (7)

$$N_t = \frac{1}{2z_{11}}\left(\alpha_1 - 2z_{12}NO_{3_t} - \frac{c_N}{p}\right). \tag{8}$$

Static Proposition I: If the cost of chemical nitrogen is sufficiently low then profit maximizing farm managers will not plant cover crops in crop rotations in the short-run.

Two steps are required to address static proposition I. First determine the impact of decreasing the cost of chemical nitrogen on the application of chemical nitrogen. Second, define the impact of changing the application of chemical nitrogen on the value marginal product of chemical nitrates.

To determine the impact of the cost of chemical nitrogen on applications of chemical nitrogen differentiate equation (8) with respects to the cost of chemical nitrogen,

$$\frac{\partial N_t}{\partial c_N} = \frac{-1}{2z_{11}p} < 0\,. \tag{9}$$

All terms in the denominator of equation (9) are positive and so the cost of chemical nitrogen increases the total application of chemical nitrogen.

To realize the impact of chemical nitrogen on the $VMP_{NO_{3_t}}$ differentiate the left hand side of equation (6a) with respects to chemical nitrogen resulting in,

$$\frac{\partial VMP_{NO_{3_t}}}{\partial N_t} = -2z_{12}a_2 p < 0\,. \tag{10}$$

Given that application of chemical nitrogen and stocks of soil nitrates are substitutes and thus $z_{12} > 0$, the effect cover crops on cover crops on soil nitrates (a_2) is positive, and the price of the crops p is positive, the derivative in equation (10) is negative, so increasing the application of chemical nitrogen reduces the $VMP_{NO_{3_t}}$.

Accordingly if the cost of chemical nitrogen decreases the application of chemical fertilizer will increase, which reduces the $VMP_{NO_{3_t}}$ relative to the marginal cost of cover cropping. Therefore reducing the cost of chemical nitrogen implies that the manager will only plant cover crops at relatively lower per unit cost of cover cropping.

Static Proposition II: A manager who observes that soil nitrates levels are low will compensate for the low soil fertility by planting cover crops, unless the cost of nitrogen fertilizer is low.

From equation (6a) and (5) notice that decreasing the stock of soil nitrates increases the $VMP_{NO_{3_t}}$ and VMP_{N_t} respectively.

$$\frac{\partial VMP_{NO_{3_t}}}{\partial NO_{3t}} = -2z_{22}a_2 p < 0 \tag{11}$$

$$\frac{\partial VMP_{N_t}}{\partial NO_{3t}} = -2z_{12}a_2p < 0 \tag{12}$$

Recall that z_{22}, z_{12}, p and a_2 are all both positive parameters.

From equation (11) the marginal impact of soil nitrates on the VMP_{NO3_t} is negative. Specifically, the affect of planting cover crops on stocks of soil nitrates (a_2) is positive and (by substitution) parameter z_{22} is also positive. Therefore as the stocks of soil nitrates decrease the VMP_{NO3_t} increases relative to the cost of cover cropping. From equation (12) decreasing the stock of soil nitrates increases the VMP_{N_t} relative to the cost of chemical nitrogen.

When the stocks of soil nitrates decrease, equation (6a) the VMP_{NO3_t} may become greater than the cost of cover cropping. In this situation there is an alternative position that increases the manager profits and there are two techniques for reaching this position.

First, equation (8) indicates that decreasing the stocks of soil nitrates will results in an increase in the optimal application of chemical nitrogen. This may be seen by differentiating equation (8) with respect to soil nitrates (see equation(13)).

$$\frac{\partial N_t^*}{\partial NO_{3t}} = -2z_{12} < 0\,. \tag{13}$$

From equation (10) we show it was proven that increasing the application of chemical nitrogen reduces the VMP_{NO3_t}. Also by differentiating equation (5) we see that increasing the application of chemical nitrogen decreases VMP_{N_t} , see equation (14) below.

$$\frac{\partial VMP_{N_t}}{\partial N_t} = -2pz_{11} < 0 \tag{14}$$

Alternatively, planting cover crops increases the stocks of soil nitrates though equation (7), so reducing VMP_{NO3_t} and VMP_{N_t} .

Given two methods by which the manager may move closer to an optimal solution through three possibilities: (i) increasing applications of chemical nitrogen, planting cover crops, (ii) planting cover crops and, (iii) increasing applications of chemical nitrogen. It is important to understand the conditions that determine which is more desirable. Recall, that the decision to plant cover crops depends on the switch condition equation (6a), so begin with this equation. In this condition the VMP_{NO3_t} may be reduced by increasing the application of N_t or by planting cover crops and increasing NO_{3t}. After such increases, the VMP_{NO3_t} may be less than the cost of cover cropping, as expressed with inequality in equation (15a). Rearrange equation (15a) so that only the application of chemical nitrogen is on the left hand side,

$$\alpha_2 - 2z_{12}N_t - 2z_{22}NO_{3t} < \frac{c_c}{pa_2} \tag{15a}$$

$$-2z_{12}N_t < \frac{c_c}{pa_2} + 2z_{22}NO_{3t} - \alpha_2$$

$$N_t > \frac{1}{2z_{12}}\left(\alpha_2 - \frac{c_c}{pa_2} - 2z_{22}NO_{3t}\right) \tag{15b}$$

Equation (9) indicates that increasing the cost of cover cropping reduces the level of chemical nitrogen that is required to make VMP_{NO3_t} less than the cost of cover cropping. By the same token equation (9) indicates decreasing the cost of chemical fertilizer increases the application of chemical nitrogen. Therefore, a manager that observes low levels of soil nitrates will apply chemical nitrogen if, the cost of chemical nitrogen is relatively lower than the cost of cover crop. Alternatively, if the cost of cover cropping is relatively smaller than the cost of chemical nitrates, the manager will use cover crops to increase soil quality.

4. DYNAMIC ANALYSIS

4.1 SYSTEMS OF DIFFERENTIAL EQUATIONS

The principal structure of the LRMD problem is identical to the SRSP problem. The objective is still to maximize profits by selecting the level of the control variables: chemical nitrogen and the management choice to plant winter cover crops or leave the ground fallow, subject to adjustments in soil nitrate levels. The difference between the static and dynamic models is that the level of soil nitrates stocks will vary each season depending on decisions in the past season and each decision impacts farm profits. The LRMD optimal control problem for all 't' seasons is,1

$$\underset{\{Nt, CCt\}}{\text{Max}} \quad \Pi(.) = \int_0^T e^{-rt}\left(p\,Y_t - c_N N_t - c_c CC_t\right)dt$$

subject to $\qquad \dot{NO}_{3t} = -a_1 NO_{3t} + a_2 CC_t,$

where $\qquad Y_t = \alpha_1 N_t + \alpha_2 NO_{3t} - \begin{bmatrix} N_t & NO_{3t} \end{bmatrix} \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} N_t \\ NO_{3t} \end{bmatrix},$

$$NO_3(0) = NO_{30},$$

$$N_t \geq 0,\ NO_{3t} > 0, \text{ and } CC_t = 1 \text{ or } 0.$$

The current value Hamiltonian for this problem may be constructed as follows:

$$H(.) = \left\{ p\left(\alpha_1 N_t + \alpha_2 NO_{3t} - \begin{bmatrix} N_t & NO_{3t} \end{bmatrix} \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} N_t \\ NO_{3t} \end{bmatrix} \right) - c_N N_t - c_c CC_t \right\} + \phi_t\left(-a_1 NO_{3t} + a_2 CC_t\right) \tag{16}$$

where ϕ_t is the costate variable for the LRDP problem, which denotes the value of the stock of soil nitrates in each season.

According to the maximum principal first order conditions for the Hamiltonian are:

1 Three general references for material developed in the remainder of this paper include Léonard and Long (1998), Kamien and Schwartz (2000), and Caputo (2005).

$$\frac{\partial H(.)}{\partial N_t} = p\left(\alpha_1 - 2z_{11}N_t - 2z_{12}NO_{3t}\right) - c_N = 0 \tag{17a}$$

$$\frac{\partial H(.)}{\partial CC_t} = -c_c + \phi_t\, a_2 = 0 \tag{18}$$

$$\begin{aligned} \dot{\phi}_t(.) = r\phi_t - \frac{\partial H(.)}{\partial NO_{3t}} &= r\phi_t - p\left(\alpha_2 - 2z_{12}N_t - 2z_{22}NO_{3t} - a_1\phi_t\right) \\ &= \left(r + a_1\right)\phi_t - p\left(\alpha_2 - 2z_{12}N_t - 2z_{22}NO_{3t}\right) \end{aligned} \tag{19a}$$

$$\dot{NO}_{3t} = -a_1 NO_{3t} + a_2 CC_t \tag{20a}$$

To solve the LRDP problem we want to first develop a system of equations that may be solved for this optimization problem. Using this system of equations we may be able to understand the switching behavior of farmers in the long-run optimization model and compare it to results from the static optimization model.

Equation (5) and (17a) are identical thus the optimal dynamic application is,

$$N_t = \frac{1}{2z_{11}}\left(\alpha_1 - 2z_{12}NO_{3t} - \frac{c_N}{p}\right). \tag{17b}$$

Substituting equation (17b) into the equation (19a) yields,

$$\dot{\phi}_t(.) = \left(r + a_1\right)\phi_t + R_1 + R_2 NO_{3t} \tag{19b}$$

where $$R_1 = -p\left(\alpha_2 - \frac{z_{12}}{z_{11}}\left[\alpha_1 - \frac{c_N}{p}\right]\right) \tag{21}$$

$$R_2 = \frac{2p}{z_{11}}\left[z_{22}z_{11} - z_{12}^2\right] \tag{22}$$

From the concavity assumption on the production function, we have the follow restriction $(z_{22}z_{11} - z_{12}^2) > 0$ and so R_2 is positive. The sign of R_1 is not so obvious, but it is possible to prove this indirectly. Later when solving this dynamic system of equations, it is necessary to identify a positive fixed point for the system of equations. This uses equation (19b) and by setting $\dot{\phi}_t(.) = 0$ which results in $\phi_t = \left(R_1 - R_2 NO_{3_t}\right) \Big/ (r + a_1)$. Notice that as R_2 is positive then the only possibility of obtaining a fixed point with a non-negative costate is if R_1 is negative. Given this assertion take a closer look at what this implies for R_1 from equation (21). If R_1 is negative, then from equation (21) it is required that,

$$\alpha_2 - \frac{z_{12}}{z_{11}}\left[\alpha_1 - \frac{c_N}{p}\right] > 0, \text{ or}$$

$$\alpha_2 > \frac{z_{12}}{z_{11}}\left[\alpha_1 - \frac{c_N}{p}\right] \tag{23}$$

Also from the first order equation (17) it is obvious that $\alpha_1 - \frac{c_N}{p} > 0$.

In conclusion the LRDP problem may be defined by the following system of equations one for the stock of soil nitrates and anther for the costate variable.

$$\dot{NO}_{3t} = -a_1 NO_{3t} + a_2 CC_t \tag{20a}$$

$$\dot{\phi}_t(.) = (r + a_1)\phi_t + R_1 + R_2 NO_{3t} \tag{19b}$$

$$\text{where } R_1 = -p\left(\alpha_2 - \frac{z_{12}}{z_{11}}\left[\alpha_1 - \frac{c_N}{p}\right]\right) < 0 \tag{21}$$

$$\text{and} \qquad R_2 = \frac{2p}{z_{11}}\left[z_{22} z_{11} - z_{12}^2\right] > 0 \tag{22}$$

$$\text{and the switch condition is } \phi_t\, a_2 \begin{array}{c} > \\ = \\ < \end{array} c_c. \tag{18}$$

The precise form of the equation of motion for soil nitrates (equation (20a)), depends on the switch condition. If the marginal value of cover cropping $(\phi_t\, a_2)$ is greater than the per unit cost of cover cropping then it is beneficial to plant cover crops,

$CC_t = 1$ and $\dot{NO}_{3t} = -a_1 NO_{3t} + a_2$. Alternatively, if the marginal value of cover cropping is less than the per unit cost of cover cropping, then $CC_t = 0$ and $\dot{NO}_{3t} = -a_1 NO_{3t}$. From these statements we see that there are two distinctly different systems of differential equations for the LRDP problem, which depend on the switch conditions. In the literature of dynamic optimization this problem is termed a bang-bang problem. The traditional bang-bang model results if the Hamiltonian is linear in a given control variable. However, in our model the traditional bang-bang results as the control is a binary variable. To analyze this problem we will first identify and characterize the fixed points and steady states for the systems of linear differential equations: first where the cover crop is not planted and second where the cover crop is planted. Next we will estimate general and specific solutions for each case. Finally, the two solutions are combined in phase diagrams using the switch condition and each of the dynamic propositions are addressed.

4.2 CLASSIFICATION AND STABILITY OF THE FIXED POINT

4.2.1 WITHOUT COVER CROPS

First consider the set of dynamic equations where cover crops are not planted $(CC_t = 0)$ given that $\phi_t a_2 < c_c$. The system of equations are defined by following equations,

$$\dot{NO}_{3t} = -a_1 NO_{3t} \tag{20b}$$

$$\dot{\phi}_t(.) \quad = (r + a_1)\phi_t + R_1 + R_2 NO_{3t} \tag{19b}$$

Notice that this is a system of linear autonomous equations and so can use theorems for linear systems of equations to characterize this solution, estimate the fixed point, and a general and specific solution. First, identify the fixed point of this linear system. Second, estimate the eigenvalues and eigenvectors of this linear system of differential equations. Third, use the theorems for linear systems of equations to determine the phase diagram a set of linear solutions to approximate the set of nonlinear equations, and estimate a specific solution.

First, identify the fixed point(s) of the linear differential equations when cover crops are not planted. In equation (20b) set $\dot{NO}_{3t} = 0$ and in equation (19b) which results in,

$$\overline{NO3} = 0\text{, and} \tag{24}$$

$$\bar{\phi}_t = \frac{-R_1}{(r + a_1)} \tag{25}$$

which is a potential steady state solution.

Second, estimate the Jacobian matrix for this linear systems of equations.

$$J\left[\dot{NO}_{3t}, \dot{\phi}_t\right]\Big|_{(\overline{NO}, \bar{\phi})} = \left[\begin{matrix} \frac{\partial \dot{NO}_{3t}}{\partial NO_{3t}} & \frac{\partial \dot{NO}_{3t}}{\partial \phi_t} \\ \frac{\partial \dot{\phi}_{3t}}{\partial NO_{3t}} & \frac{\partial \dot{\phi}_{3t}}{\partial \phi_t} \end{matrix}\right]\Bigg|_{(\overline{NO3}, \bar{\phi})}$$

$$= \begin{bmatrix} -a_1 & 0 \\ R_2 & (r + a_1) \end{bmatrix} \tag{26}$$

Third, begin identifying the eigenvalues and eigenvectors for the system of differential equations. Calculate the eigenvalues for equation (26) by subtracting λI_2 from the Jacobian matrix, taking the determinant of the resulting matrix, and setting this value equal to zero.

$$\left| J\left[\dot{NO}_{3t}, \dot{\phi}_t\right] - \lambda I_2 \right| = \begin{vmatrix} -a_1 - \lambda & 0 \\ R_2 & (a_1 + r) - \lambda \end{vmatrix} = 0$$

$$= (-a_1 - \lambda)((a_1 + r) - \lambda) - 0 \times R_2 = 0$$

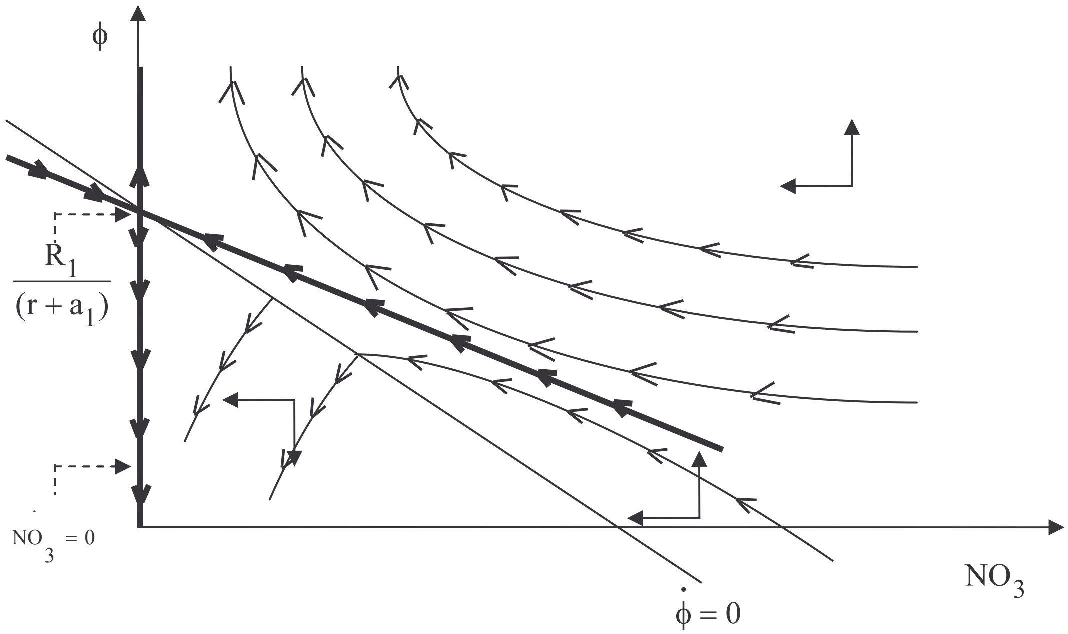
The two eigenvalues are $\lambda_1 = -a_1$ and $\lambda_2 = (a_1 + r)$. As one eigenvalue is negative and the other positive, the fixed point is a saddle point. The associated eigenvectors for the eigenvalue $\lambda_1 = -a_1$ are $v_1^1 = -(2a_1 + r)$ and $v_2^1 = \delta$, and for the eigenvalue $\lambda_2 = (a_1 + r)$ are $v_1^2 = 0$ and $v_2^2 = -(2a_1 + r)$.

We begin by plotting the two equations when the rate of change in soil nitrates and the costate variable are constant. Previously, in equation (24) it was shown that $NO_{3t} = 0$ when ($\dot{NO}_{3t} = 0$). Also if the rate of change in the costate variable equal to zero ($\dot{\phi}_t(.) = 0$) this results in $\phi_t = -(R_1 + R_2 NO_{3t}) \big/ (r + a_1)$. It may be observed that these relationships are plotted in Figure (1). Also recall that the point where these two equations intersect is the fixed point, previously calculated at $(\overline{NO}_3, \bar{\phi})$ (see equations (24) and (25)).

Differential equations (20) and (19b) are used to determine the directions of movement for the trajectories in Figure (1). These directions of movement for stocks of soil nitrates are displayed by the horizontal arrows in Figure (1). The directions of movement for the costate variable are illustrated by the vertical arrows in Figure (1).

Now to finalized drawing of the phase diagram let's include the information from the eigenvalues and eigenvectors. For a saddle point the positive eigenvalue $\lambda_2 = (r + a_1)$ represents the stable manifold and the negative eigenvalue $\lambda_1 = -a_1$ is the unstable manifold. The unstable manifold that is drawn from the fixed point in the direction of the eigenvector $v_1^2 = 0$ and $v_2^2 = -(2a_1 + r)$, hence equation $NO_{3t} = a_2 / a_1$ ($\dot{NO}_{3t} = 0$) is the unstable trajectory. In Figure (1) this information is represented by the darkened line with arrows pointing away from the fixed point: the unstable manifold. The stable manifold is drawn from the fixed point in the direction of the eigenvector $v_1^1 = -(2a_1 + r)$ and $v_2^1 = \delta$. In Figure (1) this is represented by the second bold line with a negative slope, and the arrows on this line point towards the fixed point and so represents the stable manifold.

Figure 1: Solution to the LRMD Model Without Cover Crops.



4.2.2 WITH COVER CROPS

Next consider the case where cover crops are planted $(CC_t = 1)$. The system of equations for this problem is as follows is similar to that for case 1, however in case 2 the stock of soil nitrates is increased by a fixed factor a_2 (see equation (20c)).

$$\dot{NO}_{3t} = -a_1 NO_{3t} + a_2 \tag{20c}$$

$$\dot{\phi}_t(.) \quad = \left(r + a_1\right)\phi_t + R_1 + R_2 NO_{3t} \tag{19b}$$

At this point if we calculate the Jacobian matrix for this linear system of equations then we obtain the same matrix as that in equation (26). Therefore, the linear system of differential equations for the case where cover crops are planted is a saddle point. In addition, the systems of differential equations when cover crops are planted (equations (20c) and (19b) have eigenvalues and eigenvector identical to the system of differential equations when cover crops are not planted (equations (20b) and (19b)). The only difference between these systems is they different fixed points.

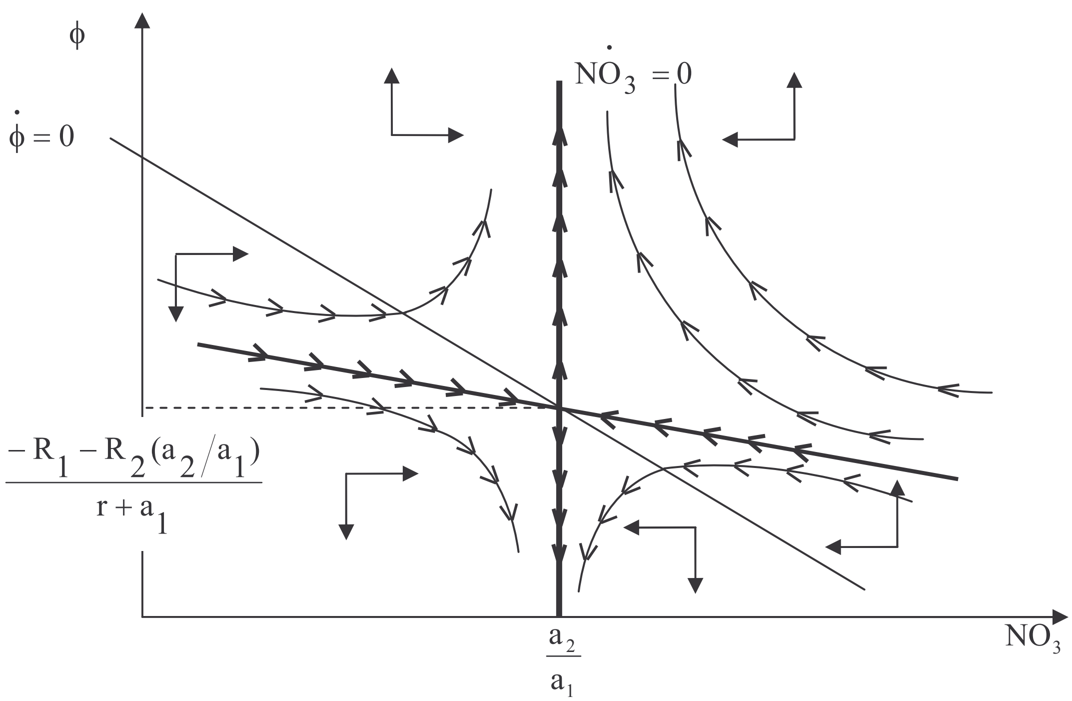
The fixed point(s) of systems of linear differential equations when cover crops are not planted are defined as follows. In equation (20c) set $\dot{NO}_{3t} = 0$ and in equation which results in equation (27) set $\dot{\phi}_t(.) = 0$,

$$\overline{NO_3} = a_2/a_1 , \tag{27}$$

$$\bar{\phi}_t = \frac{-R_1 - R_2(a_2/a_1)}{r + a_1} \tag{28}$$

If we follow through with the same techniques described in section 3.2.2a, for the model without cover crops, then we are able to plot a phase diagram for the system of differential equations where cover crops are planted. The solution is displayed below in Figure 2.

Figure 2: Phase Diagram for LRMD Problem with Cover cropping.



4.3 SOLUTIONS FOR CASES

4.3.1 WITHOUT COVER CROPS

Recall that the system of equations for this case are equations (20c) and (19b). Now lets' proceed to solve this system of equations. Equation (20c) may be rewritten as,

$$\dot{NO}_{3t} + a_1 NO_{3t} = 0 .$$

Multiply both sides of this equation by the integrating factor $e^{a_1 t}$ and integrate by parts, to generate the general equation for this system of equations.

$$\int \left(\dot{NO}_{3t} e^{a_1 t} + a_1 NO_{3t} e^{a_1 t} \right) dt = \int 0 \, e^{a_1 t} \, dt ,$$

$$NO_{3t} e^{a_1 t} = k ,$$

$$NO_{3t} = k e^{-a_1 t} .$$

Evaluate this expression at time t=0 to give $k = NO_3(0) = NO_{3_0}$ where NO_{3_0} is the initial stock of soil nitrates. Therefore the flow equation for soil nitrates may be re-written as,

$$NO_{3t} = NO_{3_0} e^{-a_1 t} \tag{29}$$

This means that when cover crops are not planted, the stocks of soil nitrates are continually decreasing through time.

Before continuing with the derivation of the costate equation and phase diagram, consider the equation that determines that optimal application levels of chemical nitrogen. Substitute equation (29) into equation (17b),

$$N_t = \frac{1}{2z_{11}} \left(\alpha_1 - 2z_{12} NO_{3_0} e^{-a_1 t} - \frac{c_N}{p} \right) \tag{30}$$

As is expected this equation provides some interesting information. First, the application of chemical nitrogen decreases each season if cover crops are not planted. Second, if there is no option to plant winter cover crops and the initial soil quality is low (NO_{3_0}) then initial applications of chemical nitrogen are high. Third, as the cost of chemical nitrogen decreases then the application of chemical nitrogen is greater for all seasons. Fourth, as the price of the cash crop increases the application of chemical nitrogen increases.

To derive an equation for the costate variable, substitute equation (29) back into equation (19b), multiply by $e^{-(r+a_1)t}$ and integrate by parts to obtain the following expression for the shadow value of soil nitrates.

$$\phi_t = \frac{R_1}{(r+a_1)} + \frac{R_2\, NO_{30} e^{-a_1 t}}{-(r+2a_1)} + k \tag{31}$$

where k is the constant of integration. A visual analysis of the phase diagram for the where cover crops are not planted indicates that the value of the costate in the final period (T) may be zero or take the value of the switch point. If $\phi(T)=0$, substitute this relation into equation (31).

$$k = -\frac{R_1}{(r+a_1)} + \frac{R_2\, NO_{30} e^{-a_1 T}}{(r+2a_1)}.$$

Next substitute the value of 'k' into equation (31) to obtain the specific solution for the costate.

$$\phi_t = \frac{R_2\, NO_{30}}{(r+2a_1)}\left[e^{-a_1 T} - e^{-a_1 t}\right] \tag{32}$$

The term in square brackets of equation (32) is negative and so the costate is positive. In addition, by analyzing the derivative of equation (32) it is proven that the costate is always increasing each season cover crops are not planted.

In conclusion the solutions to the system of differential equations when cover crops are not planted is,

$$NO_{3t} = NO_{30} e^{-a_1 t} \tag{29}$$

$$N_t = \frac{1}{2z_{11}}\left(\alpha_1 - 2z_{12} NO_{30} e^{-a_1 t} - \frac{c_N}{p}\right) \tag{30}$$

$$\phi_t = \frac{c_c}{a_2} + \frac{R_2\, NO_{30}}{(r+2a_1)}\left[e^{-a_1 T} - e^{-a_1 t}\right] \tag{32}$$

4.3.2 WITH COVER CROPS

The system of linear differential equations for the case where cover crops are planted which is as follows,

$$\dot{NO}_{3t} = -a_1 NO_{3t} + a_2 \qquad (20c)$$

$$\dot{\phi}_t(.) = (r + a_1)\phi_t + R_1 + R_2 NO_{3t} \qquad (19b)$$

Proceed in solving this system of linear equations. Equation (20c) is rewritten as,

$$\dot{NO}_{3t} + a_1 NO_{3t} = a_2 .$$

Integrate to generate the general equation for this system of equations.

$$NO_{3t} = \frac{a_2}{a_1} + ke^{-a_1 t} \qquad (33)$$

To determine the value of 'k' the constant of integration, evaluate the above equation at time t=0. This yields $k = a_2/a_1 - NO_{30}$, where NO_{30} is the initial stock of soil nitrates. Substitute the value of k back into equation (6a) obtain the solution of soil nitrates for the case where cover crops are planted.

$$NO_{3t} = \frac{a_2}{a_1} + \left(\frac{a_2}{a_1} - NO_{30}\right)e^{-a_1 t} \qquad (34)$$

This means that when cover crops are not planted, the stocks of soil nitrates are continually decreasing through time.

By substituting the solution of soil nitrates into equation (30) the optimal application of chemical nitrogen is determined.

$$N_t = \frac{1}{2z_{11}}\left(\alpha_1 - \frac{2z_{12}a_2}{a_1} - \frac{c_N}{p} + \left(\frac{-2z_{12}a_2}{a_1} + 2z_{12}NO_{30}\right)e^{-a_1 t}\right) \qquad (35)$$

This equation also points to some interesting results. First, the application of chemical nitrogen converges to a stable positive value. Second, the greater is the initial level of soil fertility the smaller is the optimal allocation of chemical nitrogen. Third, as the cost of chemical nitrogen decreases, the application of chemical nitrogen is greater for all seasons. Fourth, as the price of the cash crop increases the optimal application of chemical nitrogen increases.

Substitute equation (33) back into equation (19b), multiply by $e^{-(r+a_1)t}$, and integrate by parts to obtain an expression for the shadow value of soil nitrates.

$$\phi_t = \frac{R_1}{(r+a_1)} - \frac{R_2 a_2}{(r+a_1)a_1} - \frac{R_2}{(r+2a_1)}\left(\frac{a_2}{a_1} - NO_{30}\right)e^{-a_1 t} + k \tag{36}$$

where k is the constant of integration. To determine the value of the constant of integration, consider the phase diagram in Figure 2 that indicates that the final value of the costate may be the value of the switch point $\phi(T) = c_c / a_2$ or some large value say $\phi(T) = z$. First, impose the constraint $\phi(T) = c_c / a_2$ in equation (36) so that,

$$k = \frac{c_c}{a_2} - \frac{R_1}{(r+a_1)} + \frac{R_2 a_2}{(r+a_1)a_1} + \frac{R_2}{(r+2a_1)}\left(\frac{a_2}{a_1} - NO_{30}\right)e^{-a_1 T}$$

Next substitute this expression back into equation (36) which results in the following specific equation.

$$\phi_t = \frac{c_c}{a_2} + \frac{R_2}{(r+2a_1)}\left(NO_{30} - \frac{a_2}{a_1}\right)\left[e^{-a_1 t} - e^{-a_1 T}\right] \tag{37}$$

Equation (37) indicates that the rate of change in the costate for this case is always decreasing. Second impose the constraint $\phi(T) = z$ in equation (36) so that,

$$\phi(T) = \frac{R_1}{(r+a_1)} - \frac{R_2 a_2}{(r+a_1)a_1} - \frac{R_2}{(r+2a_1)}\left(\frac{a_2}{a_1} - NO_{30}\right)e^{-a_1 T} + k = z.$$

With similar manipulations as those completed for equation (37) this results in the following specific solution.

$$\phi_t = z + \frac{R_2}{(r+2a_1)}\left(NO_{30} - \frac{a_2}{a_1}\right)\left[e^{-a_1 t} - e^{-a_1 T}\right] \tag{38}$$

Differentiating this specific solution with respects to time indicate that the costate is always increasing.

Solutions for the system of differential equations when cover crops are planted is,

$$NO_{3t} = \frac{a_2}{a_1} + \left(\frac{a_2}{a_1} - NO_{30} \right) e^{-a_1 t} \tag{34}$$

$$N_t = \frac{1}{2z_{11}} \left(\alpha_1 - \frac{2z_{12} a_2}{a_1} - \frac{c_N}{p} + \left(\frac{-2z_{12} a_2}{a_1} + 2z_{12} NO_{30} \right) e^{-a_1 t} \right) \tag{35}$$

$$\phi\left(t/(NO_{30} > a_2/a_1)\right) = \frac{c_c}{a_2} + \frac{R_2}{(r + 2a_1)} \left(NO_{30} - \frac{a_2}{a_1} \right) \left[e^{-a_1 t} - e^{-a_1 T} \right] \tag{37}$$

$$\phi\left(t/(NO_{30} < a_2/a_1)\right) = z + \frac{R_2}{(r + 2a_1)} \left(NO_{30} - \frac{a_2}{a_1} \right) \left[e^{-a_1 t} - e^{-a_1 T} \right] \tag{38}$$

4.4 THE COMBINED BANG-BANG SOLUTION

We combine the two cases where cover crops are planted and are not planted into one phase diagram solutions. Phase diagrams and specific solutions, for soil nitrates and the costate variable, developed in Section (3.3.2) and (3.3.3) are no integrated. The switch point indicates points where the value marginal product changes from greater than to less than the cost of cover cropping, defining when it is relatively profitable to plant cover crops and not plant cover crops respectively. There are three cases that should be considered where: (i) the fixed point and intercept of $\dot{\phi}_t = 0$ is below the switch point,

$$\frac{-R_1}{(r + a_1)} < \frac{c_c}{a_2}, \tag{39}$$

(ii) the fixed point is below but the intercept of $\dot{\phi}_t = 0$ is above the switch point,

$$\frac{c_c}{a_2} < \frac{-R_1}{(r + a_1)} < \frac{-R_1 - R_2(a_2/a_1)}{(r + a_1)}, \tag{40}$$

and (iii) the fixed point is above the switch point,

$$\frac{-R_1 - R_2(a_2/a_1)}{(r + a_1)} > \frac{c_c}{a_2}. \tag{41}$$

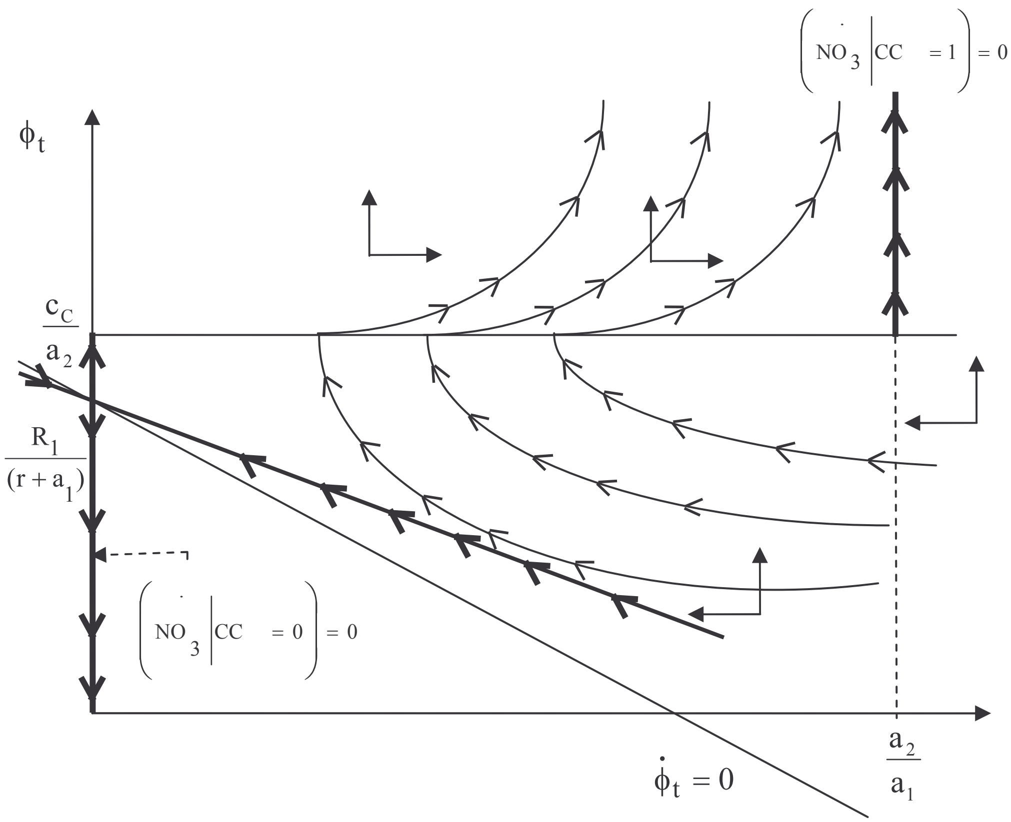
Consider case (i) and substitute the value of R_1 as defined in equation (39) into the restriction defined above,

$$\frac{p\alpha_2}{(r+a_1)} - \frac{z_{12}}{(r+a_1)z_{11}}\left[p\alpha_1 - c_N\right] < \frac{c_c}{a_2} \tag{42}$$

The above equation will hold if the price of cash crop is low relative to the cost of the cover crop. However, recall from equation (19b) that $p\alpha_1 > c_N$ therefore if the price of the cash crop is low then the relative cost of chemical fertilizers is also low. Therefore case (i) results when the price of the cash crops and cost of chemical nitrogen are relatively smaller than the cost of cover crops.

Recall that this fixed point which represented by the intersection of the flow equation where they are equal to zero and is placed below the switch point, represented by the horizontal line at $\phi_t = c_C / a_2$. Using information from Section 4.2 it is possible to draw the follow phase diagram. Where the section below the switch point is taken from Figure 1 and the diagram above the switch point is developed from Figure 2, and the relevant trajectories were identified and discussed in Section 4.3. In the below graph it may be seen that if the price of the cash crops and cost of chemical nitrogen are relatively smaller than the cost of cover crops, then the manager is able to run out the stocks and waits longer before the value of the costate moves above the switch point and cover crops are planted. However, no matter how low are the cost of nitrogen and prices it is always in the best interests of the farmer to plant cover crops in the long-run. Though there is also a stable solution that where the stock of soil nitrates is completely exhausted, which is represented by the stable manifold.

Figure 3: **Case (i): Price of Cash Crop and Cost of Chemical Nitrogen is Low Relative to the Cost of Cover Cropping.**



As the value of the price of cash crops and the cost of nitrogen increase relative to the cost of cover cropping then case (ii) and case (iii) result. To see this more closely consider case (iii) and substitute the value of R_1 and R_2 as defined by equation (21) and (22) into the above equation (41).

$$\frac{p}{(r+a_1)}\left(\alpha_2 - \frac{2a_2}{a_1 z_{11}}\left[z_{22}z_{11} - z_{12}^2\right] - \frac{z_{12}}{(r+a_1)z_{11}}\left[p\alpha_1 - c_N\right]\right) > \frac{c_c}{a_2} \tag{43}$$

From equation (43) it is clear that if the price of cash crop and the cost of nitrogen are relatively larger than the cost of cover cropping the case (iii) results and the fixed point is above the switch point. In the following phase diagram we graph this case. In this case notice that the fixed point occurs above the switch point.

Figure 4: **Case (ii): Price of Cash Crop and Cost of Chemical Nitrogen is high Relative to the Cost of Cover Cropping.**

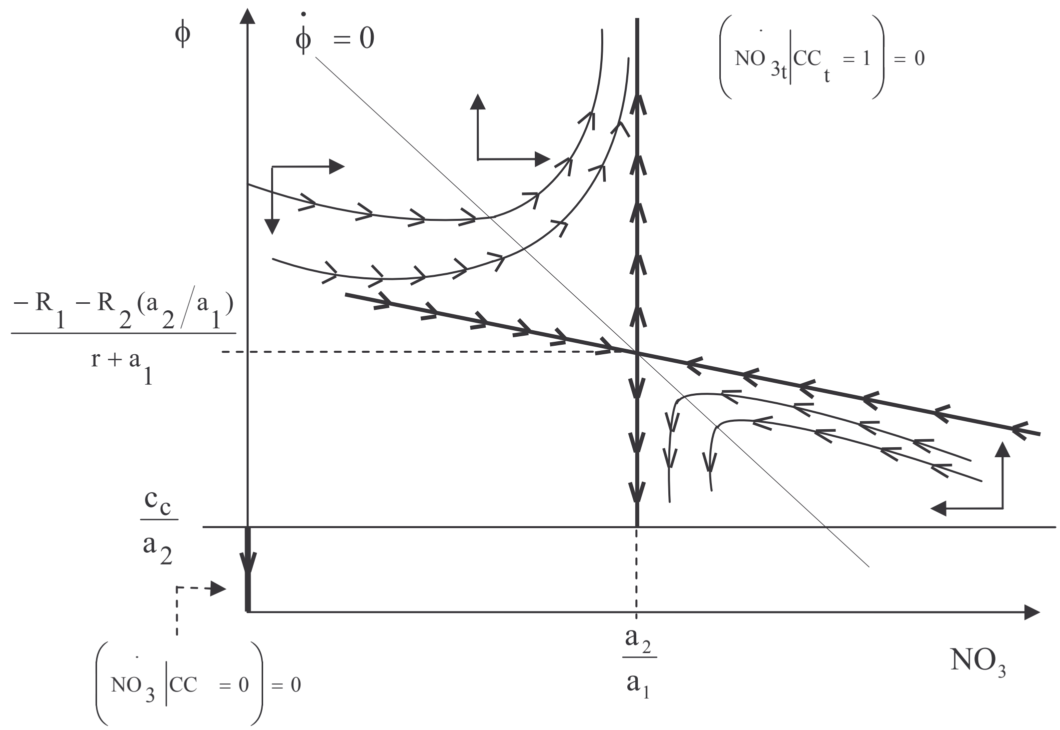


Figure 4 contains the phase diagram for case (iii) and is drawn with information from Section 4.2. In the graph below it may be seen that if the price of the cash crops and cost of chemical nitrogen are relatively smaller than the cost of cover crops, and the cover cropping increases the stock of soil nitrates, then manager will always plant cover crops. The costate is always above the switch point and the stock of soil nitrate converges to a_2/a_1. The stable solution exists at the fixed point and may be reached if the manager is on the stable manifold.

4.5 REVIEWING THE PROPOSITIONS

We begin with a briefly recap of solutions to the static propositions.

Static Proposition I: If the cost of chemical nitrogen is sufficiently low then profit maximizing farm managers will not plant cover crops in crop rotations in the short-run.

Static Proposition II: The manager who observes that soil nitrates levels are low will compensate for the low soil fertility by applying larger applications of chemical nitrogen, if the cost of nitrogen fertilizer is lower.

Dynamic Proposition I: There exists a cost of chemical nitrogen sufficiently low that profit maximizing farm managers will not plant cover crops in the long-run.

Assume for the moment that cover crops are planted. In which case take the partial derivative of the left hand side of equation (43) with respects to the cost of nitrogen (c_N),

$$\frac{\partial LHS}{\partial c_N} = \frac{z_{12}}{(r + a_1)z_{11}} > 0. \tag{44}$$

This derivative indicates that as the cost of nitrogen decreases then the right hand side of equation (43) decreases. So as the cost of nitrogen decreases at some point this will ensure that the fixed point falls below the cost of cover cropping, and so it is relatively undesirable to plant cover crops. Furthermore, if the cost of nitrogen is low relative to a the price of the cash crop, then fixed point will be less than the cost of cover cropping (the switch point). By Figure 4 it may be seen that in this case the manager is less likely to plant cover crop.

Dynamic Proposition II: Even if it is unprofitable to plant cover crops in the short-run it may be profitable to plant cover crops when maximizing long-run profits.

Return to the static solution and substitute equation (8) into the switch condition equation (6a).

$$\alpha_2 - 2z_{12}\left(\frac{1}{2z_{11}}\left(\alpha_1 - 2z_{12}NO_{3_t} - \frac{c_N}{p}\right)\right) - 2z_{22}NO_{3t} \begin{matrix} > \\ = \\ < \end{matrix} \frac{c_c}{pa_2}$$

$$\alpha_2 - \left(\frac{z_{12}}{z_{11}}\left(\alpha_1 - \frac{c_N}{p}\right)\right) + 2NO_{3t}\left(\frac{z_{12}}{z_{11}}z_{12} - z_{22}\right) \begin{matrix} > \\ = \\ < \end{matrix} \frac{c_c}{pa_2}$$

$$\alpha_2 - \left(\frac{z_{12}}{z_{11}}\left(\alpha_1 - \frac{c_N}{p}\right)\right) - 2\frac{NO_{3t}}{z_{11}}\left(z_{22}z_{11} - z_{12}^2\right) \overset{>}{\underset{<}{=}} \frac{c_c}{pa_2} \tag{45}$$

Next substitute equation (7) the equation that defines the stock of soil nitrates into equation (45). Recall that from this equation it may be concluded that cover crops are only planted if the value marginal product of soil nitrates are greater than the cost of cover cropping. Assume for the moment that cover crops are not planted and so the switch condition holds with equality.

$$\alpha_2 - \left(\frac{z_{12}}{z_{11}}\left(\alpha_1 - \frac{c_N}{p}\right)\right) - \frac{2(1-a_1)NO_{3t-1}}{z_{11}}\left(z_{22}z_{11} - z_{12}^2\right) = \frac{c_c}{pa_2} \tag{46}$$

Where NO_{3t-1} is the initial stock of soil nitrates and is the identical to NO_{30} in the LRMD model. From equation (46) it may be asserted that,

$$\alpha_2 - \frac{z_{12}}{z_{11}}\left(\alpha_1 - \frac{c_N}{p}\right) > \frac{c_c}{pa_2}. \tag{47}$$

Equation (42) defines the condition in the LRMD model the manager is less likely to plant cover crops, as the cost of nitrogen and the price of the crop are relatively low. Recall this equation,

$$\frac{p\alpha_2}{(r+a_1)} - \frac{z_{12}}{(r+a_1)z_{11}}\left[p\alpha_1 - c_N\right] < \frac{c_c}{a_2}. \tag{42}$$

However, as equation (42) is true when it is less likely that cover crops are planted in the short-run. It must be that in the long-run model we are at either case (ii) or (iii), equations (41) and (42) respectively. So it is possible that the manager will switch to cover cropping in the long-run when they were not adopted in the short-run.

Dynamic Proposition III: Applying restrictions on the maximum applications of chemical nitrogen (increases the value marginal product of cover crops) at low nitrogen costs.

First consider the SRSP model. If the maximum application of chemical nitrogen is restricted, at $N_t = \overline{N}$, below the optimal application of chemical nitrogen. Differentiate the static switch condition equation (6a) with respects to the application of chemical nitrogen. To determine the impact of reducing the application of chemical nitrogen on the decision to plant cover crops. Recall equation (6a) is,

$$\frac{\partial \Pi(.)}{\partial CC_t} = p\left(\alpha_2 a_2 - 2z_{12} a_2 N_t - 2z_{12} a_2 NO_{3t}\right) \begin{matrix} > \\ = \\ < \end{matrix} c_c \tag{6a}$$

$$\frac{\partial LHS}{\partial N_t} = -2pz_{12} a_2 \tag{48}$$

The derivative in equation (48) is negative, therefore if the application of chemical nitrogen decreases then the value marginal product of soil nitrates increases and the relative switch point for planting cover crops declines. It may be concluded that in the short-run restricting the maximum applications of chemical nitrogen below the optimal application of chemical nitrogen, increases the value marginal product of soil nitrates and reduces the switch point where the manager will plant cover crops.

Next consider the LRMD problem if the maximum application of chemical nitrogen is set at $N_t = \overline{N}$, below the optimal application of chemical nitrogen. Recall equation (19a)

$$\dot{\phi}_t(.) = (r + a_1)\phi_t - p\left(\alpha_2 - 2z_{12} N_t - 2z_{22} NO_{3t}\right) \tag{19a}$$

Differentiating equation (19a) with respects to the application of chemical nitrogen

$$\frac{\partial \dot{\phi}_t(.)}{\partial N_t} = 2pz_{12}.$$

This equation states that decreasing the application of chemical nitrogen decreases the value of $\dot{\phi}_t(.)$ and only results if the stock of soil nitrates is increasing and cover crops are planted.

5. CONCLUSIONS

With a growing interest in sustainable agriculture this paper attempts to derive theoretical conditions that define when and how cover cropping is used. We are aware of the simplifications required to obtain analytical theoretical results for this problem, but think that theoretical conclusions may be adapted to other systems of sustainable agriculture systems and more realistic production function specifications. The results demonstrate the importance of analyzing sustainable agriculture as producing capital assets. We believe that the resulting conditions explain observed farm managers' behavior more accurately than static myopic approaches. Such models can be used to provide theoretical foundation for an empirical model of cover crops which is developed in associated research. In addition, the theoretical model suggests that policies such as the recently introduced TMDL regulations on nitrogen runoff will influence the decisions to adopt cover crops.

6. REFERENCES

Kamien, M.I., and N.L. Schwartz., *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*, 2nd Edition, Elsevier Applied Science Publishers, 2000.

Léonard, D., and N.V. Long., *Optimal Control Theory and Static Optimization in Economics*, Cambridge University Press, 1998.

Caputo, M., *Foundations of Dynamic Economic Analysis : Optimal Control Theory and Applications,* Cambridge University Press, 2005.