ELSEVIER

Agricultural Economics 12 (1995) 1–9

AGRICULTURAL ECONOMICS

Approximation methods for ranking risky investment alternatives

David Bigman

Department of Agricultural Economics, Faculty of Agriculture, The Hebrew University of Jerusalem, P.O. Box 12, Rehovot 76100, Israel

Accepted 20 September 1994

Abstract

The paper develops and illustrates the application of criteria for ranking risky investment alternatives that are based on their certainly equivalent (CE) outcomes and determines expressions for approximating the CE outcomes by means of the central moments of their distribution. The paper develops criteria on the basis of the CE outcomes for determining a complete ranking of risky investment alternatives that can represent the choice of many – though not all – risk-averse agents.

The application of means and variances for ranking risky investment alternatives has long been realized in the economic and financial literature (Markovitz, 1952, 1959; Levy, 1974; Tobin, 1958; Tsiang, 1972). It has also been realized, however, that criteria that apply to these statistics are only approximations which may often be misleading (Borch, 1969; Feldstein, 1969; Meyer, 1977, 1987) or represent the choice of only a small group of agents. Nevertheless, their simplicity and intuitive appeal made the expected value-variance (E-V) criteria widely used. (See also Hadar and Russel, 1969; Hanoch and Levy, 1969; Levy and Markowitz, 1979; Samuelson, 1970; Baron, 1977.)

The theoretical work in this area has generally focused on criteria for establishing the ranking of investment alternatives that can represent the choice of *all* 'ordinary' agents – through the Stochastic Dominance (SD) rules – or criteria that represent the choice of a well defined (and necessarily limited) *subgroup* of agents whose utility structure can be clearly identified. The SD criteria are notoriously incomplete however, since they can only identify those alternatives that would *not* be selected by ordinary risk-averse agents but they cannot establish priorities between the alternatives that may be selected. Criteria that are based on the expected utility (EU) for a specific class of utility functions are often much too limiting.

This paper develops and illustrates the application of criteria for ranking risky investment alternatives that are based on their certainty equivalent (CE) outcomes and determines expressions for approximating the CE outcomes be means of the first two – or three – central moments of their distribution. Clearly, the ranking of investment alternatives by their CE outcome is consistent with their ranking by their EU. By comparing the CE outcomes, however, we can determine not only *whether* one investment alternative would be preferred over another but also *by how much* would the agent be better off in selecting the one rather than the other. In other words, these criteria provide *cardinal* measures of desirability.

SSDI 0169-5150(94)00035-2

The main goal of the paper is to develop, on the basis of the CE outcomes, criteria for determining a *complete* ranking of investment alternatives that can represent the choice of many – *though not all* – risk-average agents. These criteria will allow us to exclude from the 'efficient set' of alternatives (i.e. alternatives that cannot be ranked by the SD criteria) those alternatives that may be selected only by the *most* risk-averse or the *least* risk-averse agents, and leave in the 'choice-set' only those alternatives that are likely to be selected by most 'ordinary' agents thereby considerably reducing the choice set from which the most desirable alternative is likely to be selected by the large majority of agents.

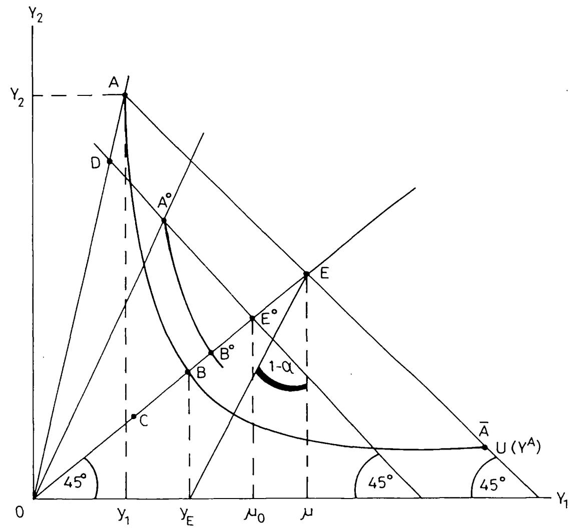

Fig. 1.

1. Cardinal ranking criteria and the approximations methods

Let $Y = (y_1, \ldots, y_n) \in \Omega^n$ be a set of possible 'outcomes' (i.e., present values of future net returns) of a given investment alternative. To simplify the notations – and without liming the generality – assume all the outcomes to be equally probable and non-negative. Let μ denote the expected outcome of (i.e., the present value of the expected return from) that investment and let y_E denote the 'certainty equivalent (CE) outcome.

To illustrate the proposed criterion and describe the method of approximation, let us turn to Fig. 1. Consider the set of (two) possible outcomes (y_1, y_2) indicated by point A. The mean value of these two (equally likely) outcomes is determined in Fig. 1 at the intersection of the (positive) 45° ray from the origin with the (negative) 45° ray that crosses through A at E. The agent's preferences are assumed to be represented by a continuously differentiable and (strictly) concave utility function that has the expected utility property. In the figure, these preferences are shown by the indifference curve that crosses through A. The intersection of that indifference curve with the (positive) 45° ray from the origin at B determines the CE outcome y_E.

A comparison of the investment alternative given by the outcomes at A with another alternative given by the outcomes at A° shows that the former has a higher mean ($\mu > \mu^o$) but also a higher spread. In the two-dimensional illustration, the higher spread is indicated by the steeper slope of the ray OA, which shows the ratio between the two outcomes, relative to that of OA°. Furthermore, the outcomes at A do not dominate the outcomes at A° in the sense of the first criterion of stochastic dominance (i.e. Pareto-dominance) since $y_1^o < y_1$ but $y_2^o < y_2$. The choice between these two investment alternatives clearly depends on the degree to which agents are risk-averse. Risk-neutral agents would select the investment alternative that offers the highest expected outcome and prefer A over A°. The difference $(\mu - \mu^o)$ expresses their potential gains from this choice in money terms. Extreme risk-averse agents would select the alternative that offers the highest outcome in the worst case and prefer A° over A. The difference $(y_1^o - y_1)$ expresses their potential gains from this choice in money terms.

In general, these two portfolios are included in the *efficient set* from which the final choice of the most desirable alternative would be made. To identify that choice we must have more knowledge on the agent's preferences and, in particular, on the degree to which he is risk-averse. Without that knowledge, we cannot determine whether the alternative with the outcomes at A would be preferred over the one at A^o, no matter how much larger y_2 is than y_2^o and how small is the difference between y_1^o and y_1. In other words, the ranking of investment alternatives on the basis of the SD criteria only, is *incomplete* and often not very helpful. These, of course, are familiar problems in the theoretical analysis of choices among risky investment alternatives which every textbook in financial analysis must encounter early on. If we have *some* knowledge on the agent's preferences, however, e.g., if we know that he is neither risk-neutral nor the most extreme (Max-Min) risk averse, or if we know the *range* within which his true degree of relative risk aversion is likely to be, we should be able to use that knowledge in order to determine a *more complete* ranking of investment portfolios and thus a smaller 'efficient' set – by excluding those alternatives that ordinary agents are *not* likely to select.

To identify that criterion, consider agents having a 'rational' preference ordering (i.e., complete, transitive and reflexive) that can be represented by an appropriate utility function having the expected utility property. To represent 'ordinary' risk-averse agents, Arrow (1965) added the following requirements:

(i) $U'(y) > 0$

(ii) $U''(y) < 0$

(iii) $\frac{\mathrm{d}}{\mathrm{d}y}\left(-\frac{U''(y)}{U'(y)}\right) \leqslant 0$

(iv) $\frac{\mathrm{d}}{\mathrm{d}y}\left(-\frac{U''(y)\cdot y}{U'(y)}\right) \geqslant 0$

Ordinary agents are thus assumed to be strictly risk-averse, having non-increasing *absolute* risk-aversion and non-decreasing *relative* risk aversion. By referring back to the figure, notice that outcomes along the (negative) 45° line that cross through A (strictly) *inside* the segment [$\bar{\text{A}}$, A] represent mean-preserving (strict) reductions in spread. If the initial spread of the outcomes at A is not 'too large', we can apply a Taylor-series approximation of Y^A around the expected outcome μ, and, assuming that we can disregard the remainder beyond the second-order (an assumption that I will relax later on), we can present the expected utility as a function of the first two central moments of the distribution:

$$\mathrm{EU}(Y^A) \approx U(\mu) + \tfrac{1}{2}U''(\mu)\cdot V(Y) \tag{1}$$

where $V(Y)$ is the variance of the outcomes at A, and $\approx$ indicates an approximation.

The (certain) outcome y_E can be expressed as a fraction of the mean, i.e., $y_E = (\alpha\cdot\mu) : \alpha \leq 1$. The smaller that fraction is, the more risk-averse is the agent. $\alpha = 1$ represents an agent who is risk-neutral, whereas $\alpha = (y_1/\mu)$ represents agents that select portfolios on the basis of the Max-Min rule. The expression $[(1-\alpha)\cdot\mu]$ represents the loss in utility, expressed in terms of money, on account of the spread – and the risk – of the outcomes at A. [1] Applying a Taylor-series approximation of the expected utility at B around E, we get;

$$U(y_E) = U(\mu) + (\alpha - 1)\cdot\mu\cdot U'(y)$$
$$y_E < y < \mu \tag{2}$$

Define the index I_A as that value which *exactly* equates:

$$U(y_E) = U(\mu) - I_A\cdot\mu\cdot U'(\mu) \tag{3}$$

With a strictly positive and strictly concave utility function: $I_A > (1-\alpha)$. By combining (1) and (3) we can express that index as:

$$I_A = \tfrac{1}{2}C^2(Y)\cdot R(\mu) \tag{4}$$

where

$$R(\mu) = -\frac{U''(\mu)}{U'(\mu)}\mu$$

is the coefficient of relative risk aversion at μ, and $C^2(Y) = [V(Y)/\mu^2]$ is the second central moment of the distribution of the outcomes. That index thus establishes an *upper* bound on the true value of $(1-\alpha)$. To establish a lower bound

[1] The fraction $(1-\alpha)$ is often referred to as Atkinson's measure of inequality (or spread) – see Atkinson (1970). Extending this analogy, notice that:

$0 \leq (1-\alpha) \leq (\mu - y_1)/\mu$

In the event that μ is taken to represent the 'poverty-line', then $(1-\alpha)$ is bounded from above by the maximum 'poverty gap'.

on that value notice that, by definition and by the properties (i) and (ii) of the utility function:

$$(1-\alpha) < (1-\alpha)\frac{U'(y)}{U'(\mu)}$$
$$= I_A < (1-\alpha)\frac{U'(y_E)}{U'(\mu)} \qquad y_E < y < \mu$$

By expanding $U'(y_E)$ about μ, assuming that $U''' \geq 0$, we therefore get:

$$(1-\alpha) < I_A < \left[(1-\alpha) + (1-\alpha)^2 \cdot R(\mu)\right] \quad (5)$$

Hence, the smaller the spread of the outcomes and the less risk-averse the agent, the more accurate the approximation of $(1-\alpha)$ by means of the index I_A. The condition $U'''(y) \geq 0$ is necessary and sufficient to secure property (iii) of the utility function. Tsiang (1972) termed this property "skewness preference", which characterises 'normal' risk-averse agents. By inserting (4) into (5), and granted that $V(Y) = 0$ implies $\alpha = 1$, we can therefore conclude that, for ordinary agents:

$$I_A \geq (1-\alpha) \geq \frac{+\sqrt{1 + 4 \cdot R(\mu) \cdot I_A} - 1}{2 \cdot R(\mu)} \quad (6)$$

and therefore:

$$\mu(1 - I_A) < y_E < \mu\left[1 - \frac{\sqrt{1 + 4 \cdot R(\mu) \cdot I_A} - 1}{2R(\mu)}\right]$$

The index I_A, which evaluates the utility losses (expressed in terms of money) on account of the risk, can be used to establish criteria for ranking alternative risky prospects, as we shall see in the following propositions. To simplify the notations, the criteria are spelled out in the propositions for agents having a constant coefficient of relative risk aversion, although in the proofs themselves the more general criteria are determined. For the same reason, I also assume at this stage that the second-order approximation is sufficiently close, and discuss only later approximations of a higher order.

1.1. Proposition 1

Consider the ranking of two risky investments Y and Y^o – where $\mu > \mu^o$ and $V(Y) > V(Y^o)$ – that is determined by 'ordinary' risk-averse agents having a constant coefficient of relative risk aversion.

– A *sufficient* condition for $\mathrm{EU}(Y) > \mathrm{EU}(Y^o)$ is:

$$\tfrac{1}{2}R(\mu)\left[\frac{V(Y)}{\mu} - \frac{V(Y^o)}{\mu_0}\right] < (\mu - \mu_0) \quad (7)$$

– A *necessary* condition for $\mathrm{EU}(Y) > \mathrm{EU}(Y^o)$ is:

$$\tfrac{1}{2}R(\mu)\left\{\left[1 - \frac{(\mu-\mu_0)}{\mu_0}R(\mu)\right]\frac{V(Y)}{\mu} - \frac{V(Y^o)}{\mu_0}\right\} < (\mu - \mu_0) \quad (8)$$

Proof. A necessary *and* sufficient condition for $U(y_E) > U(y_E^o)$ is, by definition:

$$U(\mu) - I_A \cdot \mu \cdot U'(\mu) > U(\mu_0) - I_A^o \cdot \mu_0 \cdot U'(\mu_0) \quad (9)$$

But, with a strictly concave utility function:

$$U(\mu) > U(\mu_0) + (\mu - \mu_0) \cdot U'(\mu)$$

and

$$U'(\mu) < U'(\mu_0)$$

Hence, a *sufficient* condition for (9) is:

$$(\mu - \mu_0) > I_A \cdot \mu - I_A^o \cdot \mu_0 \quad (10)$$

By inserting the corresponding approximations of I_A and I_A^o [as in (4)] into the latter inequality we get the following sufficient condition: [2]

$$(\mu - \mu_0) > \frac{1}{2}\frac{V(Y)}{\mu}R(\mu) - \frac{1}{2}\frac{V(Y^o)}{\mu_0}R(\mu_0) \quad (11)$$

[2] Notice that this condition does *not* depend on how close the approximation I_A is to the true value of $(1-\alpha)$. Rather, it depends on how close is the approximation of the index I_A itself by means of the first two central moments, which is determined (4). If the outcomes are skewed, we may have to use a third-order approximation – which I discuss later. It is then showed that (12) establishes a sufficient condition if, *in addition*: $(\mu C^3 - \mu_0 C_0^3) > 0$, where C^3 and C_0^3 are the third central moment of the corresponding distributions.

If we assume $R(\mu) = R(\mu_0)$, we get from (11) the sufficient condition (7).

To determine the *necessary* condition, insert into the inequality in (9) the following inequality:

$$U(\mu) < U(\mu_0) + (\mu - \mu_0) \cdot U'(\mu_0)$$

to obtain the necessary condition, given by:

$$(\mu - \mu_0) > I_A \cdot \mu \frac{U'(\mu)}{U'(\mu_0)} - I_A^{\circ} \cdot \mu_0 \qquad (12)$$

For an ordinary risk-averse agent, the ratio $[U'(\mu)/U'(\mu_0)]$ is strictly positive and smaller than 1. If we assume also $U''' > 0$, then the bounds of this ratio would be given by:

$$1 > \frac{U'(\mu)}{U'(\mu_0)} > 1 - \frac{(\mu - \mu_0)}{\mu_0} R(\mu_0) \qquad (13)$$

It should be noted, though, that the expression that specifies the lower bound may well be negative. In that case, that value of the lower bound does not add any relevant information since we already know that for ordinary agents this ratio must be non-negative. In other words, in that case the only relevant necessary condition is simply $\mu > \mu_0$, whereas the additional condition in (8) is redundant. In the event that the lower bound in (13) is strictly positive, however, we can insert that lower limit into (12) and obtain the following necessary condition:

$$(\mu - \mu_0) > I_A \cdot \mu d \left[1 - \frac{(\mu - \mu_0)}{\mu_0} R(\mu_0) \right] - I_A^{\circ} \cdot \mu_0 \qquad (14)$$

By inserting the corresponding values of I_A and I_A° into (14) and assuming $R(\mu) = R(\mu_0)$, we get the necessary condition in (8). Q.E.D.

Borch (1969) has warned us, however, that the criteria put forward in the proposition should be applied with caution since it may well be the case that *all* the outcomes in Y are larger than the outcomes in Y°, i.e., that Y dominates Y° according to the first criterion of SD. For the proposition to have any meaning this possibility must be considered first and only when it is ruled out can the criteria be applied. The following proposition determines necessary and sufficient conditions for $\mathrm{EU}(Y) > \mathrm{EU}(Y^{\circ})$ by a proper application of the upper and lower bounds on $(1 - \alpha)$ and thus also on the value of Y_E that has been determined in (6):

1.2. Proposition 2

Consider the ranking of two risky investments Y and Y° – where $\mu > \mu_0$ and $V(Y) > V(Y^{\circ})$ – that is determined by an ordinary risk-averse agent having a constant coefficient of relative risk aversion.

– A *sufficient* condition for $\mathrm{EU}(Y) > \mathrm{EU}(Y^{\circ})$ is:

$$(\mu - \mu_0) > \tfrac{1}{2}\mu C^2 \cdot R(\mu) - \frac{\mu_0}{2R(\mu)} \left(\sqrt{1 + 2 \cdot R^2(\mu) \cdot C_0^2} - 1 \right) \qquad (15)$$

– A *necessary* condition for $\mathrm{EU}(Y) > \mathrm{EU}(Y^{\circ})$ is:

$$(\mu - \mu_0) > \frac{\mu}{2 \cdot R(\mu)} \left(\sqrt{1 + 2 \cdot R^2(\mu) \cdot C^2} - 1 \right) - \tfrac{1}{2}\mu_0 C_0^2 \cdot R(\mu) \qquad (16)$$

Since the restriction on the value of $R(\mu)$ which is implicit in the derivation of the *necessary* condition for $\mathrm{EU}(Y) > \mathrm{EU}(Y^{\circ})$ in (8) may place undue restrictions on the scope of the analysis (i.e., on the agents group of agents for which these conditions are relevant), I will focus mainly on the sufficient condition (7) and on the two conditions (15) and (16) in Proposition 2 and leave the discussion on the necessary condition (8) mainly to the endnotes. [3] The key for determining the specific functional form of these conditions is the underlying assumption that the approximations of the indices I_A and I_A° by means of the first two central moments is indeed sufficiently close. If the outcomes are highly skewed, however, these

[3] Notice, for instance, that, if $V(Y^{\circ}) = 0$, the necessary condition (8) becomes:

$$\frac{(\mu - \mu_0)}{\mu_0} > \frac{V(Y) \cdot R(\mu)}{\left[2\mu\mu_0 + V(Y) \cdot R^2(\mu) \right]}$$

This condition thus establishes the ranking of the risky alternative Y vis-a-vis the certain alternative Y°.

approximations may no longer be adequate, and approximations by central moments of a higher order should be used, properly changing the conditions of the propositions. Thus, for example, with a constant relative risk-aversion, the approximation of the index I_A by means of the first *three* central moments would be given by: [4]

$$I_A \approx \tfrac{1}{2}C^2(Y)\cdot R(\mu) - \tfrac{1}{6}C^3(Y)\cdot R(\mu)[1+R(\mu)] \quad (17)$$

where $C^2(Y)$ and $C^3(Y)$ are the second and the third central moments of the distribution. With this approximation, the sufficient condition in Proposition 1 becomes:

$$(\mu-\mu_0) > \frac{R(\mu)}{2}\left[\frac{V(Y)}{\mu} - \frac{V(Y^o)}{\mu_0}\right] - \frac{R(\mu)\cdot[1-R(\mu)]}{6}(\mu C^3 - \mu_0 C_0^3) \quad (18)$$

Clearly, if $\mu C^3 > \mu_0 C_0^3$, the sufficient condition in (7) is still relevant – although we may lose vital information by ignoring the third central moment of the distributions.

The criteria established in the propositions for ranking risky investment alternatives by comparing their EU should obviously be identical to the criteria established by comparing their CE outcomes. For the alternatives under consideration we can make use of our earlier notations to specify these criteria in the following form:

$y_E > y_E^o$ if, and only if: $\alpha > \alpha_0\beta$, where $\beta = \mu_0/\mu$

Indeed, the sufficient condition for $U(y_E) > U(y_E^o)$ which has been established in Proposition 1 can also be obtained directly by inserting the approximations $I_A \approx (1-\alpha)$ and $I_A^o \approx (1-\alpha_0)$ into the condition $\alpha > \alpha_0\beta$, yielding: $(1-I_A) > \beta(1-I_A^o)$. In general we do not know, however, how close these approximations are, and we need the detailed procedure developed in the proofs of the propositions in order to determine that these indeed are necessary and sufficient conditions. To the extent possible, we should also make more accurate approximations of I_A and I_A^o via central moments of a higher order.

The second-order approximation of the index I_A is proportional to the index $\pi^* = \tfrac{1}{2}V(Y)\cdot R(\mu)$. That index has been termed by Pratt (1964, p. 134) "proportionate risk premium", the 'risk-premium' itself being defined as: $\pi \approx \tfrac{1}{2}[V(Y)\cdot r]$, where $r = R(\mu)\cdot\mu$ is the coefficient of *absolute risk aversion* (p. 125). I_A itself is therefore a unit-free measure of the proportionate risk-premium. The sufficient condition established by this index in (7) closely resembles – but is different from – the criterion suggested by Baumol (1963) for ranking investment portfolios, which is based on an ad-hoc expected utility function of the form: $\text{EU}(Y) = \mu - k\sigma$, where σ is the standard deviation of the outcomes and k represents the agent's risk-aversion. By this criterion, $\text{EU}(Y) > \text{EU}(Y^o)$ if, and only if:

$$\mu - k\sigma > \mu_0 - k\sigma_0$$

In contrast, the condition in (7) can be written as:

$$\mu - \tfrac{1}{2}r(\mu)\cdot V(Y) > \mu_0 - \tfrac{1}{2}r(\mu_0)\cdot V(Y^o)$$

And this condition is sufficient but not necessary for $\text{EU}(Y) > \text{EU}(Y^o)$. In fact, it may well be the case that the two criteria will determine contradicting rankings of the same investment alternatives. [5]

The following properties of the criteria in the propositions should be noted:

(i) The *sufficient* conditions (7) and (15) indicate that, up to the second-order approximation,

[4] To prove this, notice that $R'(\mu)=0$ implies:

$$\frac{U'''(\mu)}{U'(\mu)} = \frac{1}{\mu^2}R(\mu)\cdot[1+R(\mu)]$$

[5] In other words, if we define $k = \tfrac{1}{2}R(\mu)$ in order to present the two criteria in the same functional form, then it may be the case that the following inequalities hold:

$$k\left[\frac{\sqrt{V}}{\mu} - \frac{\sqrt{V_0}}{\mu_0}\right] < (\mu-\mu_0) < k\left[\frac{V}{\mu} - \frac{V_0}{\mu_0}\right]$$

In that case, $\text{EU}(Y) > \text{EU}(Y^o)$ according to Baumol's criterion whereas the inequality on the right hand side of the latter expression establishes the *necessary* condition for $\text{EU}(Y^o) > \text{EU}(Y)$.

i.e., provided that moments of a higher order can be safely disregarded, the two conditions:

$$\mu > \mu_0 \quad \textit{and} \quad \frac{V(Y)}{\mu} < \frac{V(Y^{\circ})}{\mu_0}$$

jointly, are sufficient to establish the superiority of Y over Y_0 for a *all* positive $R(\mu)$, i.e. for *all* 'ordinary' risk-averse agents having a constant coefficient of relative risk aversion. In that case we can say that, for these agents, *Y stochastically dominates Y°.*

(ii) If we use the third-order approximation, the sufficient condition (18) determines the following *joint* sufficient conditions for stochastic dominance of Y over Y°:

$$\mu > \mu_0$$

$$\frac{V(Y)}{\mu} < \frac{V(Y^{\circ})}{\mu_0} \quad \textit{and} \quad \mu C^3 > \mu_0 C_0^3$$

These conditions thus indicate that Y may have a higher mean and a lower variance but than Y° some agents may still prefer Y° over Y in the event that $\mu C^3 < \mu_0 C_0^3$, so that the sufficient condition (18) is not satisfied.

(iii) I_A rises with both $V(Y)$ and $R(\mu)$ when it is approximated by the second-order approximation. When we use the third-order approximation, we find:

$$\frac{\mathrm{d}I_A}{\mathrm{d}R(\mu)} = \tfrac{1}{2}C^2 - \tfrac{1}{6}C^3(1 + 2 \cdot R(\mu))$$

If $C^3 < 0$ then I_A still rises with $R(\mu)$; if $C^3 > 0$, however, and the agent is 'highly risk-averse' – in the sense that:

$$R(\mu) > \tfrac{1}{2}\left(\frac{3C^2}{C^3} - 1\right)$$

then a further rise in $R(\mu)$ will *lower* the value of that index. To examine the seeming 'paradox' that this may raise, consider again the sufficient conditions in (11) and (18). If we focus on the second-order approximation only and evaluate the two investment alternatives Y and Y° where $\mu > \mu^{\circ}$ *and* $\mu C^2 > \mu_0 C_0^2$, then these conditions lead to the following conclusion: If an agent, having a (constant) coefficient of relative risk aversion $\bar{\mathscr{R}}$, ranks: $\mathrm{EU}(Y) > \mathrm{EU}(Y^{\circ})$, then *all* agents having a coefficient of risk aversion *equal to or lower than* $\bar{\mathscr{R}}$, will also agree with that ranking. This, indeed, is the result we normally expect. Consider however, the third-order approximation, and assume that both $\mu C^2 > \mu_0 C_0^2$ *and* $\mu C^3 > \mu_0 C_0^3$. From (18) we can find a value of $\bar{\bar{\mathscr{R}}}$[6] such that the *less* risk-averse agents – agents having a coefficient of relative risk aversion *lower* than $\bar{\bar{\mathscr{R}}}$ – *may* rank Y° higher than Y (in the sense that the *necessary* conditions for $\mathrm{EU}(Y^{\circ}) > \mathrm{EU}(Y)$ will be satisfied) whereas the *more* risk-averse agents – having a coefficient of relative risk aversion *higher* than $\bar{\bar{\mathscr{R}}}$ – may rank Y (that has the larger variance) higher than Y°. The reason is that when the outcomes are skewed, the variance may not be an adequate indicator of the risk.

(iv) I_A is a well defined *cardinal* measure in the sense that its value does not change as an effect of linear transformations of the utility function.

(v) I_A is *scale-independent* if, and only if, $R(\mu)$ is constant.

2. Ranking investment alternatives for sub-groups of agents

The criteria determined in the previous section require some a-priori knowledge on the preferences of the economic agents – particularly on the degree to which they are risk-averse. These criteria can also be used, however, to determine the ranking of risky investment alternatives when we have *only partial* knowledge on the agent's preferences, e.g., when all we know is that the agent is not extreme risk-averse, but we still do not know exactly to what extent is he risk-averse.

To define that criterion, consider two invest-

[6] That value is given by:

$$\bar{\bar{\mathscr{R}}} > \frac{1}{2}\left\{\frac{3(\mu C^2 - \mu_0 C_0^2)}{(\mu C^3 - \mu_0 C_0^3)}\right\} - 1$$

ment alternatives Y and Y° such that $\mu > \mu_0$ and $V(Y) > V(Y^{\circ})$ and none dominates the other in the sense of the first and the second criteria of the SD. Assume also, that the agent has a constant coefficient of relative risk aversion. From the sufficient condition (7) we can calculate the *critical value* R^* of the coefficient of relative risk aversion, given by:

$$R^* = \frac{2(\mu - \mu_0)}{\left[\dfrac{V(Y)}{\mu} - \dfrac{V(Y^{\circ})}{\mu_0}\right]} \tag{19}$$

From the sufficient condition in Proposition 1 we can conclude that Y would be ranked higher than Y° by all economic agents having a (constant) coefficient of relative risk aversion *smaller* than R^*. The necessary condition in (8) determines the critical value R^{**} such that Y° would be ranked higher than Y by all agents having a (constant) coefficient of relative risk aversion *larger* than R^{**}. That value is determined in the Appendix.

When the first two central moments of the distribution do not provide an adequate approximation of the index I_A, we can use the first *three* moments and determine the critical value of R^* by inserting the approximations in (17) to the corresponding sufficient condition in Proposition 1, yielding the equality:

$$\tfrac{1}{6}\left(\mu C^3 - \mu_0 C_0^3\right) R^{*2} + \left[\tfrac{1}{6}\left(\mu C^3 - \mu_0 C_0^3\right) - \tfrac{1}{2}\left(\mu C^2 - \mu_0 C_0^2\right)\right] R^* + (\mu - \mu_0) = 0$$

from which the critical value R^* can be determined. In this case, however, it would depend on all the three central moments to determine whether this critical value constitutes an *upper* or a *lower* limit on the values of $R(\mu)$ for which $\mathrm{EU}(Y) > \mathrm{EU}(Y^{\circ})$. When $\mu > \mu_0$, sufficient conditions for R^* to determine an *upper* limit, i.e., sufficient conditions for $\mathrm{EU}(Y) > \mathrm{EU}(Y^{\circ})$ for all $R(\mu) < R^*$ are:

$$\mu C^3 < \mu_0 C_0^3 \quad \textit{and} \quad \mu C^2 > \mu_0 C_0^2$$

whereas sufficient conditions for R^* to determine a *lower* limit, i.e., sufficient conditions for $\mathrm{EU}(Y) > \mathrm{EU}(Y^{\circ})$ for all $R(\mu) > R^*$ are:

$$\mu C^3 > \mu_0 C_0^3 \quad \textit{and} \quad \mu C^2 < \mu_0 C_0^2$$

When the outcomes are skewed it may therefore be *wrong* to conclude that the *less* risk averse the agent the more likely he is to select the alternative that has the smaller variance.

The ranking of 'non-inferior' investment alternatives, i.e., alternatives that are not dominated by any of the others in the sense of the first or the second SD criteria, cannot be uniquely established for *all* (risk-averse) agents, since that ranking depends on the *extent* to which agents are risk-averse, and thus it may differ form one risk averse agent to another. A unique ranking can be established, however, for *sub-groups* of agents by means of the critical value R^* that can be calculated from either one of the sufficient conditions in Proposition 1. That critical value R^* can be calculated by means of the central moments of the distribution of the outcomes and it does not require any subjective, agent-specific information. If, by comparing any two alternatives, the critical value R^* is either very small (but still positive) or very large (but still finite) we can safely conclude that one alternative is *likely* to be ranked higher than the other by *most* agents – with the possible exception of the most risk-averse or the least risk-averse agents (depending on the case). By *excluding* the alternatives that may be selected by only a small minority of the agents, we can considerably narrow down the choice set from which the preferred alternative is likely to be selected by *most* risk-averse agents.

Appendix

From the necessary condition (8), that critical value can be determined from the following equation:

$$(R^{**})^2 \cdot \frac{(\mu - \mu_0)\cdot V(Y)}{\mu \cdot \mu_0} - R^{**}\left[\frac{V(Y)}{\mu} - \frac{V(Y^{\circ})}{\mu_0}\right] + 2(\mu - \mu_0) = 0 \tag{A-1}$$

To determine the range of values of $R(\mu)$ for which the conditions of the Proposition will be satisfied, we must examine first whether the expression on the left-hand side of the latter equation rises or falls with $R(\mu)$. After some algebra, it can be proved that this expression falls with R if, and only if:

$$R^{**} > \frac{1}{2}\frac{\mu_0}{(\mu-\mu_0)}\left\{\frac{\left[\frac{V(Y)}{\mu}-\frac{V(Y^{\circ})}{\mu_0}\right]}{\frac{V(Y)}{\mu}}\right\} \tag{A-2}$$

We can therefore conclude, for example, that Y° would be ranked higher than Y by all agents having a coefficient of relative risk aversion larger than R^{**}, provided that R^{*} is larger than the value determined in (A-2).

References

Arrow, K.J., 1965. Aspects of the Theory of Risk-Bearing. Lectures, Helsinki.

Atkinson, A.B., 1970. On the measurement of inequality. J. Econ. Theory, 2: 244–263.

Baron, D.P., 1977. On the utility theoretic foundations of mean-variance analysis J. Finance, 32: 1683–1697.

Baumol, W.J., 1963. An expected gain confidence limit criterion for portfolio selection. Manage. Sci., 10: 174–182.

Borch, K., 1969. A note on uncertainty and indifference curves. Rev. Econ. Stud., 36: 1–4.

Feldstein, M.S., 1969. Mean-variance analysis in the theory of liquidity preference and portfolio selection. Rev. Econ. Stud., 36.

Hadar, J. and W. Russel, 1969. Rules for ordering uncertain prospects. Am. Econ. Rev., 59: 25–34.

Hanoch G. and H. Levy, 1969. The efficiency analysis of choices involving risk. Rev. Econ. Stud., 36: 335–341.

Levy, H., 1974. The rationale of the mean-standard deviation analysis: Comment. Am. Econ. Rev., 64: 434–441.

Levy, H. and H. Markowitz, 1979. Approximating expected utility by a function of mean and variance. Am. Econ. Rev., 69: 308–317.

Markowitz, H., 1952. Portfolio selection J. Finance, 72: 77–91.

Markowitz, H., 1959. Portfolio selection: Efficient Diversification of Investments. New York.

Meyer, J., 1977. Choice among distributions. J. Econ. Theory, 14: 326–336.

Meyer, J., 1987. Two-moment decision models and expected utility maximization. Am. Econ. Rev., 77: 421–430.

Pratt, J., 1964. Risk aversion in the small and in the large. Econometrica, 32: 122–136.

Samuelson, P.A., 1970. The fundamental approximation theorem of portfolio analysis in terms of means, variances and higher moments. Rev. Econ. Stud., 37: 152–166.

Tobin, J., 1958. Liquidity preference as behavior towards risk. Rev. Econ. Stud., 28: 65–86.

Tsiang, S.C., 1972. The rationale of the mean-standard deviation analysis, skewness preference, and the demand for money. Am. Econ. Rev., 62: 354–371.