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BOUNDED RECURSIVE STOCHASTIC SIMULATION - a simple and efficient method for pricing complex American type options -

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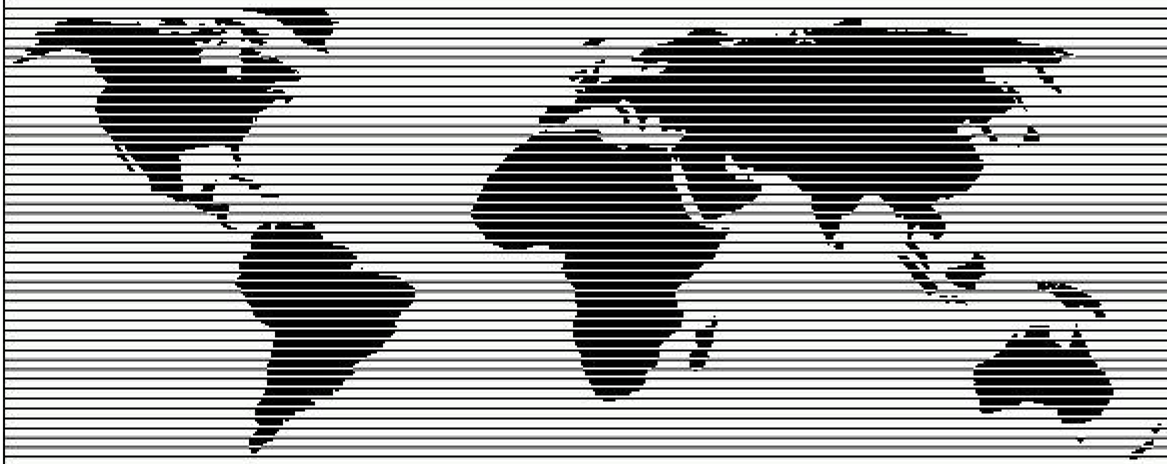

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BOUNDED RECURSIVE STOCHASTIC SIMULATION

– a simple and efficient method for pricing complex American type options –

Summary

This paper gives an overview of simulation based procedures, which have proved to be efficient in valuing American options and therefore real options. Many of them integrate sequential stochastic simulations in the backward recursive programming approach to determine the early-exercise frontier. They subsequently value the option by initiating a Monte-Carlo simulation from the valuation date of the option. It turns out that one approach (Grant et al., 1997) is especially simple. We are able to enhance its efficiency by stripping it of some time consuming but unnecessary simulation steps. Our simplified approach could be called "Bounded Recursive Stochastic Simulation".

1 Introduction

Capital investments represent an equally important and difficult part of entrepreneurial decision-making. On the one hand, the consequences of investment decisions extend far into the future and may determine the strategic position of a firm for a long time. This is due to the fact that any investment is at least partly irreversible. On the other hand, there is always uncertainty with regard to future returns. However, it should be noted that even though (potential) investors run a risk, they also enjoy some sort of flexibility. They have for instance the choice of different dates to carry out an investment. Therefore, in the context of investment planning two questions have to be answered: (1) What is the expected value of an investment? (2) What expected present value of returns should trigger immediate investment?

Traditionally the net present value (NPV) has been used to answer these questions. According to this approach, the value of an investment opportunity is represented by the expected present value of investment cash flows minus investment costs. The investment should be carried out as soon as the investment costs are covered. However, the "New Investment Theory" (or: Real Options Approach) posits that the flexibility to defer an investment may have a positive value. This value will be lost once the investment is carried out. Consequently, this value, in addition to the investment costs, has to be covered by the present value of investment returns[1]. Intuition is quite direct: Given a deferrable investment opportunity, it is not simply a question of whether to invest or not. Rather we have to make a choice between several mutually exclusive opportunities over time. The values of subsequent investment opportunities represent opportunity costs to an immediate investment. These opportunity costs, and therefore the monetary value of the flexibility of waiting, can be quantified by using procedures derived from the pricing of financial options. This is feasible because there is a close analogy between investment opportunities (real options) and financial options: The opportunity to delay an investment can be compared to a call option. The investor has the right, but *not* the obligation to buy a (real) asset at a given "strike price" (i.e. investment costs) and to receive hereby a stochastic value (i.e. expected present value of investment returns). The pe-

[1] Cf. e.g. Dixit and Pindyck (1994), Trigeorgis (1996), McDonald and Siegel (1986) or Amram and Kulatilaka (1999).

riod for which the decision to invest can be delayed can be interpreted as the lifetime of the option. Since investment options can be usually exercised at several dates during their lifetime, real options are usually of American type. Thus the relatively simple valuation of European style options is of minor importance for real options.

One difficulty in applications of the real options approach is caused by the fact that there rarely is a closed form analytical solution for complex investment options. Furthermore, most numerical valuation techniques known from finance option theory are regularly not applicable either. There are two main problems concerning these procedures: Some of them lack sufficient flexibility with regard to the form of stochastic processes, the number of stochastic variables and possible interactions between different real options (analytical methods, finite difference procedures, lattice approaches). Others are, on their own, not capable of pricing American style options at all (standard simulation procedures)[2]. For the valuation of simple financial options (on stocks) these restrictions will not often cause problems, because (1) the assumption of Geometric Brownian Motion (GBM)[3] appears plausible and because (2) stock prices are observed directly and presumably represent the only source of uncertainty. It is therefore possible to use lattice approaches or difference procedures to price regular American type financial options. In order to price complex financial options or real options, however, one must search for different valuation methods.

The value of a real asset over time (e.g. the expected present value of the investment returns of an industrial plant) cannot be observed directly. Therefore, we must consider the factors which determine the value of the real asset, i.e. the ”disaggregate state variables“ (e.g. gross margins or revenues and variable costs of the production activity). Since multiple (stochastic) state variables are common in real options problems, possible correlations between these variables have to be considered as well. Furthermore, the assumption that the value of real assets or their disaggregate components follow GBM is hardly realistic (see Lund, 1993). GBM implicitly assumes constant relative changes (e.g. yield rates) and the non-occurrence of negative values of the stochastic variable. Neither assumption may hold for real options: disaggregate components of the value of real assets such as gross margins may oscillate around a more or less constant level; that is to say, they will approach the original level again after a stochastic deviation, which may also have resulted in a temporary negative value. Such behaviour cannot be represented by GBM. One has to resort to mean reversion processes[4]. Recent research shows that the kind of stochastic process has a decisive influence both on option prices and critical early-exercise values (see e.g. Odening et al., 2001).

It has long been noted that stochastic simulation procedures can handle multiple stochastic variables, alternative stochastic processes etc. quite easily. However, they are not directly applicable to American type options. The most successful way to handle American type options is to integrate a stochastic simulation of the state variable(s) into a more complex, back-

[2] A survey of various analytical and numerical option pricing procedures can be found in Hull (2000, chapter 16). The methods described are originally designed for pricing financial options. An application of these methods to real options is given by Trigeorgis (1996, chapter 10).

[3] GBM is a non-stationary Markov-Process. Consequently, the future value of a stochastic variable following GBM only depends on the last observed value, i.e. the current value represents all relevant information of the past.

[4] The identification of stochastic processes can be based on statistical test procedures. For example, unit-root tests can be used for testing whether a stochastic variable follows a random-walk (e.g. GBM) or a stationary process (e.g. mean-reversion processes) (see Pindyck and Rubinfeld, 1998). Time series models (e.g. ARIMA-processes) can be identified using a Box-Jenkins test (Box and Jenkins, 1976).

ward-recursive framework of option pricing. This general approach is the subject of this paper which studies the pricing of American type options in the context of real options. However, since real options are always complex options, the findings are equally useful for the pricing of complex financial options.

An outline of the overall problem concerning the valuation of American type options is presented in **Section 2**. Given the important role stochastic simulation plays in more sophisticated methods of option pricing, **Section 3** briefly sets forth the basics of stochastic simulation in the context of option pricing. It then describes and classifies the different methods developed up to now which integrate - in one way or another - stochastic simulation into a more complex framework to value American type options. **Section 4** describes one especially simple and straightforward method in greater detail. This method integrates sequential stochastic simulations in a backward recursive programming approach to determine the critical early-exercise path. It will hereafter be called Bounded Recursive Stochastic Simulation (BRSS). Exemplary results are demonstrated in **Section 5**. For validation purposes, we use a straightforward exemplary valuation problem for an American type option which can be solved by binomial tree methods as well. The paper closes with an outlook emphasizing the need for still more complex (agent based) valuation methods, especially if one is to include competition into the model by modelling price dynamics endogenously (**Section 6**). However, agent-dependent option pricing is a problem only of interest in the context of real options and in cases where stochastic price processes cannot be identified.

2 Problems in Valuing American Style Options

In the case of European type options there is only one question to be answered: (1) What is the value of the option? The exercise strategy is known: The option should be exercised at the expiry date if the difference between the (market) value of the underlying and the strike price is positive. In the case of American type options, which can also be exercised before maturity, there is an additional question to be answered: (2) At a given time, at which price of the underlying (i.e. critical early-exercise value) should the option be exercised? Hence, with regard to real options we have to answer both questions.

We first present the principal problem and background of pricing American type (real) options: Let I be the (constant) investment cost or purchase price of a real asset which generates a stochastic expected present value of investment cash flows V. The option to realize the investment is given in a period $[0, T]$ at discrete potential exercise dates τ, $\tau = 0 \cdot \Delta\tau, 1 \cdot \Delta\tau, ..., \Gamma \cdot \Delta\tau$, [5]. The number $\Gamma+1$ of potential exercise dates is determined by the time between two potential exercise dates $\Delta\tau$ ($\Gamma = T/\Delta\tau$). The investment costs are completely sunk once the investment is carried out[6]. We seek both the value of the investment option and the critical early-exercise values V_τ^* which would trigger an immediate investment at any given exercise date τ. According to traditional investment theory, the value of this investment opportunity at every potential exercise date equals the positive net present value i_τ:

5 For financial options with continuous exercising opportunities, option pricing based on discrete exercise points would only approximate the true value of the option and lead to a low bias because it underestimates the flexibility (see Balmann and Mußhoff, 2002). However, it is realistic to assume that real options can only be exercised at a limited number of dates.

6 If investment costs are not completely sunk, follow-up real options, e.g. the option to abandon etc. will arise which will have to be valued simultaneously because of their interactions.

$$i_\tau = \max(0, V_\tau - I) \tag{1}$$

Furthermore, traditional investment theory recommends that the investment should be carried out as soon as the expected present value of the investment cash flows V_τ exceeds the investment costs *I*, i.e. as soon as there is a positive net present value (NPV).

However, using the theory of option pricing we know that the NPV-calculus only takes into account one part of the option value, namely the intrinsic value. But an option does not have to be exercised at date τ. Therefore, the option also has a continuation value f_τ, which represents the discounted expectation value of the option, if it is not exercised at time τ, but the decision is put off until the next early-exercise date $\tau+\Delta\tau$:

$$f_\tau = \hat{E}\left(F_{\tau+\Delta\tau}\right)\cdot e^{-r\cdot\Delta\tau} \tag{2}$$

In this equation *F* denotes the value of the option, *r* the continuously compounded risk free interest rate and $\hat{E}$ the risk neutral expectation operator. The use of the risk neutral expectation operator and the risk-free interest rate follows the risk-neutral-valuation principle (see Cox and Ross, 1976). The fact that we take into account both the intrinsic value and the continuation value expresses the fact that the decision is regarded as a choice between two alternatives: (1) immediate investment and (2) delaying the investment decision. As a normative rule the investment option should be exercised if the intrinsic value equals or exceeds the continuation value. Therefore the value of the option is calculated as follows:

$$F_\tau = \max\left(i_\tau, f_\tau\right) \tag{3}$$

The binary decision problem between exercising and waiting can be understood as a specific stopping problem. Equation (3) is equivalent to the Bellman-equation (see Dixit and Pindyck, 1994, p. 109). It can be shown that under certain regularity conditions[7] there is an optimal exercise path which separates the stopping region from the continuation region. This exercise path or frontier consists of boundary values for the underlying V_τ^* which indicate the values in time where a decision-maker is indifferent with regard to early-exercise or continuation. In this case, the intrinsic value $i_\tau\left(V_\tau^*\right) = i_\tau^*$ must equal the continuation value $f_\tau\left(V_\tau^*\right) = f_\tau^*$ (identity- or value-matching condition):

$$i_\tau^* = f_\tau^* \tag{4}$$

In Figure 1 (left) the graphs of the intrinsic value and the continuation value as functions of the expected present value of the investment cash flows are given for *one* potential exercise date τ (non dividend-protected American type call option on a dividend paying underlying[8]). Graphically speaking, the option should *not* be exercised if V_τ is situated to the left of the intersection of the graphs of the intrinsic value and continuation value. It should be exercised immediately if V_τ is situated to the right. In other words: A profit maximizing decision-maker should exercise an investment option immediately if $V_\tau \geq V_\tau^*$. Note that for the sake of more convenient formulation we define in Figure 1 and for the rest of this paper $\Delta\tau = 1$.

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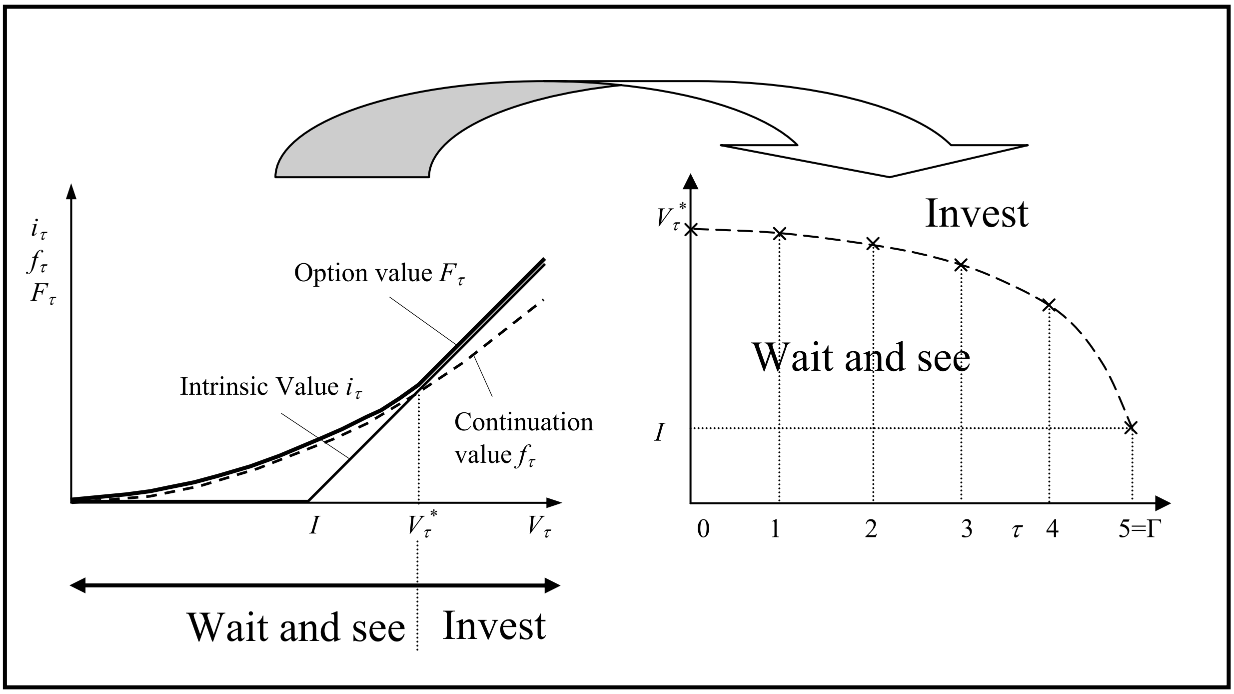
7 The regularity-conditions demand that (1) the intrinsic value and the continuation value are monotone functions of the value of the underlying and that (2) the distribution function of the underlying in $\tau+\Delta\tau$ will shift to the right (left) side, if the value in τ increases (decreases), i.e. a positive persistence of the stochastic process (see Dixit and Pindyck, 1994, p. 129).

8 Note that the early-exercise of an American type call option without dividends of the underlying is never optimal. Therefore the option value corresponds to the value of an equivalent European style call option (see Merton, 1973).

Figure 1: Relationship between intrinsic value and continuation value (on the left) and exercise frontier (on the right) [a]



[a] Depicted for a non-dividend-protected American call option with six potential exercise dates on a dividend paying underlying; $\Delta\tau = 1$.

While the left hand side of Figure 1 refers to *one* potential exercise date, the right hand side shows the *entire* critical early-exercise path over time for an American call option with six potential exercise dates. The critical early-exercise path defines the optimal exercising strategy for all potential exercise dates. Referring to the graph, one should immediately invest above the critical early-exercise path. Below the critical early-exercise path one should wait and see how the expected present value of the investment cash flows develops in the future. One characteristic of the critical early-exercise path is its negative exponential slope which expresses the reduction of flexibility in time. The shorter the residual lifetime of the option, the more ready an investor will be to carry out an investment. At the last possible exercise date Γ there is no more temporal flexibility to further delay the investment. Then the classical investment theory is valid and $V_\Gamma^* = I$.

Since the option price is the value of an option to a person who uses the optimal early-exercise strategy, the determination of both the option price and the early-exercise strategy are closely associated. Numerous procedures are available for the computation of option values. They all have restrictions and can be divided in two main classes: (1) Analytical solutions of the stopping problem which require solving a partial differential equation[9]. Such closed form solutions are only available for simple valuation problems[10]. (2) Various numerical approximation methods for more complex valuation problems. Note that even a simple American option with a finite lifetime belongs to this second class. Because of the special character of real options, the most commonly used numerical methods for pricing financial options (finite

9 For a detailed representation see Dixit and Pindyck (1994, chapter 4).
10 An extensive overview of option pricing formulas can be found in Haug (1998).

difference- and lattice methods) are not applicable. Only simulation-based approaches provide enough flexibility for successful valuation of complex options and therefore real options.

3 Simulation-Based Methods for Valuing American Type Options

Using a single *standard* stochastic simulation procedure, only European style options can be priced. Nevertheless, stochastic simulation is important for American type options because it can be successfully embedded in a more sophisticated methodological framework to value such options. That ranges from its use in a simple Black-approximation (Black, 1975) to its use in the simulation-based options pricing methods outlined in subsection 3.2. The difference between the various methods arises from their diverse ways of determining the critical early-exercise path, before actually valuing the option. It is a common feature of all the methods that they simulate the stochastic development of one or several state variables.

3.1 The Starting Point: The Classical Stochastic Simulation

Boyle (1977) was the first to use stochastic simulation to value European type options. The basic idea is to simulate the value of the underlying (price path) in accordance with a plausible stochastic process in a risk neutral world up to the expiry date *T*. Then the expected present value of the option price is computed using the risk neutral discount factor. In order to be able to do this, one has to guarantee, that - in accordance with risk neutral valuation principles - the risk neutral drift rate is used for the simulation instead of the real growth rate[11]. At the expiry date *T*, for every simulated value of the underlying ${}^{s}V_T$ (*s* is a simulation run out of a total of *S*) the payoff of the option ${}^{s}i_T$ can be calculated. For a call option we would write:

$$ {}^{s}i_T = \max(0, {}^{s}V_T - I) \tag{5} $$

The current option value ${}^{s}F_0$ for a given price path and therefore simulation run is the payoff ${}^{s}i_T$ discounted with the risk free interest rate *r*:

$$ {}^{s}F_0 = {}^{s}i_T \cdot e^{-r \cdot T} \tag{6} $$

The risk neutral expected value for the option price F_0 can subsequently be computed as the average over all simulated ${}^{s}F_0$:

$$ F_0 = \sum_{s=1}^{S} {}^{s}F_0 \cdot \frac{1}{S} \tag{7} $$

The advantages and disadvantages of stochastic simulation in comparison to other valuation procedures can be summarized as follows:

Advantages:

1. Almost any stochastic process can be simulated in a realistic way. This is especially important in the case of stochastic processes which are non-Markov in nature (i.e. future values depend not only on the presently observed value, but also on previous values; e.g. Autoregressive Integrated Moving-Average (ARIMA)-processes). It is almost impossible to handle such processes within the framework of lattice- or finite difference methods.

[11] The risk-neutral drift rate equals the risk free interest rate minus dividends of the underlying (see Luenberger, 1998, p. 357). In the context of real options the so called convenience yield can be interpreted as a dividend. The convenience yield represents monetary advantages the owner of an asset enjoys compared to the owner of the option.

2. It is relatively easy to consider several stochastic variables simultaneously. Stochastic simulation is considerably more efficient than other numerical procedures such as lattice approaches if two or more stochastic variables (or a great number of potential exercise dates) must be considered. With stochastic simulation the computational requirements increase only linearly with the number of the variables whereas with other methods it increases exponentially.

3. The accuracy of option valuation can be estimated by computing confidence intervals for the option price.

Disadvantages:

1. A very high number S of simulation runs is necessary in order to guarantee a sufficiently precise option valuation. Consequently, for simple valuation problems the computational speed is lower than that of lattice- or finite difference methods. For example, Haug (1998, p. 40) stipulates carrying out at least 10 000 simulation runs. Fortunately, with any given number of simulation runs, one can improve the stability of the solution by using so called variance reduction methods (e.g. antithetic variables technique) without a great increase of computational time[12].

2. Using a single standard stochastic simulation procedure only European style options can be priced. The problem in the case of American style options is that with a simple forward moving simulation of the price path it is not clear at potential exercise dates whether waiting or exercising represents the optimal strategy. In other words: The critical early-exercise path is not known in advance.

Because of these disadvantages, for a long time stochastic simulation was not believed to be feasible for the valuation of American type options at all (see e.g. Hull, 1993, p. 363; Briys et al., 1998, p. 62). Accordingly, it was as well deemed unsuitable for the valuation of real options (see e.g. Trigeorgis, 1996). However, due to the great flexibility of simulation-based approaches, many attempts have been made to integrate stochastic simulation(s) in a more complex valuation framework in order to overcome the disadvantage of the simple forward moving simulation.

Despite the fact that some of these valuation methods provide accurate results and are relatively simple to use, the potential of stochastic simulation for the valuation of American type options, whether they be real options or complex financial options, is still not widely appreciated (see e.g. Hull, 2000, p. 408).

3.2 Overview of Simulation-Based Methods

The different methods which use simulation procedures to determine the early-exercise path and the price of American type options may be grouped as follows:

A: Simulation of one finite sample of price paths starting at time 0 and subsequent stratification of the state space

1. Tilley (1993) uses a so-called bundling algorithm whereby, once and for all, starting from time 0 a large number of price paths are simulated. At each potential early-exercise date

[12] An overview of various variance reduction procedures can be found in Hull (2000, chapter 16.7).

the paths are ordered according to the stock price level at that date and divided into “bundles” of the same size. Next, starting from the expiry date of the option the average of the continuation values of prices in a bundle is taken as the single continuation value for that bundle. This backwards-recursive procedure yields an early-exercise strategy for each simulated stock price path. The ordering of the prices enables the critical early-exercise value to be determined through the ”identity condition“; that is, the early-exercise value lies where the intrinsic value coincides with the continuation value. However, this is based on the assumption that the average described above is an accurate estimate of the continuation value of the prices in a bundle, and also that there is a small distance between the bundles. In other words: Hence we need an extremely high number of paths in each bundle and a high number of bundles as well. In fact, there is a transition zone between holding and exercising in which the early-exercise strategy is determined by a pragmatic rule. Finally the option price is obtained as the average of the discounted payoffs for the initially simulated stock price paths according to the early-exercise strategy.

2. Barraquand and Martineau (1995) also reduce the dimensionality of the valuation problem by grouping simulated paths at any point in time into a limited number of ”bins”. They then determine transition frequencies between successive bins by another simulation and finally solve backwards like in a multinomial tree.

3. Raymar and Zwecher (1997) similarly design a grid of ”bucket” regions and simulate paths through that grid. Therefore, at any point in time, the realized outcome will be assigned to one bucket. They then determine (1) transition frequencies into/out-of each bucket at every date t to each bucket at the next point in time $t+1$ and (2) average realized values in each bucket. Eventually, they determine the average payoff in each bucket at the date of expiration and iterate as in a multinomial tree to compute the current value of the American option.

All three methods are mimicking the standard binomial tree by stratifying the state space and putting the simulated paths into groups which are called ”bundles” by Tilley, ”bins” by Barraquand/Martineau and ”buckets” by Raymar/Zwecher. Indeed, Garcia (2000, p. 3) also puts all three methods in one group: “The papers by [Tilley, Barraquand/Martineau, Raymar/Zwecher …] incorporate different aspects of the usual backwards induction algorithm by stratifying the state space and finding the optimal exercise decision in each subset of the state variables.“ However, unlike Barraquand/Martineau or Raymar/Zwecher, Tilley does not calculate transitions probabilities between successive bundles and solve as in a multinomial tree. Instead, he uses a path-wise determination of the exercise strategy.

B: Simulation of one finite sample of price paths starting at time 0 and subsequent backward-recursive estimation of a continuation value function

4. In a discussion of the Tilley-paper, Carriere (1996) describes Tilley’s bundling algorithm as a “regression method, albeit crude“. In a publication (1996) of his own he develops the regression method. Like Tilley, he first simulates the stock price movement a large number of times. Subsequently however, assuming that the option has not been exercised before, he determines the value functions which describe at any given date the value of the option depending on the basis values in a backward recursive fashion. At the expiry date

this value function is just the intrinsic value. After having determined the value function at a certain exercise date, he approximates a continuation value function at the previous exercise date by carrying out a piecewise polynomial regression of the already determined values against the basis values at the previous exercise point. He then takes the value function as the maximum of the continuation value function obtained by this regression and the intrinsic value function. Using the continuation value function its intersection with the intrinsic value function can be calculated in order to determine the critical early-exercise value. In sum, by different regression methods he arrives at a "comparable performance to Dr. Tilley's method". It should be noted, that although the critical exercise value is calculated, it is *not* used for a downstream simulation to determine the option price. The option price is rather determined as the average of the discounted cash flows of all the price paths according to their respective early-exercise strategies.

5. The method proposed by Longstaff and Schwartz (2001) also proceeds in backward-recursive fashion to obtain at each discrete exercise date the continuation value function depending on the basis value. This is achieved through use of the simple Least Squares method. They subsequently determine for each basis value whether exercising or holding of the option leads to the higher value. This results in a certain exercise strategy for each price path . However, the critical exercise value is *not* explicitly calculated and again, no downstream simulation to determine the option price takes place. The option price is rather found as the average of the discounted payoffs of all the paths simulated at the beginning according to their respective early-exercise strategies.

C: Multiple simulations and determination of the early-exercise strategy through maximization of the option value with regard to parameters of an exercise function over time

6. Bossaerts (1989) "develops simulation estimates of American option prices by parameterise the stopping rule (i.e. exercise function over time) and then solving for the parameters that maximize the value of the option".

7. Fu and Hu (1995), Fu and Hill (1997) and Fu et al. (2000) likewise parameterise the exercise boundary, and then maximize the expected discounted payoff with respect to the parameters. "[...] no dynamic programming is involved, i.e., the procedure simultaneously optimises all parameters by iteration instead of sequentially by backward recursion. [...] It is "mimicking steepest-descent algorithms from the deterministic domain of non linear programming" (see Fu et al., 2000, p. 13).

8. Garcia (2000) also tries to find a suitable parameterisation of the exercise boundary by using an optimisation algorithm to determine those values of the parameters which yield the maximum option value.

D: Backward-recursive determination of critical exercise values using sequential simulation of price paths starting from the respective exercise dates

9. Grant et al. (1997) suggest a backward-recursive procedure whereby at each possible early-exercise date the stochastic development of the basis value (state variable) is simulated starting from different discrete test-values. For each test-value, the intrinsic values can be directly derived. Knowing the future exercise strategy, the respective stochastic continuation value is computed by using the expectation operator over all corresponding

sample paths. Generally speaking, the zeros of the "identity function" $f_{id} = i(V_\tau) - f(V_\tau)$ can be approximated by using a root finding algorithm such as the bisection or secant method, then bracketing the root of f_{id} to a required tolerance and interpolating linearly to obtain an estimate for the root. Grant et al. predefine the bracket of the root by the set interval between the different discrete test-values where simulations are started.

10. Ibanez and Zapatero (1998) also suggest a backwards-recursive procedure whereby at each possible early-exercise time the stochastic development of the basis value and thereby the continuation value is simulated. However, the intersection of the easily calculated intrinsic value i_τ and the simulated continuation value f_τ (Value-Matching condition) is determined by several iterations of Newton's method starting from an arbitrary basis value.

Actually the approaches of Grant et al. (1997) and Ibanez and Zapatero (1998) are quite similar. There is one main difference. Grant et al. find two values of the asset at which the "identity function" has opposite signs and then use linear interpolation to approximate the boundary value (this is actually just one step of the secant method). Ibanez and Zapatero, in contrast, use several iterations of Newton's method to find the zero of the identity function. It would seem that a lot of work could be saved here by using the secant method as a root finding algorithm rather than Newton's. The secant method converges almost as fast as Newton's method but with the advantage that one avoids the rather cumbersome evaluation of the derivative of the identity function.

E: Multiple simulations and determination of the critical exercise path through maximization of the option value with regard to a heuristically varied set of exercise path values

11. Dias (2001) or Balmann and Mußhoff (2002) maximize the option value with regard to a complete exercise path containing a set of critical values which is gradually optimised by means of genetic algorithms. First, *S* simulation runs, each starting from a different initial price in time 0, are carried out. Then, average option prices are computed for a number of test early-exercise paths (genomes) which had been randomly selected. These test exercise paths are ordered by the level of the option value (fitness) they generate respectively. The application of the genetic algorithm (selection, recombination, mutation) determines the composition of the test exercise paths in the next test run (the next generation). This process, which generates increasingly fitter exercise paths by mimicking natural evolution, is repeated until all exercise paths of a generation are nearly identical. At this point, they also hardly differ from those of the previous generation and an improvement of the fitness is no longer possible (maximal option value).

An overview of the basic characteristics of the different simulation-based methods to value American type options which are described above is given in table 1.

Table 1: Classification of different simulation-based methods

		nonrecurring simulation	stratification of state space	backward-recursive determination of exercise strategy within known paths	combination with lattice method	estimation of continuation function	explicit backward-recursive determination of critical values	multiple simulations for determination of free boundary	downstream simulation for determination of option value	maximization of option value with regard to parameters of exercise function	maximization of option value with regard to heuristically varied set of critical values
1	TILLEY (1993)	X	X	X			X				
2	BARRAQUAND and MARTINEAU (1995)	X	X		X						
3	RAYMAR and ZWECHER (1997)	X	X		X						
4	CARRIERE (1996)	X		X		X	X				
5	LONGSTAFF and SCHWARTZ (2001)	X		X		X					
6	BOSSAERTS (1989)							X	X	X	
7	FU and HU (1995); FU and HILL (1997); FU et al. (2000)							X	X	X	
8	GARCIA (2000)							X	X	X	
9	GRANT et al. (1997)						X	X	X		
10	IBANEZ and ZAPATERO (1998)						X	X	X		
11	DIAS (2001); BALMANN and MUßHOFF (2002)							X	X	X [a]	X [a]

[a] Genetic Algorithms can be used to determine an optimal set of critical values or to optimise the parameters of an exercise function over time.

Many – albeit not all – of these procedures are either not satisfying in regard to the precision of the option valuation or require so much programming effort and computational time that they hardly appear to be practical. This may or may not explain why they are, in general and without much differentiation, regarded by many experts as difficult to implement (see e.g. Hull, 2000, p. 408). Another obstacle for widespread applications seems to be that some of the procedures offer little intuition or require a lot of deep mathematics. Indeed, some procedures may be unnecessarily complicated compared to others for their intended application. At the same time, some of the more complex and cumbersome methods are justified because they permit the modelling of special complexities inherent in some options. Agent-based simulations (genetic algorithms), for example, are sometimes necessary because they allow for the endogenous modelling of price dynamics within the framework of a real option pricing model (see e.g. Balmann and Mußhoff, 2002). It should be noted, however, that a common feature of all real options is their complexity. A suitable valuation method has to be flexible enough to take account of all real world qualities of an option. They may for instance arise from non-Markov processes, multiple stochastic variables, correlations of stochastic variables, interactions of different options etc. The following discussion focuses on the potential of the valuation procedure suggested by Grant et al. (1997). After some modifications it is well suited for the valuation of non-agent dependent, complex American type options with all the above mentioned features including multiple stochastic variables. Additionally, its performance can be improved by slight further modifications. The method combines high accuracy and good intuition with simple implementation (see section 5.2). According to our classification, it belongs to the group of methods which integrate sequential simulations of basis values for respective exercise dates into a backward-recursive procedure which in turn determines the critical exercise path.

4 Description of a Simple Recursive Stochastic Simulation Approach

The literature usually refers to all procedures that determine the critical early-exercise frontier by a backward-moving recursion (backward induction) as "dynamic programming approaches" (see e.g. Fu et al., 2000). This is due to the fact that the binary decision problem between exercising or waiting is regarded as a specific stopping problem (see Bellman, 1957). In this sense, lattice approaches also represent discrete approximations of the dynamic programming principle. Accepting this terminology, those simulation-based procedures which determine the critical early-exercise frontier by a backward-moving recursion have to be subsumed under "dynamic programming" as well. The approach of Grant et al. (1997) belongs to this group. It will be shown that it represents a straightforward and fast way to determine the critical early-exercise path and the option price of an investment option or an American style call option on a dividend paying underlying. Its advantages can be expanded by adopting some slight but effective modifications. This is already true if we consider a single stochastic variable. For the sake of clarity we choose this case for a detailed description of operational procedures in section 4.1. It is, however, equally valid if we consider multiple stochastic variables. The extensions which are necessary in this case are shown in section 4.2. In both cases, we could label this modified method as "Bounded Recursive Stochastic Simulation" (BRSS) due to the specific characteristics of the process used to determine the critical early-exercise values.

4.1 Valuing a call Option with a Single Stochastic State Variable

Before we describe the BRSS-procedure step by step for the case of a single stochastic variable, we summarize the modifications which we have made to improve the effectiveness of the approach of Grant et al. (1997):

1. In all simulations starting from different test-values with each one consisting of S paths, we use the same sequence of random numbers for the stochastic process. Technically speaking, we only have one Monte Carlo simulation and we save a lot of time by *simultaneously* simulating all price paths starting from different values.

2. We always use the (already known) critical exercise-value of the subsequent period as a *lower bound* for the test-values we start simulating from in order to determine the critical early-exercise value as a free boundary. We make an educated guess at an upper bound, obtaining thereby an interval which is divided into equal subintervals of a length which is already deemed sufficiently small for interpolation. Using the exercise strategy of the subsequent period as a lower bound turns out to be even more advantageous when dealing with multiple stochastic variables and therefore more cumbersome exercise functions at any one point in time. The procedure of using a predefined sequence of test-values has an additional advantage because it facilitates the automation of consecutive work steps. We do not use a manual criterion (i.e. "stop, if intrinsic value exceeds the continuation value for the first time") which would tell us when to stop simulating paths starting from still another test-value. Instead we predefine an interval where we expect the identity function to be zero and program the determination of the critical value (see Figure 4).

Step 1: Determination of the critical exercise value V_Γ^*

The critical exercise value V_Γ^* at the expiry date Γ of the option is the starting point of any backward-recursive valuation. Since there is no temporal flexibility at the last potential exercise date, the investment should be carried out as soon as the investment payoff V_Γ covers the investment costs *I*. Therefore, V_Γ^* equals *I*. The knowledge of V_Γ^* is the precondition for calculating $V_{\Gamma-1}^*$. Let the length of the time period between two potential exercise dates be $\Delta\tau = 1$.

Step 2: Determination of the critical early-exercise value $V_{\Gamma-1}^*$

The critical early-exercise value $V_{\Gamma-1}^*$ is the present value of the investment which yields an identical intrinsic value and continuation value. We calculate the intrinsic value $i_{\Gamma-1}(V_{\Gamma-1})$ and the continuation value $f_{\Gamma-1}(V_{\Gamma-1})$ for a set of different test-values ${}_nV_{\Gamma-1}$, with $n = 1, 2, ..., N$. For each test-value the intrinsic value can be directly derived. The corresponding continuation value is estimated after running a stochastic simulation with S runs starting from the given test-value. We proceed as follows (see Figure 2):

Step 2.1: Definition of test-values (test present value of the investment)

The lowest test-value ${}_1V_{\Gamma-1}$ we start simulating from is the theoretically known lower bound for date Γ-1 which is given by the critical exercise-value of the subsequent period, i.e. V_Γ^*. Then we make an educated guess at a preliminary upper bound. The interval between the lower and upper bound (parameterization interval) is divided into *N*-1 equal subintervals whose length we deem sufficiently small for interpolation. The endpoints of these subintervals give us a total of *N* test-values ${}_nV_{\Gamma-1}$ to start test-simulations from. The critical value $V_{\Gamma-1}^*$ falls between those two test-values the larger of which is the first test-value for which the intrinsic value is higher than the continuation value.

Step 2.2: Determination of continuation values for each test-value by means of stochastic simulation

S runs or paths starting from the lower bound ${}_1V_{\Gamma-1} = V_\Gamma^*$ are simulated resulting in S values of the investment ${}_n^sV_\Gamma$ at date Γ. Simultaneously, *S* paths are simulated starting from the other test-values ${}_nV_{\Gamma-1}$. The simulations starting from different test-values are indeed one simultaneous simulation because we use the same sequence of random numbers for all S runs. This reduces computation time significantly. Knowing V_{Γ}^{*} we calculate the continuation values ${}_{n}^{s}f_{\Gamma-1}$ for all paths starting from any given test-value ${}_{n}V_{\Gamma-1}$ as the discounted payoff of the option:

$$ {}_{n}^{s}f_{\Gamma-1} = \max(0,\ {}_{n}^{s}V_{\Gamma} - I)\cdot e^{-r\cdot 1} \tag{8} $$

The expected value for the continuation value ${}_{n}f_{\Gamma-1}$ is the average value of all ${}_{n}^{s}f_{\Gamma-1}$:

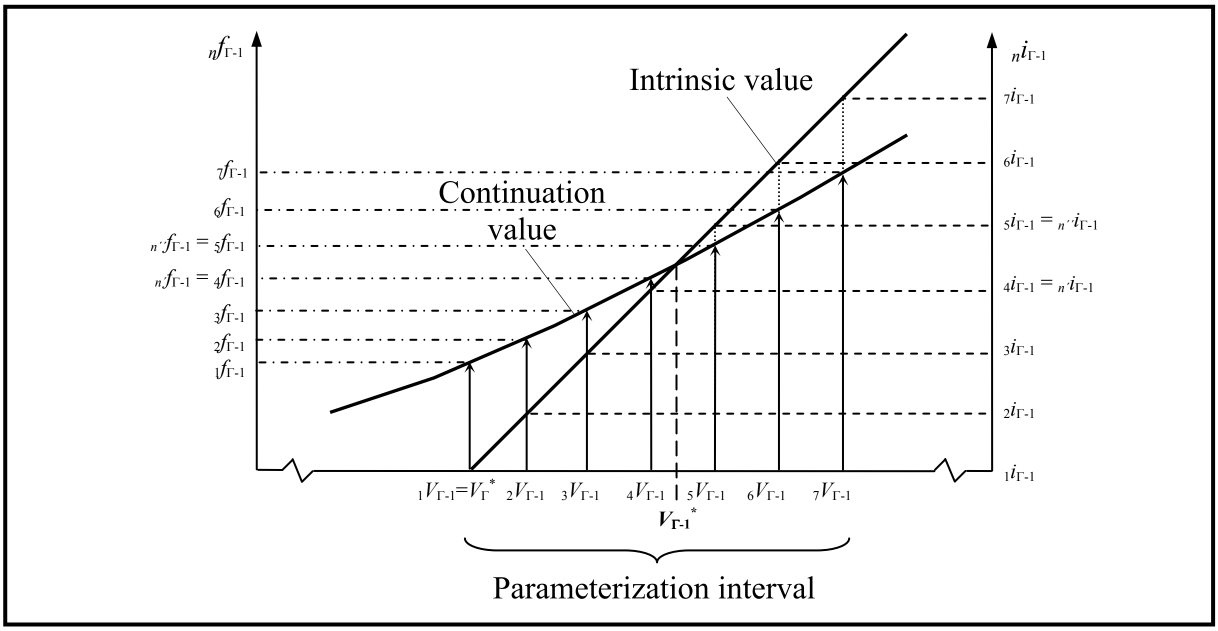
$$ {}_{n}f_{\Gamma-1} = \sum_{s=1}^{S} {}_{n}^{s}f_{\Gamma-1}\cdot\frac{1}{S} \tag{9} $$

Step 2.3: Calculation of intrinsic values for each test-value

In order to compare the possible strategies "invest" and "wait", we must also calculate the intrinsic value. The intrinsic value ${}_{n}i_{\Gamma-1}$ for each test-value ${}_{n}V_{\Gamma-1}$ can be directly derived as:

$$ {}_{n}i_{\Gamma-1} = \max(0,\ {}_{n}V_{\Gamma-1} - I) \tag{10} $$

Figure 2: Determination of a critical early-exercise value of a (dividend paying) American call option using BRSS [a]



[a] Depicted for $N = 7$.

Step 2.4: Approximation of the critical early-exercise value $V_{\Gamma-1}^{*}$ by means of linear interpolation

It is very unlikely that the intrinsic value and continuation value will coincide at one of the pre-defined test-values. In most cases, the critical value will fall between two predefined test-values, those which yield a change of sign of the difference of intrinsic value and continuation value. They will be denoted by n' and n'', where it does not matter which one is the smaller. The respective intrinsic values (${}_{n'}i_{\Gamma-1}$ and ${}_{n''}i_{\Gamma-1}$) and continuation values (${}_{n'}f_{\Gamma-1}$ and ${}_{n''}f_{\Gamma-1}$) are used for linear interpolation (equivalent to one step of the secant method):

$$V^*_{\Gamma-1} = {}_{n''}V_{\Gamma-1} + \frac{{}_{n'}V_{\Gamma-1} - {}_{n''}V_{\Gamma-1}}{({}_{n'}i_{\Gamma-1} - {}_{n'}f_{\Gamma-1}) - ({}_{n''}i_{\Gamma-1} - {}_{n''}f_{\Gamma-1})} \cdot \left[-\left({}_{n''}i_{\Gamma-1} - {}_{n''}f_{\Gamma-1}\right)\right] \tag{11}$$

$i^*_{\Gamma-1}$ and $f^*_{\Gamma-1}$ denote the intrinsic value and the continuation value of the critical exercise-value $V^*_{\Gamma-1}$. Note that $i^*_{\Gamma-1} - f^*_{\Gamma-1} = 0$ (identity condition).

In the example presented in Figure 2, one must interpolate between the values ${}_{n'}V_{\Gamma-1} = {}_4V_{\Gamma-1}$ and ${}_{n''}V_{\Gamma-1} = {}_5V_{\Gamma-1}$.

Control step: Improving the interpolation

In order to improve the approximation, we could reduce the length of the initial interval (see step 2.1). We would thereby also shorten the subintervals on which we have to interpolate by rerunning steps 2.2 to 2.4. Note that the initially chosen interval must be enlarged if it did not include a test-value yielding an intrinsic value higher than the continuation value.

Step 3: Determination of the critical early-exercise value $V^*_{\Gamma-2}$

In order to determine the critical early-exercise value $V^*_{\Gamma-2}$ one has to take into account the fact that the option may be exercised both at Γ-1 and at Γ. We can again use stochastic simulation to determine continuation values for a given set of test-values ${}_nV_{\Gamma-2}$, because we already know $V^*_{\Gamma-1}$ and V^*_{Γ} and therefore the future exercise strategy. The procedure to determine $V^*_{\Gamma-2}$ is analogous to the procedure described in step 2 above. Only the computation of the continuation value for each path has to modify according to the optimality of exercising either at Γ or Γ-1:

$$ {}^s_nf_{\Gamma-2} = \max(0,\ {}^s_nV_\kappa - I) \cdot e^{-r\cdot[\kappa-(\Gamma-2)]} \text{ with}$$
$$\kappa = \begin{Bmatrix} \Gamma-1, \text{ if } {}^s_nV_{\Gamma-1} \geq V^*_{\Gamma-1} \\ \Gamma, \quad \text{otherwise} \end{Bmatrix} \tag{12}$$

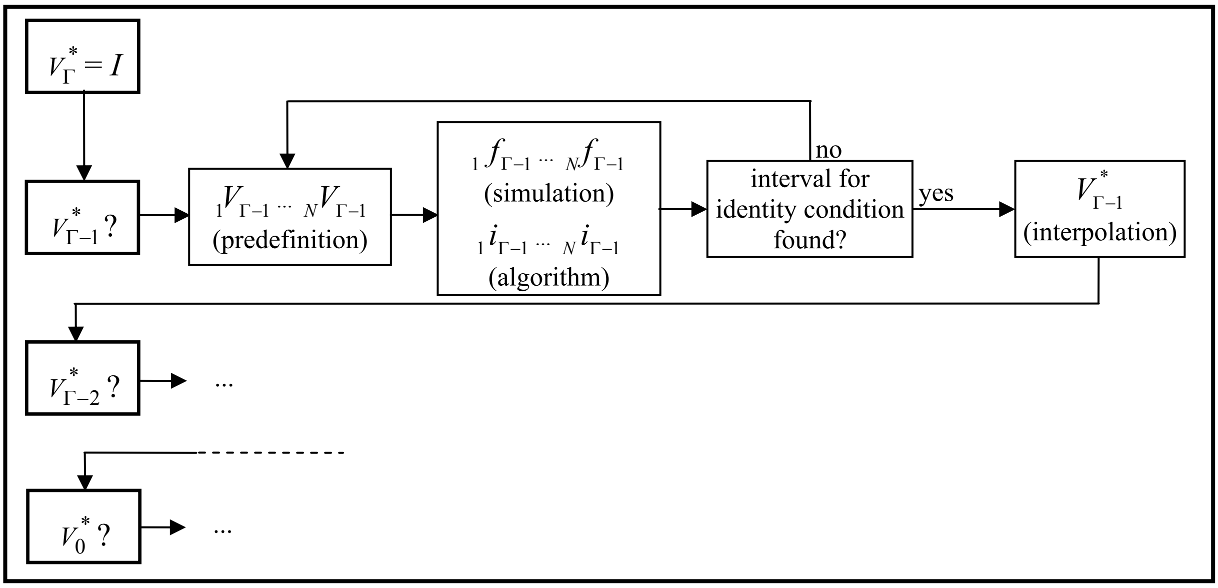
Step 4: Definition of the critical early-exercise values $V^*_{\Gamma-3}, V^*_{\Gamma-4}, ..., V^*_0$

The procedure described above is applied backwards until all critical early-exercise values are known. An increasing "length" of future exercise strategy increases the complexity of determining ${}^s_nf_\tau$ according to the optimality of exercising at future dates. Equation (12) has therefore to be generalized as follows:

$$ {}^s_nf_\tau = \max(0,\ {}^s_nV_\kappa - I) \cdot e^{-r\cdot(\kappa-\tau)} \text{ with}$$
$$\kappa = \begin{Bmatrix} \tau+1, \text{ if } {}^s_nV_{\tau+1} \geq V^*_{\tau+1} \\ \tau+2, \text{ if } ({}^s_nV_{\tau+2} \geq V^*_{\tau+2}) \wedge ({}^s_nV_{\tau+1} < V^*_{\tau+1}) \\ : \\ \Gamma, \quad \text{otherwise} \end{Bmatrix} \tag{13}$$

Figure 3 gives a graphical representation of the basic procedure to determine the critical early-exercise path.

Figure 3: Basic procedure to determine the critical early-exercise path using BRSS



Step 5: Determination of the option value

F_0 is the maximum of intrinsic value i_0 and continuation value f_0 . After having determined the optimal strategy as a free boundary, one has to initiate one last simulation starting from the actual present value of investment cash flows V_0. Then, the option value F_0 can be determined as the expected value of all simulation runs S by determining i_0 and f_0 analogous to steps 2.2 and 2.3. Straightforward stochastic simulation can be applied because the optimality of exercising (i.e. the early-exercise path as a whole) is already known.

Using the BRSS method, only Γ simulations and $\Gamma \cdot S$ simulation runs are needed to determine the early-exercise strategy because all paths starting from different test-values are simulated using the same random numbers. In order to value the option another simulation with S runs is necessary.

4.2 Valuing an Option with Multiple Stochastic State Variables

When pricing financial options we very often assume that the value of the underlying is the only stochastic state variable. However, some financial option pricing models consider additional stochastic variables, such as a stochastic variance and/or a stochastic risk free interest rate. With real options there are still more sources of uncertainty to be taken account of. This is, in part, due to the fact that real options do not represent contractual rights. A good example is the investment costs which are a stochastic variable, even though they are analogous to the (contractually fixed) strike price of financial options. Another necessity for integrating additional stochastic variables may arise from a disaggregation of the state variable. A disaggregation is necessary if we value compound options with one or several follow-up options such as options to switch use or to switch operating mode etc. The modelling of a choice between different outputs and/or inputs requires the use of revenues and variable costs (or even more disaggregated variables such as input or output prices) instead of the aggregated value of the underlying. Hence, we can summarize three reasons why we have to take account of multiple variables in real option pricing.

1. Several factors which can be considered as being stochastic in the case of financial options may *also* be stochastic in the case of real options (e.g. variance of the underlying)
2. Several factors which are contractually fixed in the case of financial options may represent *additional* stochastic variables in the case of real options (e.g. strike price)
3. Several factors which arise from a disaggregation of the state variable may *replace* this stochastic state variable in the case of real options

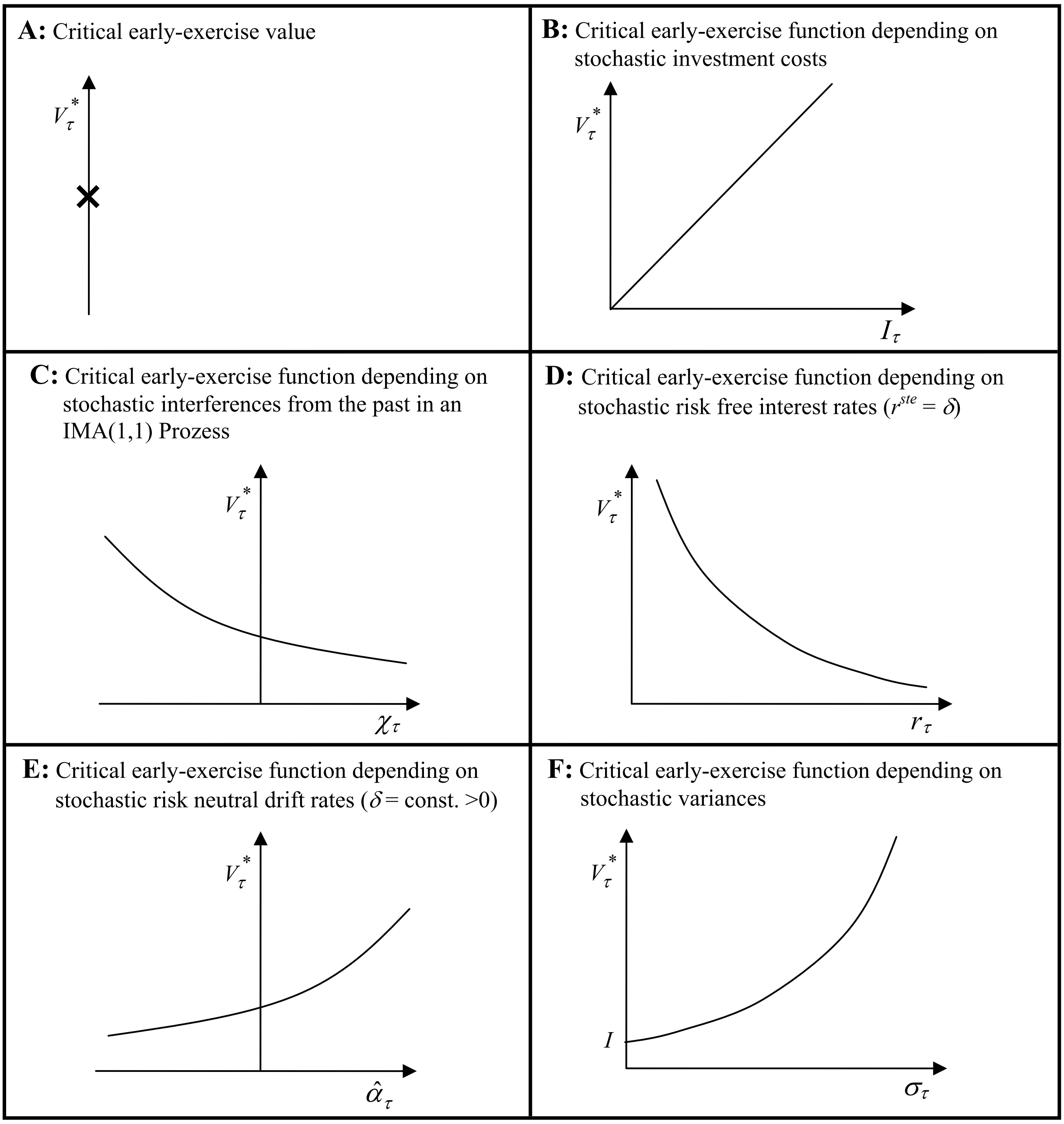
Option pricing based on more than one stochastic variable has to take account of correlations. Correlations between stochastic variables will alter the stochastic development of these variables and lead to a different option value. The modelling of correlations is straightforward in simulation-based option pricing procedures. Let ${}^{X}\varepsilon_{\tau}$ and ${}^{Z}\varepsilon_{\tau}$ be two non correlated random numbers belonging to a standard normal distribution and $\rho_{X,Y}$ the correlation between two random numbers ${}^{X}\varepsilon_{\tau}$ and ${}^{Y}\varepsilon_{\tau}$. Consequently the correlated random number ${}^{Y}\varepsilon_{\tau}$ is to be generated as follows:

$$ {}^{Y}\varepsilon_{\tau} = \rho_{X,Y} \cdot {}^{X}\varepsilon_{\tau} + \sqrt{1-\rho_{X,Y}^{2}} \cdot {}^{Z}\varepsilon_{\tau} \tag{14} $$

Stochastic simulation procedures can generally handle multiple stochastic variables quite easily. Therefore, not many adjustments to option pricing procedures have to be made when determining the value of European options depending on multiple stochastic variables. It should be noted, however, that time discrete versions of stochastic processes cannot be used in the case of stochastic variances, risk free interest rate etc, because they imply a constant (i.e. non-stochastic) value of these parameters over time. That is why we have - even in the case of European options - to resort to a sufficiently fine discretisation of time when we simulate the price path of the underlying.

Contrary to European options, the determination of the early-exercise strategy and the value of American options depending on multiple stochastic variables is quite complicated. With one stochastic variable, we had, at any one point in time, to determine one critical exercise value forming, in turn, a one-dimensional early-exercise strategy over time (see Figure 1 and Figure 5 A). Now we have, at any one point in time, to determine critical combinations of values for different stochastic variables (early-exercise function) forming in turn a multidimensional early-exercise strategy over time. For the sake of clarity and feasibility of graphical representation we subsequently consider only a stochastic underlying and one additional stochastic variable at a time, namely: stochastic investment costs (see Figure 5 B), an additional interference from the past emanating from an IMA(1,1)-process (C), stochastic risk free interest rates (D), stochastic risk neutral drift rates (E) and stochastic variances (F).

Figure 4: Diagrammatic representation of the critical early-exercise function at any point in time depending on two stochastic variables



In the case of stochastic investment costs, the problem of multiple stochastic variables can be reduced to determining one critical value m_τ^* defining the constant positive gradient of the linear early-exercise function (see Figure 4 B).

$$V_\tau^* = m_\tau^* \cdot I_\tau \qquad (15)$$

In contrast, an additional stochastic interference term from the past χ_τ emanating from an IMA(1,1)-process as well as stochastic risk free interest rates generate early-exercise functions with negative and exponentially decreasing gradients (see Figure 4 C and D).

Stochastic risk neutral drift rates and stochastic variances, in turn, generate early-exercise functions with positive and exponentially increasing gradients (see Figure 4 E and F).

We were already able to show in the previous part of this paper, that the powerful stochastic simulation procedures can be used for pricing American options by embedding them in a broader backward-recursive framework to value such options. Now, dealing with the problem of multiple stochastic variables and therefore early-exercise functions instead of early-exercise values, we are again able to integrate operational procedures solving this problem into the broader methodological framework of simulation-based option pricing. The following steps specify which additional operational procedures are needed within a framework of option pricing which is based on a backward recursive sequential stochastic simulation of price paths.

- Taking account of all correlations we simulate *S* paths of each stochastic variable and store them separately in columns, before we actually value the option.
- At expiration, the critical exercise function is reduced to the well known critical exercise value *I*.
- In the last early-exercise date we use the free boundary approach. Instead of determining one critical value, however, we determine a discrete critical value of the underlying for a given value of the second stochastic variable using the value matching condition. By a subsequent systematic variation of the second variable, we find critical values of the underlying depending on the second stochastic variable (critical combinations).
- In most cases subsequently simulated price paths will not coincide exactly with these selected (discrete values). We will therefore either have to use linear interpolation or we have to estimate, at any one point in time, an explicit critical early-exercise function in order to determine the exercise strategy.
- By proceeding recursively backwards, all the other critical combinations (respectively early-exercise functions) are determined because at any one exercise point the future strategy is known. In order to save time, we always use the (already known) critical combination of values of the subsequent period as a *lower bound* for the test-values we start simulating from.
- ...
- After the determination of this multidimensional early-exercise strategy the value of the option can again be calculated by one simple Monte Carlo simulation starting from the presently observed value.

5 Validation

In this section, five different methods to determine the critical early-exercise paths and the option value are compared and validated: (1) the initial approach by Grant et al. (1997), (2) the BRSS-method derived above, (3) the binomial tree method, (4) the approach by Ibanez and Zapatero (1998) and (5) genetic algorithms. For this purpose, we use a simple investment option which allows for the binomial solution as an additional benchmark.

5.1 Model Assumptions

A present value of investment cash flows V_0 of 110 T€ is expected [13]. The investment costs I are 100 T€. They are completely sunk once the investment is carried out. Investment opportunities are given at dates τ, $\tau = 0, 1, ..., \Gamma$ with $\Gamma = 5$. The lifetime of the option is $T = 5$ years. The length of a time period between two potential exercise dates is $\Delta\tau = T/\Gamma = 1$ year. Therefore, there are $\Gamma+1 = 6$ potential exercise opportunities. The standard deviation of the stochastic process for the expected present value of the investment cash flows σ is 20% p.a. Additionally, there is a continuous convenience yield (dividend payment) δ of 5.83%. The continuous risk free interest rate r is also 5.83 %[14]. It is assumed that σ, r and δ are constant. Any uncertainty concerning these parameters, however, could be easily implemented within the framework of simulation-based procedures. The state variable V follows GBM (see Hull, 2000, p. 407):

$$dV = (r-\delta)\cdot V\cdot dt + \sigma\cdot V\cdot dz \tag{16}$$

dz describes a Wiener process. We use the discrete-time version of a GBM for the simulation:

$$V_{\tau+\Delta\tau} = V_\tau \cdot e^{\left[\left(r-\delta-\frac{\sigma^2}{2}\right)\cdot\Delta\tau+\sigma\cdot\varepsilon_{\tau+\Delta\tau}\cdot\sqrt{\Delta\tau}\right]} \tag{17}$$

ε is a standard normally distributed random number. With $\Delta\tau = 1$, we write:

$$V_{\tau+1} = V_\tau \cdot e^{\left[\left(r-\delta-\frac{\sigma^2}{2}\right)+\sigma\cdot\varepsilon_{\tau+1}\cdot\right]}$$

We are to answer the following questions: (1) What is the value of the investment option? and (2) What expected present value of investment cash flows would trigger an immediate investment?

5.2 Results and Validation

The results for both the early-exercise path and the value of the option are shown in Table 2. According to the BRSS-method, the investment option should be immediately exercised if the expected present value of investment cash flows exceeds 145.47 T€. Looking at the critical values at subsequent dates one sees very easily that the critical exercise path decreases exponentially with the reduction of the lifetime of the option. That was to be expected from theoretical insight (see right illustration in Figure 1). The value of the investment option according to the BRSS-method is 19.91 T€.

13 Disaggregated values (e.g. revenues and variable costs of the production) can be considered in principle. This would only demand minor modifications to the simulation model, binomial or even analytical solutions, however, would not be feasible.

14 This is equivalent to a discrete interest payment of 6 % p.a.

Table 2: Comparison of different valuation procedures

τ	Bounded recursive stochastic simulation (BRSS)	Approach by Grant et al.	Binomial tree method [a]	Approach by Ibanez and Zapatero	Genetic algorithm [b]
Critical early-exercise value V_τ^*					
0	145.47	145.43	145.27	145.39	144.57
1	142.43	141.53	142.50	142.66	140.69
2	138.55	137.78	138.96	138.55	134.24
3	132.53	133.73	133.56	133.42	128.59
4	124.34	124.48	125.11	124.94	121.94
5	100.00	100.00	100.00	100.00	100.00
American style option value F_0					
	19.91	19.88	19.86	19.77	19.67
Confidence interval for the "true" option value $\widetilde{F}_0$ (with 5% error probability)					
	$19.70 < \widetilde{F}_0 < 20.12$	$19.67 < \widetilde{F}_0 < 20.09$	-	$19.56 < \widetilde{F}_0 < 19.97$	$19.47 < \widetilde{F}_0 < 19.88$
Time required for programming the model					
	small	small	small	high	very high
Time required for computation of V_τ^* and F_0 [c]					
	approx. 30 min	approx. 2 h	approx. 5 min	approx. 8 h	approx. 12 – 24 h [d]

[a] The discretisation of the development of the state variable (value of investment cash flows) is 0.05 years.
[b] 100 generations with in each case 50 000 simulation runs.
[c] Computing time with direct programming in MS-EXCEL. For the simulation-based methods, 50 000 simulation runs are carried out; 1 400 MHz-processor.
[d] Highly dependent on random numbers.

The results of alternative numerical methods are also given in Table 2. The approach of Grant et al. (1997), the binomial tree method (as a non-simulation-based procedure), the approach of Ibanez and Zapatero (1998) and the genetic algorithms employed by Diaz (2001) and Balmann and Mußhoff (2002) are used for the valuation of the example real option. It is apparent that all procedures yield almost identical values for the critical early-exercise path. Only the value found by the genetic algorithms approach deviates a little bit from the others. But the option prices F_0 found by the different methods, including the one found by genetic algorithms, are virtually the same. Apparently the option price is not very sensitive to small deviations of the assumed early-exercise path. As a result of the validation one can state, that the BRSS yields highly accurate results.

The performance of the different numerical methods from the practical point of view (i.e. the amount of work involved) is also summarized in Table 2. For this simple problem, the binomial tree would clearly be the least cumbersome, both with regard to programming effort and computational time. It must be emphasised, however, that lattice methods in general do not show enough flexibility to integrate complex stochastic processes, multiple stochastic variables etc. in option pricing. On the other hand, genetic algorithms feature the highest programming and computation requirements and represent the most flexible option pricing method. They even allow for agent-based simulations and therefore facilitate, in contrast to *all* other methods, the direct integration of competition into an option pricing model. This can be important for the pricing of real assets, since real options are often only exclusive "to a certain degree". Consequently, real option pricing must often be set in a framework of game theory where we have to consider different "players" and where we must look for an equilibrium-strategy (Nash-equilibrium). However, since genetic algorithms require multiple programming effort and computational time compared to other simulation approaches, they should be used only if they are really essential. In all other cases, where we do not have to deal with competition, but only with complex (American type) options, the BRSS-method is to be preferred. It provides the fastest solution of all simulation-based procedures used in the test bed and requires the smallest programming effort.

6 Summary and Conclusion

Quite contrary to a belief still prevailing even in the "professional world" (see e.g. Hull, 2000, p. 408; Trigeorgis, 1996) American type options can be quite easily priced by methods using Monte Carlo simulation. In the last decade, a great number of simulation-based procedures have been proposed. They are flexible enough to value even complex options and therefore, in particular, real options which are often characterised by complex stochastic processes, multiple stochastic factors, correlation between different options etc. In other words: Pricing of complex financial options as well as realistic applications of the new investment theory and therefore the valuation of entrepreneurial flexibilities are made feasible through the integration of stochastic simulations in option pricing methods. Some of the procedures proposed so far suffer from an unsatisfactory flexibility or accuracy and/or a high programming and computational demand. Others, in contrast, are particularly appealing because of their accurateness, simplicity, flexibility and intuition. This is particularly true for the modification of the approach of Grant et al. (1997) we propose in this paper and call "Bounded Recursive Stochastic Simulation" (BRSS). Although this modification appears to be rather marginal, it allows for a significant reduction of computational time, without loss of applicability.

Corresponding to Grant et al. (1997), the BRSS integrates a sequential stochastic simulation of price paths in a backward recursive programming approach to determine the critical early-exercise path. It then values the option by initiating a simple Monte-Carlo simulation from the valuation date of the option. The determination of the critical early-exercise values is straightforward: Starting from the end and moving backward, for every exercise date, the critical value is determined by systematically simulating sample paths for the underlying emanating from different test-values at the respective date. The critical value falls between those test-values which yield a change of sign of the difference of intrinsic value and continuation value. It can be estimated by linear interpolation.

The BRSS is to be preferred to other simulation-based valuation procedures, which offer the same flexibility, due to its simplicity, intuition and ease of implementation. Furthermore, comparing its results to those of the binomial tree method shows that it yields highly accurate results. However, in real options problems, there may be the need for even more flexible valuation methods. This is for instance the case when we want to model competition and therefore price dynamics endogenously instead of using a given stochastic price process as an input for the option pricing model. Consequently, option pricing will have to take account of decisions of agents or "players" and will be set in a framework of game theory where we have to look for an equilibrium-strategy (Nash-equilibrium). This can be implemented by integrating genetic algorithms into simulation procedures (agent-based simulation procedures). Yet, it should be stated, that these methods are more complex and require much more computational time. Whenever possible, one should therefore use simpler procedures.

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