# A Bayesian Implementation of the Standard Optimal Hedging Model: Parameter Estimation Risk and Subjective Views

by

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**Abstract**

We propose a Bayesian implementation of the standard optimal hedging model that effectively and practically accommodates estimation errors and subjective views regarding both the expectation vector and the covariance matrix of asset returns. Numerical examples show that subjective views have a substantial impact on a hedger's optimal position and that the impact of views regarding the direction of future price changes far outweighs that of views regarding the standard deviation of future price changes.

**keywords**: optimal hedging, parameter estimation risk, subjective views, Bayesian Decision Theory.

**A Bayesian Implementation of the Standard Optimal Hedging Model: Parameter Estimation Risk and Subjective Views**

The standard optimal hedging model (Johnson, 1960; Stein, 1961; Anderson and Danthine, 1980) has been the preferred theoretical model of normative hedging behavior for some time. In empirical applications, the model is often implemented with a Parameter Certainty Equivalent (PCE) procedure, which directly substitutes sample estimates for the model's parameters in determining the optimal hedging position. However, the PCE procedure completely ignores parameter estimation risk, i.e., the estimation errors in the expectation vector and covariance matrix of returns on the assets involved in the hedging decision. Furthermore, the PCE procedure cannot accommodate hedgers' subjective views, which refer to hedgers' opinions ("views") regarding the direction of market returns of the assets involved in hedging decisions.

The problem of decision making in the presence of parameter uncertainty has long been recognized and has been analyzed in a Bayesian decision theory framework. Within a portfolio optimization context, the Bayesian framework has been used to accommodate parameter estimation risk and subjective views (e.g., Brown, 1979; Jorion, 1985, 1986; Frost and Savarino, 1986; Black and Litterman, 1990, 1992; Polson and Tew, 2000; Pastor, 2000). Since optimal hedging can be considered a special case of portfolio optimization, the Bayesian framework can be applied to optimal hedging to accommodate the problems identified with the PCE procedure. In the first formal hedging applications, Lence and Hayes (1994a,b) develop a Bayesian optimal hedging model that can effectively accommodate parameter estimation risk. However their model can only accommodate subjective views under restrictive and unrealistic assumptions due to its underlying "pure" Bayesian approach. The pure Bayesian approach requires hedgers to calibrate their prior distribution with non-sample information including subjective views. However, in practice most hedgers are unlikely to have subjective views on more than one or two parameters of the prior distribution or only the relative relation of the parameters of the prior distribution. Thus, it is unlikely that hedgers can calibrate the entire prior distribution with only non-sample information.

Shi and Irwin (2005) argue that the Bayesian framework should be implemented with an "empirical" Bayesian approach when applied to optimal hedging. The reason is that with an empirical Bayesian approach hedgers calibrate the prior distribution with sample data, which, compared with non-sample information, should contain enough information regarding all the parameters of the prior distribution. Furthermore, with an empirical Bayesian approach the number and type of subjective views that hedgers can express is quite flexible. For example, hedgers can have one or more subjective views that may be in the form of "absolute" or "relative" views regarding expected asset returns. However, Shi and Irwin (2005) only consider estimation risk and subjective views regarding the expectation vector of asset returns, ignoring those regarding the covariance matrix of asset returns.

Empirical work clearly indicates that time-varying volatility prevails in many economic and financial time series, and conditional volatility models such as GARCH/ARCH-type models have been widely used in estimating the volatilities and correlations of asset returns. In addition, other methods such as implied volatility, factor models, exponential weighting methods and Bayesian shrinkage estimators have also been developed to estimate the covariance matrix. Given the large array of estimation methods and potentially different data sets available, hedgers may obtain estimates of volatility and correlation quite different from those obtained by most other market participants (market consensus). The differences in covariance matrix estimates may have a significant impact on hedgers' optimal hedging positions. For example, Myers (1991), Lien and Luo (1994), and Kroner and Sultan (1993) model the behavior of spot and futures prices with bivariate GARCH models and propose various dynamic hedging strategies.

In this study, we propose a Bayesian implementation of the standard optimal hedging model that accommodates estimation risk and subjective views regarding both the expectation vector and the covariance matrix of asset returns. The Bayesian framework is applied to optimal hedging with an empirical Bayesian approach in order to accommodate subjective views in a practical and realistic manner. Compared with Bayesian models proposed in previous studies, the new Bayesian model solves the problems identified with the PCE procedure in a more satisfactory and complete manner.

**Theoretical Model**

To begin, we assume that (1) the hedger has no other investment opportunities and does not borrow or lend, (2) markets are frictionless, which means no commissions, no margin requirements and no lumpiness due to standardization of futures contracts, (3) asset price changes follow a multivariate normal distribution, (4) the hedger has a long position in the spot ("cash") market and hedges the spot position with a futures contract matching the hedging horizon, and (5) the hedger's objective is to maximize a mean-variance function of his/her end-period profit/loss.[1] These simplifying assumptions are made here for ease of exposition and explicitness of results when these assumptions do not impair the features of the model that we seek to emphasize.

The multivariate normality assumption for spot and futures price changes is denoted as:

$$\mu = \begin{bmatrix} \mu_s \\ \mu_f \end{bmatrix} \text{ and } \mathbf{\Sigma} = \begin{bmatrix} \sigma_s^2 & \sigma_{sf} \\ \sigma_{fs} & \sigma_f^2 \end{bmatrix} \tag{1}$$

where $\mu$ and $\mathbf{\Sigma}$ are the expectation vector and the covariance matrix of price changes, respectively. Subscripts $s$ and $f$ denote spot and futures, respectively. Then, the mean-variance maximization of the hedger's end-period profit/loss is:

$$\max_{Y_f} \begin{bmatrix} Y_s & Y_f \end{bmatrix} \begin{bmatrix} \mu_s \\ \mu_f \end{bmatrix} - \frac{\tau_a}{2} \begin{bmatrix} \sigma_s^2 & \sigma_{sf} \\ \sigma_{fs} & \sigma_f^2 \end{bmatrix} \tag{2}$$

where $\tau_a$ is the absolute Arrow-Pratt risk aversion coefficient and $Y_s$ and $Y_f$ are the hedger's spot and futures positions, respectively. The optimal hedging position is determined via the first order condition (FOC) of the mean-variance maximization:

$$Y_f^* = \frac{\mu_f}{\tau_a \sigma_f^2} - Y_s \frac{\sigma_{sf}}{\sigma_f^2} \tag{3}$$

where the first and second terms are the speculative and the pure hedging components of the optimal hedging position, respectively.

In practice, the standard optimal hedging model is often implemented with the Parameter Certainty Equivalence (PCE) procedure, which directly substitutes sample estimates for the true but unknown parameters. Thus, the optimal hedging position according to the PCE procedure is:

$$Y_f^* = \frac{\hat{\mu}_f}{\tau_a \hat{\sigma}_f^2} - Y_s \frac{\hat{\sigma}_{sf}}{\hat{\sigma}_f^2} \tag{4}$$

[1] Myers and Thompson (1989) show that optimal hedging positions (ratios) can be determined via mean-variance maximization of the end-period wealth, profit/loss (price change in units of the underlying asset), and portfolio rate of return. They argue that the difference in maximization objectives depends on different underlying assumptions of the model concerning the stochastic process of asset prices.

where $\hat{\mu}_f$,$\hat{\sigma}_f^2$ and $\hat{\sigma}_{sf}$ are respectively, the sample estimates of $\mu_f$, $\sigma_f^2$ and $\sigma_{sf}$.

**The Bayesian Implementation**

The Bayesian portfolio optimization framework only requires that optimal hedging positions be determined via mean-variance optimization conditioned on the predictive expectation vector and covariance matrix of asset returns. When it is implemented, the framework can be customized to fit specific problems and goals. For example, researchers may choose different prior distributions and/or implement the framework with either a pure Bayesian or empirical Bayesian approach depending on their goals. To accommodate the problems identified with the PCE procedure, we implement the Bayesian framework with an empirical Bayesian approach and consider estimation risk and subjective views regarding both the expectation vector and covariance matrix of asset returns.

To incorporate estimation errors regarding both the expectation vector and covariance matrix of asset returns, the prior is specified as a normal-inverse-Wishart distribution, which is conveniently parameterized in terms of hyperparameters

$$\mu|\mathbf{\Sigma} \sim N(\mu_{\mathbf{0}}, \kappa_0^{-1}\mathbf{\Sigma}), \text{ where } \mathbf{\Sigma} \sim \mathbf{W}_{\nu_{\mathbf{0}}}^{-\mathbf{1}}(\mathbf{\Sigma_0}) \tag{5}$$

where $\mu$ follows a multivariate normal distribution conditional on $\mathbf{\Sigma}$, which follows an inverse-Wishart distribution denoted by $\mathbf{W}^{-\mathbf{1}}$. Notice that $\kappa_0$ measures the confidence level associated with the expectation $\mu_{\mathbf{0}}$, with a larger $\kappa_0$ implying a higher confidence level, and $\nu_0$ measures the confidence level associated with the covariance matrix $\mathbf{\Sigma_0}$, with again a larger $\nu_0$ implying a higher confidence regarding the covariance matrix estimate. The prior distribution is a hierarchical and conjugate distribution because the probability distribution of $\mu$ is conditioned on $\mathbf{\Sigma}$ and the posterior distribution is also a normal-inverse-Wishart distribution (Gelman et al., 2004, p.87-88)

The hedger's possible subjective views regarding expected asset returns are specified as in Black and Litterman (1990, 1992) and Shi and Irwin (2005). Since the number of the views must be equal to or smaller than the number of assets, the maximum number of views allowed is two within the context of the standard optimal hedging model because only two assets are considered. In practice hedgers often have just one view regarding the expectation vector. The view is denoted by:

$$\mathbf{P}\mu \sim N_1(\mathbf{q}, \kappa_1^{-1}\mathbf{P\Sigma P}^{'}) \tag{6}$$

where $\mathbf{P}$ is a $1 \times n$ "weight" matrix, each row of which designates the weights of the assets in the "view" portfolio and $\mathbf{q}$ is a quantity that specifies the expected return of the "view" portfolio. The term $\kappa_1^{-1}\mathbf{P\Sigma P}^{'}$ measures the confidence level of the subjective view. The scalar $\kappa_1$ calibrates the confidence level of the subjective view against the matrix $\mathbf{P\Sigma P}^{'}$. Equation (6) denoting the subjective view can be interpreted as a regression model, where the dependent variable is $\mathbf{q}$, the explanatory variable is $\mathbf{P}$ and $\mathbf{u}$ is the slope coefficient. Since the regression model is under-determined due to $1 < n$, the least square solution of the regression model is $\mu_1 = \mathbf{P}^{'}(\mathbf{PP}^{'})^{-1}q$ and its volatility as $\kappa_1\mathbf{P}^{'}(\mathbf{P\Sigma P'})^{-1}\mathbf{P}$. This alternative interpretation of the view is implied in Black and Litterman (1990, 1992).

The hedger's possible subjective views regarding the covariance matrix of asset returns are denoted by:

$$\mathbf{\Sigma} \sim \mathbf{W}_{\nu_1}^{-1}(\mathbf{\Sigma_1}) \tag{7}$$

where $\mathbf{\Sigma_1}$ is the view and $\nu_1$ measures the confidence level of the view. We assume that $\mathbf{\Sigma_1}$ is in the form of $\mathbf{D\Omega D}^{'}$, where $\mathbf{D}$ is the diagonal matrix of standard deviations of asset returns and $\mathbf{\Omega}$ is the correlation matrix of asset returns. This decomposition of the covariance matrix makes it possible for hedgers to express their views separately on the standard deviations of asset returns and correlations across the returns yet preserve the positive semi-definitiveness of the covariance matrix (Litterman, 2003).

We combine the prior distribution with subjective views using the Bayesian updating method outlined in Gelman et al. (2004, p.87-88). Because an empirical Bayesian approach is adopted, the prior distribution is calibrated with sample data while subjective views are treated as new information. The Bayesian updating combines the two sources of information into the posterior distribution, which is also a normal-inverse-Wishart distribution due to the conjugateness of the prior distribution. The resulting posterior distribution is parameterized in terms of hyperparameters as

$$\begin{aligned}
\kappa_2 \boldsymbol{\mu_2} &= \kappa_0 \mu_0 + \kappa_1 \boldsymbol{\mu_1} \\
\kappa_2 &= \kappa_0 + \kappa_1 \\
\nu_2 \boldsymbol{\Sigma}_2 &= \nu_0 \boldsymbol{\Sigma}_0 + \nu_1 \boldsymbol{\Sigma}_1 + \frac{\kappa_0 \kappa_1}{\kappa_0 + \kappa_1} (\mu_1 - \mu_0)(\mu_1 - \mu_0)' \\
\nu_2 &= \nu_0 + \nu_1
\end{aligned} \tag{8}$$

where $\boldsymbol{\mu_2}$ and $\kappa_2$ are, respectively, the posterior expectation vector conditioned on the covariance matrix $\boldsymbol{\Sigma}$ and the confidence level attached to it, and $\boldsymbol{\Sigma_2}$ and $\nu_2$ are, respectively, the posterior covariance matrix and the associated confidence level. Finally, the predictive probability distribution is computed as the integration of the product of the likelihood function of asset returns (equation 1) and the posterior distribution (equation 5) over the uncertain parameters ( $\mu$ and $\boldsymbol{\Sigma}$). The predictive distribution is a Student-t distribution specified as:

$$\mathbf{r_{t+1}} | \mathbf{I_t}, \mathbf{X_t} \sim t_{\nu_2 - d + 1} \left( \boldsymbol{\mu_2}, \frac{(\kappa_2 + 1)\nu_2 \boldsymbol{\Sigma_2}}{\kappa_2(\nu_2 - d + 1)} \right) \tag{9}$$

with the following predictive expectation vector and the predictive covariance matrix:[2]

$$\begin{aligned}
E(\mathbf{r}_{t+1} | \mathbf{I}_t, \mathbf{X}_t) &= \mu_2 \\
\mathrm{Cov}[\mathbf{r}_{t+1} | \mathbf{I}_t, \mathbf{X}_t] &= \frac{(\kappa_2 + 1)\nu_2}{\kappa_2(\nu_2 - d - 1)} \boldsymbol{\Sigma}_2
\end{aligned} \tag{10}$$

and $\mathbf{I}_t$ and $\mathbf{X}_t$ denoting, respectively, the prior and the views (new information). We can then substitute the elements of the predictive expectation vector and the predictive covariance matrix for the model's parameters and determine the optimal hedging position.

### Optimal Hedging Applications

In the first scenario, we analyze the impact on the optimal hedging position of a single directional view regarding the expected change of the futures price. The single directional view regarding the expected change of the futures price is expressed with equation (6), where $P = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $q = [q]$. In addition, we assume that there is no view regarding the covariance matrix, i.e., $\nu_1 = 0$.

[2] Gelman et al. (2004, p.576-577) provide an account of the expectation vector and covariance matrix of the multivariate Student-t distribution.

With the view input into the Bayesian model above, we obtain the predictive expectation vector and the predictive covariance matrix as

$$
\begin{aligned}
E\left[\mathbf{r_{t+1}}|\mathbf{I_t},\mathbf{X_t}\right] =& \frac{\kappa_0}{\kappa_0+\kappa_1}\mu_0 + \frac{\kappa_1}{\kappa_0+\kappa_1}\mu_1 = \begin{bmatrix} \frac{\kappa_0}{\kappa_0+\kappa_1}\mu_{s0} + \frac{\kappa_1}{\kappa_0+\kappa_1}\cdot 0 \\ \frac{\kappa_0}{\kappa_0+\kappa_1}\mu_{f0} + \frac{\kappa_1}{\kappa_0+\kappa_1}\cdot q \end{bmatrix} \\
\mathrm{Cov}\left[\mathbf{r_{t+1}}|\mathbf{I_t},\mathbf{X_t}\right] =& \left(\frac{\nu_0}{\nu_0-3}\cdot\frac{\kappa_0+\kappa_1+1}{\kappa_0+\kappa_1}\right) \\
&\times\left(\mathbf{\Sigma_0} + \frac{\kappa_0\kappa_1}{(\kappa_0+\kappa_1)\nu_0}\begin{bmatrix} \mu_{s0}^2 & -\mu_{s0}(q-\mu_{f0}) \\ -\mu_{s0}(q-\mu_{f0}) & (q-\mu_{f0})^2 \end{bmatrix}\right)
\end{aligned} \tag{11}
$$

where the predictive expectation vector is an average of the prior ($\mu_\mathbf{0}$) and subjective view ($\mu_\mathbf{1}$) weighted by their corresponding precisions ($\kappa_0$ and $\kappa_1$) and the predictive covariance matrix is the sum of the prior information ($\mathbf{\Sigma_0}$) and a correction term measuring the difference between the subjective view and the prior regarding the expectation vector($\mu_1 - \mu_0$).

With the elements of the predictive expectation vector and the predictive covariance matrix plugged into the standard optimal hedging model (equation 3), we obtain the optimal hedging position according to the Bayesian procedure as:

$$
\hat{Y}_f^* = \frac{\frac{\kappa_0}{\kappa_0+\kappa_1}\hat{\mu}_{f0} + \frac{\kappa_1}{\kappa_0+\kappa_1}q}{\tau_a\left[\frac{\nu_0}{\nu_0-3}\cdot\frac{\kappa_0+\kappa_1+1}{\kappa_0+\kappa_1}\right]\left[\hat{\sigma}_{f0}^2 + \frac{\kappa_0\kappa_1}{(\kappa_0+\kappa_1)\nu_0}(q-\hat{\mu}_{f0})^2\right]} - Y_s\frac{\hat{\sigma}_{sf0} - \frac{\kappa_0\kappa_1}{(\kappa_0+\kappa_1)\nu_0}\hat{\mu}_{s0}(q-\hat{\mu}_{f0})}{\hat{\sigma}_{f0}^2 + \frac{\kappa_0\kappa_1}{(\kappa_0+\kappa_1)\nu_0}(q-\hat{\mu}_{f0})^2} \tag{12}
$$

This result shows that the single directional view has an impact not only on the speculative component but also on the pure hedging component of the optimal hedging position. In Shi and Irwin's (2005, equation 23) the directional view only has an impact on the speculative component. The reason for the difference is that the changes in the elements of the covariance matrix due to the single view are no longer proportional as in Shi and Irwin's model.

In the second scenario, we analyze the impact on the optimal hedging position of a single view regarding the standard deviation of futures price changes. The hedger expresses this single view using equation (7). We assume the hedger's view can be a decreasing (increasing) view, which means decreasing (increasing) the standard deviation. We further assume no view regarding the expectation vector.

With the view input into the Bayesian model, we then obtain the predictive expectation vector and the predictive covariance matrix as:

$$\begin{aligned} E(\mathbf{r_{t+1}}|\mathbf{I_t}, \mathbf{X_t}) &= \mu_{\mathbf{0}} \\ \mathrm{Cov}[\mathbf{r_{t+1}}|\mathbf{I_t}, \mathbf{X_t}] &= \left(\frac{\nu_0+\nu_1}{\nu_0+\nu_1-3}\cdot\frac{\kappa_0+1}{\kappa_0}\right)\left(\frac{\nu_0}{\nu_0+\nu_1}\mathbf{\Sigma_0}+\frac{\nu_1}{\nu_0+\nu_1}\mathbf{\Sigma_1}\right) \end{aligned} \tag{13}$$

where the predictive covariance matrix is an average of the prior information and the view weighted by their respective confidence levels. Similar to the first example, with the elements of the predictive expectation vector and the predictive covariance matrix plugged into the standard optimal hedging model (3), we obtain the optimal hedging position according to the Bayesian procedure as:

$$\hat{Y}_f^* = \frac{\hat{\mu}_{f0}}{\tau_a\left(\frac{\nu_0+\nu_1}{\nu_0+\nu_1-3}\cdot\frac{\kappa_0+1}{\kappa_0}\right)\left(\frac{\nu_0}{\nu_0+\nu_1}\hat{\sigma}_{f0}^2+\frac{\nu_1}{\nu_0+\nu_1}\hat{\sigma}_{f1}^2\right)} - Y_s\frac{\frac{\nu_0}{\nu_0+\nu_1}\hat{\sigma}_{sf0}}{\frac{\nu_0}{\nu_0+\nu_1}\hat{\sigma}_{f0}^2+\frac{\nu_1}{\nu_0+\nu_1}\hat{\sigma}_{f1}^2} \tag{14}$$

where $\hat{\sigma}_{f0}^2$ and $\hat{\sigma}_{f1}^2$ are, respectively, the sample estimate of the volatility of futures price changes and the hedger's view on volatility. The equation (13) shows that the standard deviation view enters the optimal hedging position through the predictive covariance matrix and has an impact on both the speculative component and the pure hedging component of the optimal position.

In the third scenario, we combine together the view scenarios analyzed in the previous two scenarios and consider how a hedger's optimal position should vary when he/she has a directional view as well as a view regarding the standard deviation of futures price changes. In this scenario, we investigate the interplay of those two views in determining a hedger's optimal position.[3]

With the view input into the Bayesian model, we then obtain the predictive expectation

[3]It is possible that a hedger may have other types of views, for example, the hedger may have a directional view regarding the expected change of futures price, or a view regarding the correlation between the futures price change and spot price change. The impact of those views on optimal hedging position can also be analyzed within the theoretical framework proposed in this paper, however is not presented for the sake of brevity.

vector and the predictive covariance matrix as:

$$
\begin{aligned}
E\left[\mathbf{r_{t+1}}|\mathbf{I_t},\mathbf{X_t}\right] &= \begin{bmatrix} \frac{\kappa_0}{\kappa_0+\kappa_1}\mu_{s0} \\ \frac{\kappa_0}{\kappa_0+\kappa_1}\mu_{f0}+\frac{\kappa_1}{\kappa_0+\kappa_1}q \end{bmatrix} \\
\text{Cov}\left[\mathbf{r_{t+1}}|\mathbf{I_t},\mathbf{X_t}\right] &= \left(\frac{\nu_0+\nu_1}{\nu_0+\nu_1-3}\cdot\frac{\kappa_0+\kappa_1+1}{\kappa_0+\kappa_1}\right) \\
&\times\left(\frac{nu_0}{nu_0+nu_1}\mathbf{\Sigma_0}+\frac{nu_1}{nu_0+nu_1}\mathbf{\Sigma_1}\right. \\
&+\left.\frac{\kappa_0\kappa_1}{(\kappa_0+\kappa_1)}\cdot\frac{1}{\nu_0+\nu_1}\begin{bmatrix} \mu_{s0}^2 & -\mu_{s0}(q-\mu_{f0}) \\ -\mu_{s0}(q-\mu_{f0}) & (q-\mu_{f0})^2 \end{bmatrix}\right)
\end{aligned}
\tag{15}
$$

Similarly, with the elements of the predictive expectation vector and the predictive covariance matrix plugged into the standard optimal hedging model (3), we obtain the optimal hedging position according to the Bayesian procedure as:

$$
\begin{aligned}
\hat{Y}_f^* =& \frac{\frac{\kappa_0}{\kappa_0+\kappa_1}\hat{\mu}_{f0}+\frac{\kappa_1}{\kappa_0+\kappa_1}q}{\tau_a\cdot\frac{\nu_0+\nu_1}{\nu_0+\nu_1-3}\cdot\frac{\kappa_0+\kappa_1+1}{\kappa_0+\kappa_1}\left[\frac{\nu_0}{\nu_0+\nu_1}\hat{\sigma}_{f0}^2+\frac{\nu_1}{\nu_0+\nu_1}\hat{\sigma}_{f1}^2+\frac{\kappa_0\kappa_1}{(\kappa_0+\kappa_1)}\cdot\frac{1}{\nu_0+\nu_1}(q-\hat{\mu}_{f0})^2\right]} \\
&-Y_s\frac{\frac{\nu_0}{\nu_0+\nu_1}\hat{\sigma}_{sf0}+\frac{\nu_1}{\nu_0+\nu_1}\hat{\sigma}_{sf1}-\frac{\kappa_0\kappa_1}{(\kappa_0+\kappa_1)}\cdot\frac{1}{\nu_0+\nu_1}\hat{\mu}_{s0}(q-\hat{\mu}_{f0})}{\frac{\nu_0}{\nu_0+\nu_1}\hat{\sigma}_{f0}^2+\frac{\nu_1}{\nu_0+\nu_1}\hat{\sigma}_{f1}^2+\frac{\kappa_0\kappa_1}{(\kappa_0+\kappa_1)}\cdot\frac{1}{\nu_0+\nu_1}(q-\hat{\mu}_{f0})^2}
\end{aligned}
\tag{16}
$$

where $\hat{\sigma}_{sf1}$ is the estimated covariance of price changes of futures and spot and is computed with equation (7). Notice that the formula above is a combination of the results of the previous two scenarios.

**Numerical Examples**

We illustrate the impact of the views on optimal hedging positions using numerical examples. The hedging setup is adopted from Shi and Irwin (2005). Specifically, on February 19, 2004 in the North Central region of Illinois, a farmer hedger has a long spot position in corn and expects to offset the position on July 8, 2004, 20 weeks later. To reduce price risk exposure, the farmer wants to hedge the spot position with Chicago Board of Trade (CBOT) July 2004 corn futures. We also assume that: (1) the hedger is a farmer who produces corn on 500 acres of land and has a net wealth of \$662,752, (2) his/her relative risk-aversion coefficient is 4,[4] and (3) the yield is assumed to be 150 bushels/acre, thus the hedger has a long spot

[4]We also compute the optimal hedging positions for the case where the hedger's relative risk aversion coefficient is either 2 or 6. The results suggest that when the hedger is less (more) risk-averse, his/her speculative component becomes bigger (smaller) correspondlinly, nevertheless, the hedger still should adjust

position of 75,000 bushels of corn.

The expectation vector and covariance matrix of the spot and futures price changes (in dollar per bushel) over the 20-week holding periods are estimated using historical data[5],

$$\begin{bmatrix} \mu_s \\ \mu_f \end{bmatrix} = \begin{bmatrix} 0.1089 \\ 0.0005 \end{bmatrix} \text{ and } \begin{bmatrix} \sigma_s^2 & \sigma_{sf} \\ \sigma_{fs} & \sigma_f^2 \end{bmatrix} = \begin{bmatrix} 0.1792 & 0.1751 \\ 0.1751 & 0.1948 \end{bmatrix}. \tag{17}$$

With these parameter estimates, we calibrate the prior distribution (equation 5) and assign $\kappa_0$ and $\nu_0$ equal to 10 similar to that in Shi and Irwin (2005).

We assume that the hedger may have a single directional view and/or a single standard deviation view regarding the futures price change ($\mu_f$). The single direction view is either a bullish view (+0.10\$/bushel) or a bearish view (-0.10\$/bushel) and is coupled with either a high confidence level ($\kappa_1 = 8$) or a low confidence level ($\kappa_1 = 2$).[6] The case of no view regarding expectation vector is considered for comparison. Similarly, the single standard deviation view is either a deceasing or increasing view, which means decreasing the standard deviation of futures price change by 50% or increasing it by 100%, and is coupled with either a high confidence level ($\nu_1 = 8$) or a low confidence level ($\kappa_1 = 2$). The case of no view regarding the covariance matrix is considered for comparison. Consequently, the different combinations of directional view and standard deviation view yield 25 different view scenarios.

The results (table 1) shows that subjective views have a substantial impact on a hedger's optimal position and that the impact of a directional view far outweighes that of a standard deviation view because the directional view can substantially alter the speculative component and thus a hedger's overall optimal position. For example, if the hedger has a bullish (bearish) directional view coupled with a high confidence level and has no view regarding the covariance matrix, the the speculative component of the hedger's optimal position is

his/her optimal hedging position according his/her subjective view. Theses results are available from the authors upon request.

[5]The historical dataset consists of weekly spot corn prices (Thursdays) for the North Central region of Illinois during the post-harvest periods from 1976 to 2003 and corresponding settlement prices of CBOT July corn futures. The data are obtained from the AgMAS Project at University of Illinois at Urbana-Champaign. For illustration purposes, we estimate the means and covariance matrix using standard sample estimation methods, even though more sophisticated econometric methods could be implemented. For each year during 1976 through 2003, we observe a pair of spot and futures price changes over the 20-week hedging period, thus the sample size is 28.

[6]The directional view is not extreme given the fact that the magnitude of the bullish (bearish) view is about one quarter of the estimated standard deviation of the futures price changes.

24,668 ($-24,351$) bushels and the pure hedging component is $-64,113$ (-67,727) bushels. In contrast, if the hedger has no view regarding the expectation vector but has a decreasing (increasing) view regarding the standard deviation of futures price change and coupled with a high confidence level, then the speculative component of the hedger's position is 384 (297) bushels and the pure hedging component is $-73,122$ ($-61,401$) bushel. In another example, when the directional view is bullish with a high confidence level, the optimal hedging positions are $-39,445$ bushels when there is no view regarding the covariance matrix, and the optimal hedging position is $-35,613$ ($-35,053$) when there is a decreasing (increasing) view regarding the standard deviation of the futures price change that is associated with a high confidence level. In contrast, when the directional view is bearish with a high confidence level, the optimal hedging positions are $-92,078$ bushels when there is no view regarding the covariance matrix, and the optimal hedging position is $-108,020$ ($-86,155$) when there is a decreasing (increasing) view regarding the standard deviation of the futures price change that is associated with a high confidence level.

**Conclusions**

In this study, we propose a Bayesian implementation of the standard optimal hedging model that can effectively and practically accommodate subjective views and estimation risk regarding both the expectation vector and the covariance matrix of asset returns. The new model is based on an empirical implementation of the Bayesian portfolio optimization framework. We apply the model to analysis of the impact of subjective views on a hedger's optimal position and consider the optimal hedging positions under three types of view scenarios: a single view regarding the expected change of the futures price, a single view regarding the standard deviation of futures price change, and a combination of the two. Numerical examples show that subjective views have a substantial impact on a hedger's optimal position and that the impact of views regarding the direction of future price changes far outweighs that of views regarding the standard deviation of future price changes. This study provides further evidence on the influence of subjective views on hedging behavior and contributes to explaining the large cross-sectional and time series variation of hedging positions often observed in practice.

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**Table 1. Optimal Hedging Positions under Different Subjective View Scenarios**

| Scenario of Subjective Views | | | | Optimal Hedging Position | | |
|---|---|---|---|---|---|---|
| Directional View | Confidence Level | Standard Deviation View | Confidence Level | Speculative Component | Hedging Component | Optimal Position |
| no view | NA | no view | NA | 271 | -67,415 | -67,145 |
| | | decreasing | high | 483 | -78,651 | -78,168 |
| | | | low | 331 | -70,626 | -70,294 |
| | | increasing | high | 138 | -41,733 | -41,595 |
| | | | low | 193 | -52,434 | -52,241 |
| bullish | high | no view | NA | 24,668 | -64,113 | -39,445 |
| | | decreasing | high | 44,214 | -75,682 | -31,468 |
| | | | low | 30,238 | -67,410 | -37,172 |
| | | increasing | high | 12,801 | -41,071 | -28,270 |
| | | | low | 17,795 | -50,767 | -32,972 |
| | low | no view | NA | 9,310 | -66,160 | -56,850 |
| | | decreasing | high | 16,648 | -77,525 | -60,876 |
| | | | low | 11,405 | -69,403 | -57,999 |
| | | increasing | high | 4,781 | -41,484 | -36,703 |
| | | | low | 6,675 | -51,804 | -45,129 |
| bearish | high | no view | NA | -24,351 | -67,727 | -92,078 |
| | | decreasing | high | -43,648 | -78,700 | -122,350 |
| | | | low | -29,850 | -70,854 | -100,700 |
| | | increasing | high | -12,641 | -41,949 | -54,590 |
| | | | low | -17,570 | -52,799 | -70,368 |
| | low | no view | NA | -8,854 | -67,534 | -76,388 |
| | | decreasing | high | -15,834 | -78,670 | -94,504 |
| | | | low | -10,846 | -70,712 | -81,559 |
| | | increasing | high | -4,547 | -41,814 | -46,362 |
| | | | low | -6,349 | -52,572 | -58,921 |

Note: We assume that the hedger could have a bullish, bearish or no view regarding the expected futures price change. The bullish (bearish) view means that the hedger expects the futures change to be 0.10$/bushel (−0.10$/bushel) instead of 0.0005$/bushel assumed by the market consensus. We assume that the hedger's confidence level of the prior is ten ($\kappa_0$=10), and the hedger could possibly have a high confidence level ($\kappa_1$=8) or a low confidence level ($\kappa_1$=2) associated with the subjective view. We also assume that the hedger could have a decreasing, increasing or no view regarding the standard deviation of the change of futures price. The decreasing (increasing) view means that the hedger projects the standard deviation to be 50% smaller (100% larger) compared with the market consensus. We assume that the hedger's confidence level of the prior is ten ($\nu_0$=10), and the hedger could possibly have a high confidence level ($\nu_1$=8) or a low confidence level ($\nu_1$=2) associated with the subjective view. We assume that the relative risk-averse coefficient equals 4 for this example.