

**Reservation Price Announcement in
Sealed Bid Auctions: Comment**

John R. Schroeter*
Associate Professor
Department of Economics
Iowa State University

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*273 Heady Hall/ISU/Ames, IA 50011-1070
515-294-5876
johns@iastate.edu

ABSTRACT: This comment corrects some errors of analysis contained in a 1993 paper by Carey in the *Journal of Industrial Economics*. **KEYWORDS:** auction, reservation price

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In a 1993 paper in the *Journal of Industrial Economics*, Carey studies the problem of reservation price announcement in sealed bid procurement auctions. Assuming that the auctioneer/monopsonist has a self-supply option and is motivated by the desire to minimize the expected total cost of procurement, she investigates the auctioneer's decision to announce a reservation price and the level at which it should be set. Carey's analysis includes some errors, however. This comment proposes corrections. I begin by describing the auction setting.

The auctioneer seeks to purchase a single unit of a good or service from one of N bidders. Bidders' supply costs; denoted c_1, c_2, \dots, c_N ; are independently, identically distributed $U[0,1]$ random variables. In addition, the seller has the option of self-supply at a cost of c_0 which is also drawn from the $U[0,1]$ distribution independently of the other c_j s. The model incorporates the "private value" assumption in that each player, whether a bidder or the auctioneer, is assumed to know the actual value of his or her own supply cost but only the joint distribution from which the other players' costs are drawn. Bidders enter sealed bids with the objective of maximizing expected profit given the understanding that the contract will be awarded to the low bidder at a price equal to the low bid provided that it is no greater than a (possibly unannounced) reservation price.

For each auction mechanism considered, the solution is a symmetric, Nash equilibrium bid strategy, b^* , that can be thought of as a mapping of the range of possible supply costs into bids: $b^*: [0,1] \rightarrow \mathbb{R}$. The method for deriving b^* is the customary one used to solve for Nash equilibria: Consider a representative bidder, the i^{th} , who selects a bid, b_i , to maximize expected profit, π_i , assuming that all other bidders

play strategy b^* . By requiring that optimal play for bidder i likewise be consistent with b^* ; that is, by requiring that $b_i = b^*(c_i)$; the first order condition for expected profit maximization becomes a differential equation in $b^*(\cdot)$ which, when solved subject to a boundary condition dictated by the nature of the auction mechanism, yields a unique solution.

Expected profit for the i^{th} bidder is given by Carey's equation (1):

$$\pi_i(b_i, c_i) = (b_i - c_i) G(b_i), \quad (1)$$

where $G(b_i)$ is the probability that bidder i will win with a bid of b_i assuming other bidders play b^* . Differentiating with respect to b_i and setting the result to zero yields Carey's equation (2):

$$b_i = c_i - G(b_i) / G'(b_i). \quad (2)$$

In the first auction mechanism considered, the "no-reservation-price" auction, the auctioneer renounces the self-supply option and commits to purchase from the lowest bidder.¹ For this case

$$\begin{aligned} G(b_i) &= \text{Prob}\{ b_j \leq b^*(c_j) \text{ for } j = 1, 2, \dots, N, j \neq i \} \\ &= \text{Prob}\{ b^{*-1}(b_i) \leq c_j \text{ for } j = 1, 2, \dots, N, j \neq i \} \\ &= [1 - b^{*-1}(b_i)]^{N-1} \end{aligned} \quad (3)$$

where the second equality uses the assumption that $b^*(\cdot)$ is strictly increasing on $[0, 1]$ and the third reflects the particular assumed form of the distribution of the c_j s.²

Substituting into (2), imposing $b_i = b^*(c_i)$, and dropping the "i" subscript yields

$$b^{*'}(c) - \frac{N-1}{1-c} b^*(c) = \frac{-c}{1-c} (N-1). \quad (4)$$

Carey's solution obtains by solving this equation subject to the boundary condition

$$b^*(1) = 1.^3$$

$$b^*(c) = c + (1 - c)/N. \quad (5)$$

In the second auction considered, the "unannounced-reservation-price" or "secret-reservation-price" auction, the auctioneer solicits bids with the understanding that the contract will be awarded to the low bidder unless the low bid is greater than the auctioneer's own supply cost. If it is, the low bid will be rejected and the auctioneer will self-supply. In this case, $G(b_i)$ is given by

$$\begin{aligned} G(b_i) &= \text{Prob}\{ b_j \leq b^*(c_j) \text{ for } j = 1, 2, \dots, N, j \neq i; \text{ and } b_i \leq c_0 \} \\ &= [1 - b^{*-1}(b_i)]^{N-1} (1 - b_i). \end{aligned} \quad (6)$$

This specification reflects the fact that obtaining the contract requires that the i^{th} bidder beat all rival bidders' bids and beat the auctioneer's "bid" of c_0 . Substituting into (2), imposing $b_i = b^*(c_i)$, and dropping the "i" subscript yields

$$b^{*'}(c) = \frac{(N - 1)(1 - b^*(c))(b^*(c) - c)}{(1 - c)(1 - 2b^*(c) + c)} \quad (7)$$

For this auction mechanism, the boundary condition $b^*(1) = 1$ is clearly appropriate. No bidder would bid less than own cost under any circumstances, so $b^*(1) \geq 1$. But because the auctioneer's self-supply cost is confined to $[0,1]$, bids greater than 1 can be excluded as having no chance of winning. As can be verified, by direct substitution, the solution to (7) subject to $b^*(1) = 1$ is

$$b^*(c) = c + (1 - c)/(N + 1). \quad (8)$$

Carey also reaches this conclusion, but apparently by accident. Her analysis

mistakenly treats the auctioneer as an $N + 1^{\text{st}}$ symmetric bidder, so her equilibrium bid strategy is the solution to equation (4) subject to $b^*(1) = 1$, but with $N + 1$ replacing N . However the unannounced-reservation-price auction with N bidders is not equivalent to the no-reservation-price auction with $N + 1$ bidders for an important reason that was foreshadowed by the expression for $G(\cdot)$ in equation (6): To win the unannounced-reservation-price auction, a representative bidder must beat $N - 1$ bidders playing b^* plus an additional "bidder," the auctioneer, who "bids" not $b^*(c_0)$, but c_0 itself. That the two distinct games have identical equilibrium strategies appears to be merely an artifact of the particular distribution Carey uses. In any case, Carey's subsequent analysis of expected buyer costs in the unannounced-reservation-price case is correct as are the figures in the top half of her Table I.

In the third auction mechanism considered, the "announced-reservation-price" auction, the auctioneer announces a maximal acceptable bid, \bar{b} , prior to bidding. The contract is awarded to the low bidder if the low bid is less than or equal to \bar{b} . If no qualifying bid is submitted, the auctioneer engages in self-supply. Typically, not all bidders will submit qualifying bids in this case and, from the perspective of the representative bidder, the number who will is a random variable. Carey's analysis assumes that the representative bidder will behave as if facing a non-random number of bidders equal to the expected value of the actual distribution. But there is no need to resort to this sort of *ad hoc* assumption about behavior. The equilibrium bid strategy will be strictly increasing on $[0, \bar{b}]$ and bidders with supply costs greater than \bar{b} will decline to bid. Declining to bid can be thought of as submitting any bid greater than \bar{b} .

So in order to win, the i^{th} bidder facing $N - 1$ competitors who play strategy b^* must submit a bid no bigger than $b^*(c_j)$ for $j = 1, 2, \dots, N, j \neq i$. This means that $G(b_i)$ is exactly as given in (3). The only feature that distinguishes the analysis of this case from that of the no-reservation-price case is the boundary condition. Since bidders will never bid less than cost and only bids less than or equal to \bar{b} can win, $b^*(\bar{b}) = \bar{b}$ is the appropriate boundary condition to use in solving (4). The result is

$$b^*(c) = \begin{cases} c + \frac{1}{N}(1 - c) \left(1 - \left(\frac{1 - \bar{b}}{1 - c} \right)^N \right) & \text{for } c \leq \bar{b} \\ \text{no bid} & \text{for } c > \bar{b} \end{cases} \quad (9)$$

That Carey's proposed solution, her equation (8), fails to satisfy (4) can be confirmed by direct substitution.

Calculation of the expected buyer cost and the optimal reservation price for the announced-reservation-price model is easier within the context of a clever dual approach due to Riley and Samuelson. The following presentation simply adapts their analysis to the case of a monopsonist's auction rather than a monopolist's auction.

To simplify notation and to slightly generalize the analysis, use $F(\cdot)$ and $f(\cdot)$ to denote, respectively, the cumulative distribution and density functions of the distribution of supply costs on $[0, 1]$. As before, consider the problem from the perspective of the i^{th} bidder facing $N - 1$ rivals who play b^* . Assuming that $b^*(\cdot)$ is strictly increasing, the choice of a bid, b_i , is equivalent to the choice of $x_i \equiv b^{*-1}(b_i)$. Establishing that b^* is the optimal strategy for the i^{th} bidder amounts, in this case, to showing that the optimal choice of x_i is c_i .

Let $P(x_i)$ denote the expected payment to bidder i given that he "bids" x_i and let $\Pi(x_i, c_i)$ denote the expected profit for bidder i given that he "bids" x_i and has cost c_i .

Then

$$\begin{aligned}\Pi(x_i, c_i) &= P(x_i) - c_i \text{Prob}\{i \text{ wins the auction}\} \\ &= P(x_i) - c_i [1 - F(x_i)]^{N-1},\end{aligned}$$

where the second equality uses the fact that winning with a "bid" of x_i requires $b^*(x_i) < b^*(c_j)$ for all $j \neq i$; or $x_i < c_j$ for all $j \neq i$. Differentiating $\Pi(\cdot)$ with respect to x_i , evaluating at $x_i = c_i$, and setting the result to zero yields

$$P'(c_i) = - (N - 1) c_i [1 - F(c_i)]^{N-2} f(c_i). \quad (10)$$

This differential equation must hold for all c_i less than or equal to the maximal cost associated with non-negative expected bidder profit. This cost level, denoted \bar{c} , is implicitly defined by

$$\Pi(\bar{c}, \bar{c}) = P(\bar{c}) - \bar{c} [1 - F(\bar{c})]^{N-1} = 0. \quad (11)$$

To solve the differential equation subject to the boundary condition, integrate both sides of (10) over the interval $c_i \in [c, \bar{c}]$ and substitute for $P(\bar{c})$ from (11) to get an expression for the equilibrium expected payment to a bidder with supply cost c :

$$P(c) = c[1 - F(c)]^{N-1} + \int_c^{\bar{c}} [1 - F(x)]^{N-1} dx \quad \text{for } c \leq \bar{c}. \quad (12)$$

The maximal supply cost consistent with non-negative expected bidder profit is simply \bar{b} , the announced reservation price. Given \bar{b} , the auctioneer's expected payment to an individual bidder with random supply cost is

$$p(\bar{b}) = \int_0^{\bar{b}} P(c)f(c)dc.$$

Substitution from (12) and integration by parts yields⁴

$$p(\bar{b}) = \int_0^{\bar{b}} (c f(c) + F(c)) [1 - F(c)]^{N-1} dc.$$

The auctioneer's expected total cost of procurement of the good or service is simply N times the expected payment to a representative bidder plus the self-supply cost weighted by the probability that no qualifying bids are submitted:

$$T(\bar{b}) = N p(\bar{b}) + c_0 [1 - F(\bar{b})]^N \quad (13)$$

To find the optimal reservation price, minimize $T(\bar{b})$ with respect to \bar{b} . The solution, denoted \bar{b}^\dagger , is implicitly defined by⁵

$$\bar{b}^\dagger = c_0 - F(\bar{b}^\dagger) / f(\bar{b}^\dagger). \quad (14)$$

For the particular case of a $U[0,1]$ distribution of supply costs, $F(\bar{b}) = \bar{b}$, and $f(\bar{b}) = 1$. Making these substitutions in (14) and solving yields $\bar{b}^\dagger = c_0/2$; the optimal reservation price is independent of N . Thus the entries in the top half of Carey's Table II are wrong. Each entry should simply be one half of the value for self-supply cost that heads the corresponding column. The expected costs to the auctioneer associated with choosing the reservation price to be c_0 or to be the optimal value, $c_0/2$, are found by substitution into equation (13):

$$T(c_0) = \frac{2}{N+1} [1 - (1 - c_0)^{N+1}] - c_0(1 - c_0)^N, \quad (15)$$

$$T(c_0/2) = \frac{2}{N+1} [1 - (1 - c_0/2)^{N+1}]. \quad (16)$$

The correct entries for the lower halves of Carey's Tables I and II are generated by formulas (15) and (16), respectively. Corrected versions are presented as Tables I and

II here.⁶

As Carey claimed, announcing a reservation price equal to self-supply cost is always inferior, from the auctioneer's perspective, to keeping the reservation price secret: The entries in my Table I are all larger than the corresponding entries in the top half of Carey's Table I. Also as Carey claimed, the choice between the strategies of a secret reservation price and an optimal announced reservation price depends on parameter values. Carey found that the secret reservation price is always superior for sufficiently few bidders and the optimal announced reservation price is always superior for sufficiently many bidders. I find that the optimal announced reservation price is preferred for a mid-range of bidder numbers when c_0 is in the neighborhood of 0.5 or 0.6, and the secret reservation price strategy is preferred in all other cases.⁷

One additional comment deserves mention. Both Carey's original analysis of the auctioneer's choice of auction mechanism and my "correction" of it are flawed in that they assume limited rationality on the part of bidders. In particular, they do not credit bidders with the ability to draw appropriate inferences from the auctioneer's choice of auction design. If the auctioneer elects to keep the reservation price secret, for example, bidders should use this information to update their priors on the auctioneer's self-supply cost because they know that a secret reservation price would not be rational for certain values of c_0 in the $[0,1]$ interval. But this updating of priors is not reflected in the derivation of bid strategy (8) nor in the calculation of the numbers in the top half of Carey's Table I. This feature of the problem warrants further study.

Notes

¹In this analysis, I adopt an assumption that McAfee and McMillan identify as a common feature of conventional auction theories: The auctioneer has the ability to commit to an announced set of procedures. In the no-reservation-price auction model, for example, I rule out the possibility that the auctioneer, upon receiving bids, might renege and reject the low bid.

²The assumption of strict monotonicity of $b^*(\cdot)$ is later validated by the result in equation (5).

³Actually, it is not clear that a boundary condition of $b^*(1) = 1$ is appropriate for this model. Because bidders would never bid below cost, it makes sense to require $b^*(1) \geq 1$. Bids in excess of the maximal value for the seller's supply cost cannot be ruled out, however, if the seller has credibly renounced the self-supply option. At least it can be said that Carey's solution is the unique equilibrium for an auction subject to an announced reservation price of $\bar{b} = 1$.

⁴This result is a counterpart to Riley and Samuelson's Proposition 1. An alternative route to the expression for the optimal bid strategy begins with the observation that

$$P(c) = b^*(c) [1 - F(c)]^{N-1}.$$

Substituting from (12) and solving yields

$$b^*(c) = c + \frac{\int_c^{\bar{b}} [1 - F(x)]^{N-1} dx}{[1 - F(c)]^{N-1}} \quad \text{for } c \leq \bar{b}$$

This is a counterpart to Riley and Samuelson's Proposition 2. With $F(x) = x$, as is appropriate for the $U[0,1]$ distribution, the above equation reduces to equation (9).

⁵This is a counterpart to Riley and Samuelson's Proposition 3 which should read

$$v_* = v_0 + [1 - F(v_*)]/F'(v_*)$$

⁶For large values of N and/or c_0 , the Tables report "equal" pairs of entries in many cases. This appearance is due to rounding error; the optimal reservation price always yields a strict global minimum of expected buyer cost.

⁷Entries in my Table II are less than the corresponding entries in the top half of Carey's Table I for $c_0 = 0.5$ with $N = 5, 6, 7, 8,$ and 9 ; and for $c_0 = 0.6$ with $N = 4$ and 5 . In all other cases, my Table II entries exceed their counterparts in the top half of Carey's Table I. A similar pattern emerges for c_0 values between 0 and 0.5 . For example, when $c_0 = 0.3$, the optimal reservation price is superior to the unannounced reservation price for $N = 7, 8, 9, 10,$ and 20 , while the unannounced reservation price is preferred for the other values of N in the Table.

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