Cardinal criteria for ranking uncertain prospects

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ABSTRACT

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The purpose of this paper is to develop criteria for comparing and ranking uncertain prospects when we have some information on the extent to which agents are risk-averse. The basis for these comparisons is the value of the *certainty equivalent* outcome of the corresponding uncertain prospects. Clearly, the ranking established by the values of the certainty equivalent outcome is identical to the ranking established by the expected utility of the outcome. By comparing the former values, however, we can determine not only the *ranking* of the uncertain prospects under consideration but we can also determined by *how much* one prospect would be more valuable than the other in terms of money – for that particular agent.

The paper develops expressions for approximating the values of the certainty equivalent outcomes on the basis of the central moments of their distribution and the value of the underlying coefficient of variation. These criteria are then applied for comparing alternative crop rotation and irrigation practices of wheat in Israel.

AN APPROXIMATION OF THE RISK PREMIUM

Let Y be a risky prospect (random variable) with a (known) cumulative probability distribution $F(Y)$. To simplify the notations – and without limiting the generality – I will assume that the random variable is discrete, taking the values $(y_1, \ldots, y_n)$, and that all outcomes are equally probable and all non-negative. [1] Let μ denote the expected prospect (or the ex-

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[1] Endnote on p. 29.

pected return on a given portfolio), and let y_E denote the 'certainty equivalent' (CE) prospect.

To identify that ranking, consider an agent having a 'rational' preference ordering (i.e., complete, transitive and reflexive) that can be represented by an appropriate utility function having the expected utility property. To represent 'ordinary' risk-averse agents, Arrow (1965) added the following requirements:

(i) $U'(y) > 0$

(ii) $U''(y) < 0$

(iii) $\dfrac{\mathrm{d}}{\mathrm{d}y}\left(-\dfrac{U''(y)}{U'(y)}\right) \leqslant 0$

(iv) $\dfrac{\mathrm{d}}{\mathrm{d}y}\left(-\dfrac{U''(y) \cdot y}{U'(y)}\right) \geqslant 0$

Ordinary agents are thus assumed to be strictly risk-averse, having non-increasing *absolute* risk aversion and non-decreasing *relative* risk aversion. The standard analysis assumes also the expected utility from a risky prospect to be a function of the first two central moments of the distribution, and, by applying a Taylor-series approximation – disregarding the remainder beyond the second order – presents the expected utility as:

$$\mathrm{EU}(Y) \approx U(\mu) + \tfrac{1}{2}U''(\mu) \cdot V(Y) \tag{1}$$

where $V(Y)$ is the variance of the outcomes, $\approx$ indicates an approximation, and (μ) represents the vector $(\mu, \ldots, \mu) \in \Omega^n$. The CE outcome can be expressed as a fraction of the mean, i.e., $y_E = \alpha \cdot \mu$, where $\alpha \leqslant 1$. Obviously, the smaller the fraction α is, for a given prospect, the more risk-averse is the agent whose preferences are represented by that utility function. Thus, $\alpha = 1$ indicates that the agent is risk-neutral, whereas $\alpha = \mathrm{Min}\{y_i\}/\mu$ represents an agent that ranks prospects on the basis of the maximin rule. The term $(1 - \alpha) \cdot \mu$ represents the loss in utility, expressed in terms of money, on account of the spread of the outcomes – and thus the risk of the prospect. The fraction $(1 - \alpha)$ itself expresses that loss as a proportion of the mean and is often referred to as the Atkinson's measure of inequality or spread (Atkinson, 1970). By applying a Taylor-series approximation of the CE prospect around the expected prospect, we get:

$$U(y_E) = U(\mu) + (\alpha - 1) \cdot \mu \cdot U'(\mu) + \tfrac{1}{2}(1 - \alpha)^2 \cdot \mu^2 \cdot U''(\mu) + \ldots \tag{2}$$

Hence, assuming that the remainder beyond the second order of the series is negligible, we can express the expected utility from the risky prospect as:

$$\mathrm{EU}(Y) \approx U(\mu) + (\alpha - 1) \cdot \mu \cdot U'(\mu) + \tfrac{1}{2}(1 - \alpha)^2 \cdot \mu^2 \cdot U''(\mu) \tag{3}$$

Combining the results in (1) and (3) we therefore get: [2]

$$R(\mu) \cdot \mathrm{cv}^2(Y) \approx 2(1 - \alpha) + (1 - \alpha)^2 \cdot R(\mu) \tag{4}$$

where

$$R(\mu) = -\frac{U''(\mu)}{U'(\mu)}\mu$$

is the coefficient of relative risk aversion at μ, and $\mathrm{cv}^2(Y) = V(Y)/\mu^2$ is the square of coefficient of variations of the outcomes. We can now introduce the two criteria for evaluation:

(1) Define the index I_A as that value that (exactly) equates:

$$\mathrm{EU}(Y) = U(\mu) - I_A \cdot \mu \cdot U'(\mu) \tag{5}$$

By combining this definition and (1), we get a definition of risk premium that is similar (but not identical) to the one put forward by Pratt: [3]

$$I_A \approx \tfrac{1}{2}R(\mu) \cdot \mathrm{cv}^2(Y) \tag{6}$$

(2) Define the index I_B as that value that (exactly) equates:

$$\mathrm{EU}(Y) = U(\mu) - I_B \cdot \mu \cdot U'(\mu) + \tfrac{1}{2}I_B^2 \cdot \mu^2 \cdot U''(\mu) \tag{7}$$

By combining this definition and (1), adding the restriction that $I_B = 0$ whenever $V(Y) = 0$, we obtain a second-order approximation of the risk premium:

$$I_B \approx \frac{\sqrt{1 + R^2(\mu) \cdot \mathrm{cv}^2(Y)} - 1}{R(\mu)} \tag{8}$$

I_B assumes positive values when agents are risk-averse and negative values when agents are risk preferring. It is equal to zero when agents are risk-neutral.

My objective in this paper is to examine the application and compare the performance of these two indices, I_A and I_B – which I will refer to as first and second degrees of approximations of the risk premium – for determining a consistent ranking of risky prospects. Before turning to that, let us examine some of the properties of these indices:

(1) The two indices are, by definition, invariant with respect to linear transformations of the utility function, and, in that sense, well defined *cardinal* measures, which approximate the utility losses due to risk as the percentage loss in the expected return. [4]

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(2) For risk-averse agents, I_A establishes one upper bound on the (true) value of $(1-\alpha)$ as well as on the second-order approximation I_B, and, for 'normal' agents, having 'skewness preference', i.e. $U''' > 0$ (this terminology is after Tsiang, 1972), I_B itself establishes an upper bound on $(1-\alpha)$. [5] Hence, $I_A \geqslant (1-\alpha)$ and $I_A \geqslant I_B$, with strict inequality holding if agents are strictly risk-averse, and $I_A > I_B > (1-\alpha)$ for strictly risk-averse agents having skewness preference.

(3) I_A and I_B are (money) scale-independent since (nominal money) scale changes leave both the coefficient of variation and the coefficient of relative risk aversion unchanged. The two indices remain constant with changes in *real* assets for agents having utility functions with constant coefficient of relative risk aversion, e.g. agents having log-linear utility functions.

(4) For ordinary risk-averse agents, both I_A and I_B strictly rise with a rise in the variance that leaves the mean unchanged, and strictly fall with a rise in the mean that leaves the variance unchanged. [6] Both rise with a rise in the coefficient of relative risk aversion.

With these approximations of the risk premium, the certainty equivalent outcome itself can be approximated by:

$$y_E(1-I) \cdot \mu \tag{9}$$

where I can be either I_A or I_B. Clearly, that approximation depends on the value of the coefficient of relative risk aversion, which represents the preferences of the agent under consideration. The ranking established by means of the approximations will therefore represent the ranking of agents whose preferences are represented by that value of the coefficient of relative risk aversion. When that ranking remains unchanged for an entire *range* of values of $R(\mu)$, however, it can then be viewed as the ranking of an *entire group* of agents whose preferences are represented by coefficients of relative risk aversion within that range. When that ranking remains unchanged for *all* values of $R(\mu)$ it then represents the ranking of *all* risk-averse agents and in that case a second-degree stochastic dominance is established.

Consider, as an illustration, two risky prospects Y^1 and Y^2 having means μ_1 and μ_2 and coefficients of variation $\mathrm{cv}(Y^1)$ and $\mathrm{cv}(Y^2)$, respectively, such that $\mu_1 > \mu_2$ and $\mathrm{cv}(Y^1) > \mathrm{cv}(Y^2)$. Under the first-degree approximation I_A, Y_E^1 would be larger than Y_E^2 if and only if:

$$(\mu_1 - \mu_2) > \tfrac{1}{2}R(\mu) \cdot \left(\mu_1 \cdot \mathrm{cv}^2(Y^1) - \mu_2 \cdot \mathrm{cv}^2(Y^2)\right) \tag{10}$$

[5,6] Endnotes on pp. 30, 31.

Under the second-degree approximation, y_E^1 would be larger then y_E^2 if and only if:

$$(\mu_1 - \mu_2) > \frac{\mu_1\sqrt{1 + R^2(\mu) \cdot \mathrm{cv}^2(Y^1)} - \mu_2\sqrt{1 + R^2(\mu) \cdot \mathrm{cv}^2(Y^2)}}{[1 + R(\mu)]} \qquad (11)$$

From these inequalities we can determine the range of values of $R(\mu)$ within which the sign of the inequality remains unchanged. We can also determine from these inequalities the corresponding certainty equivalent outcome for assumed values of $R(\mu)$.

AN EMPIRICAL ILLUSTRATION

For an empirical illustration of these criteria, I will evaluate the choice of crop rotation and irrigation practices in wheat production in the northern Negev in Israel. The climate in that region is typical semi-arid, characterized by low and very unstable levels of precipitation – both from year to year and within each year. The government implements in that region an extensive drought compensation program, and the data presented here were collected in a comprehensive study aimed at evaluating that program. [7] Table 1 identifies the main characteristics of these alternatives.

In addition to these practices I examined two other options: Under one option, denoted as practice no. 5, no supplementary irrigation is given and the decision whether or not to grow wheat is determined each year on the basis of the moisture in the soil on a predetermined date, usually taken to be 15 December. The tenth option is not to grow wheat at all. I assume that this option yields no profits (and no losses), although farmers may be able to make some (certain) profits by leasing the lands for grazing. Table 2 summarizes the main profit characteristics of the ten alternative cultivation practices over a period of 20 years, assuming that the Drought Compensa-

TABLE 1

Crop rotation and irrigation regimes of alternative wheat production practices

| | Fallow once every n years | | | |
	$n = 2$	$n = 3$	$n = 4$	$n = 5$
No irrigation	1	2	3	4
Supplementary irrigation	6	7	8	9

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[7] Endnote on p. 31.

TABLE 2

Profit characteristics under different crop rotation and irrigation practices (in NIS per dunam in 1988 prices)

Performance criterion	Crop rotation and irrigation practices									
	1	2	3	4	5	6	7	8	9	10
Mean (μ)	4.09	4.26	4.62 **	4.78 *	3.66	3.99	4.22	4.40	4.60	0
Variance (V)	16.87	8.63	19.15	21.16	12.01 **	15.01	16.26	17.01	18.65	0 *
Coefficient of variation	1.004	1.013	0.947	0.962	0.947	0.971	0.955	0.937 **	0.939	0 *
$\dfrac{V}{\mu}$	4.12	4.37	4.15	4.43	3.28 **	3.76	3.85	3.87	4.05	0 *
$\mu - 0.5 \times \dfrac{V}{\mu}$	2.03	2.08	2.52	2.57 **	2.02	2.11	2.30	2.46	2.58 *	0
$\mu - 1 \times \dfrac{V}{\mu}$	0.04	−0.11	+0.44	+0.35	+0.38	+0.23	+0.37	+0.53 **	*0.53	0

* First choice under the criterion.
** Second choice under the criterion.

tions program is implemented. None of these alternatives 'dominates' any of the others on the basis of the first or the second criteria of stochastic dominance. Hence, different risk-averse growers will have different preferences with respect to these alternatives, and the choice of the most desirable cultivation practice depends not only on its profit characteristics (the mean and the variance) but also on the extent to which growers are risk-averse.

Table 2 examines five different criteria for that choice: (a) Maximize μ; (b) Minimize V; (c) Minimize the coefficient of variation; (d) Minimize the ratio V/μ; and (e and f) Maximize a 'Baumol (1963) type' ad hoc expected utility function of the form $\text{EU}(Y) = \mu - \frac{1}{2}RV/\mu$ for $R = 1$ in (e) and $R = 2$ in (f).

Table 2 shows that risk-neutral growers are likely to select the more extensive crop rotation practices that fallow the lands only once every 5 years (the first choice) or 4 years (the second choice), and apply no irrigation. Risk-averse producers, in contrast, are likely to be deterred by the high variability of production practices that apply no supplementary irrigation. The reduction in the variance of the profits that can be achieved through the application of less extensive crop rotation practices may not compensate, however, for the reduction in the average annual profits unless the growers are highly risk-averse.

Table 3 presents the ranking of the alternative crop rotation and irrigation practices, which is determined by means of the two approximations I_A and I_B and the corresponding values of the CE outcomes, for different values of $R(\mu)$. The table shows that, for relatively small values of the coefficient of relative risk aversion, the ranking of the alternative production practices by means of the first-degree approximation I_A is consistent with the ranking established by means of the second-degree approximation I_B, and the difference between the calculated CE outcomes under the two methods is negligible. For higher values of the coefficient $R(\mu)$, the rankings established by the two approximations are no longer consistent, and there are very large differences between the values of the calculated CE outcomes. By comparing the rankings established by the second-degree approximation I_B for different values of $R(\mu)$, we can observe that the more risk-averse growers are likely to opt for supplementary irrigation. In the event that irrigation will not be permitted, they will be encouraged to fallow their lands once every 3 years rather than once every 4 years – despite the reduction of some 3.3% in their average profits. A further reduction in the intensity of production through fallow once every 2nd year will further reduce the variability of the annual profits, but the resulting decrease in the average annual profits by 7.8% will not make this reduction desirable even for the more risk-averse growers.

TABLE 3

Ranking of alternative crop rotation practices by means of the first and second-degree approximations of the CE outcome

		Crop rotation									
		1	2	3	4	5	6	7	8	9	10
$R(\mu) = \frac{1}{2}$	I_A	0.252	0.257	0.226	0.232	0.224	0.236	0.228	0.219	0.220	0
	$\hat{y}_E$	3.06	3.17	3.56	3.67	2.84	3.05	3.26	3.43	3.59	0
	Rank	7	6	3	1	9	8	5	4	2	10
	I_B	0.238	0.242	0.215	0.220	0.213	0.223	0.216	0.209	0.209	0
	$\hat{y}_E$	3.12	3.23	3.61	3.73	2.88	3.01	3.31	3.48	3.64	0
	Rank	7	6	3	1	9	8	5	4	2	10
$R(\mu) = 1.0$	I_A	0.504	0.513	0.448	0.463	0.448	0.471	0.456	0.439	0.441	0
	$\hat{y}_E$	2.03	2.07	2.55	2.57	2.02	2.11	2.30	2.47	2.57	0
	Rank	8	7	3	1–2	9	6	5	4	1–2	10
	I_B	0.417	0.423	0.377	0.388	0.377	0.394	0.383	0.370	0.372	0
	$\hat{y}_E$	2.38	2.46	2.88	2.93	2.28	2.42	2.61	2.77	2.89	0
	Rank	8	6	3	1	9	7	5	4	2	10
$R(\mu) = 2.0$	I_A	1.01	1.03	0.90	0.93	0.90	0.94	0.912	0.878	0.882	0
	$\hat{y}_E$	−0.03	−0.11	+0.48	+0.36	+0.38	+0.23	+0.37	+0.54	+0.54	0
	Rank	9	10	3	6	4	7	5	1–2	1–2	8
	I_B	0.622	0.630	0.575	0.581	0.570	0.592	0.578	0.562	0.564	0
	$\hat{y}_E$	1.57	1.58	1.96	1.99	1.56	1.63	1.78	1.93	2.01	0
	Rank	8	7	3	2	9	6	5	4	1	10
$R(\mu) = 4.0$	I_A	2.016	2.052	1.794	1.851	1.794	1.89	1.82	1.76	1.76	0
	$\hat{y}_E$	−4.16	−4.45	+3.67	−4.07	−2.91	−3.53	−3.45	−3.33	−3.51	0
	Rank	9	10	7	8	2	6	4	3	5	1
	I_B	0.785	0.793	0.729	0.744	0.729	0.753	0.737	0.720	0.722	0
	$\hat{y}_E$	0.88	0.88	1.25	1.22	0.99	0.99	1.11	1.23	1.28	0
	Rank	8	9	2	4	6	7	5	3	1	10

Table 3 also demonstrates the large differences between the ranking of the alternative production practices and the values of the risk premium and the CE outcome determined by these two approximations. Even for $R(\mu) = 1$, the rankings are different and the estimate of the risk premium by means of the approximation I_A is larger than the estimate by means of I_B by some 20%. As a result, the CE outcome estimated by I_A is lower by some 15% than the outcome estimated by I_B. When $R(\mu)$ rises to 2.00, these differences rise to 50–60% for the risk premium and 400% and (much) more for the CE outcome. For risk-averse growers having skewness preference (for which $I_A > I_B > (1 - \alpha)$) the estimate by means of I_A grossly overstates the risk premium and grossly understates the CE outcome.

CONCLUDING REMARKS

The paper develops a second-order approximation of the risk premium, which can be expressed as a function of the coefficient of variation and the coefficient of relative risk aversion. This expression can then be used for estimating the certainty equivalent outcome and thus for ranking risky alternatives. The paper analyses the properties of this approximation and compares it with the widely practiced approximation of the risk premium, which is similar to the one suggested by Pratt (1964). The paper then applies these approximations for evaluating the choice of crop rotation and irrigation practices in wheat production in Israel. It shows that even for relatively low values of the coefficient of relative risk aversion, the two approximations will generate different rankings among the alternative practices and determine different estimates of the certainty equivalent outcomes. These differences grow very rapidly with the rise in the value of the coefficient of relative risk aversion.

ACKNOWLEDGEMENT

ENDNOTES

[1] In other words, the ranking of risky prospects does not depend on the arbitrary choice of the 'zero' point. Making use of Arrow's (1965) symbolism, we can denote this as:

$$\left(\frac{1}{n} : y_1, \ldots, y_n \right)$$

[2] By equating (1) and (2), and dividing both sides by $U'(\mu)$, we get:

$$\frac{1}{2}\left(-\frac{U''(\mu)}{U'(\mu)}\right)V(Y) = (\alpha - 1)\cdot\mu + \tfrac{1}{2}(1-\alpha^2)\cdot\mu^2\left(\frac{-U''(\mu)}{U'(\mu)}\right)$$

Dividing both sides by μ^2 and inserting the definition of $R(\mu)$ and $\mathrm{cv}(Y)$, we get (4). Notice that this equation can also be written as

$$A^{(\mu)}\cdot V(Y) = 2(1-\alpha)\cdot\mu + (1-\alpha)^2\cdot\mu^2\cdot A^{(\mu)}$$

where $A^{(\mu)}$ is the measure of absolute risk aversion.

[3] Pratt (1964) shows the risk premium to be: $\pi \approx \tfrac{1}{2}VA$, where V is the variance and A is absolute risk aversion (p. 125). He also defines the proportionate risk premium as $\pi^* = \pi/\mu = \tfrac{1}{2}V\cdot R(\mu)$ (p. 134). Clearly, π^* has the same dimension as V whereas I_A is dimension-free.

[4] By the mean-value theorem (granted that the utility function is continuously differentiable) the fraction $(1-\alpha)$ is *exactly* given by:

$$(1-\alpha) = \frac{U(\mu) - \mathrm{EU}(Y)}{\mu\cdot U'(y)} \qquad y_E \leqslant y \leqslant \mu$$

and this fraction is therefore invariant with respect to linear transformations of the utility function. It is therefore a well defined *cardinal* measure of risk.

[5] By definition:

$$U(y_E) = U(\mu) + (\alpha - 1)\cdot\mu\cdot U'(y) = U(\mu) - I_A\cdot\mu\cdot U'(\mu)$$

where $y_E \leqslant y \leqslant \mu$. Hence, for (strictly) risk averse agents: $U'(y) \geqslant U'(\mu)$, and therefore also $I_A \geqslant (1-\alpha)$, with strict inequality holding if and only if U is strictly concave. By equation (5) and (7) we get (after simplifying the expression):

$$I_A = I_B + \tfrac{1}{2}I_B^2\cdot R(\mu) \geqslant I_B$$

with strict inequality holding if U is strictly concave. Finally, by equating (3) and (7) we get:

$$I_B + \tfrac{1}{2}I_B^2\frac{\mu\cdot U''(\mu)}{U'(\mu)} = (1-\alpha) - \tfrac{1}{2}(1-\alpha)^2\frac{\mu\cdot U''(y^*)}{U'(\mu)}$$

where $y_E \leqslant y \leqslant y^* \leqslant \mu$. If agents have skewness preference such that $U'' \geqslant 0$ then $U''(y^*) < U''(\mu) < 0$ and therefore:

$$I_B + \tfrac{1}{2}R(\mu)\cdot I_B^2 > (1-\alpha) + \tfrac{1}{2}R(\mu)\cdot(1-\alpha)^2$$

Granted that both $(1-\alpha)$ and I_B are *strictly* positive for strictly risk-averse agents, the latter inequality thus implies that $I_B > (1-\alpha)$. As is well known, for agents having decreasing or constant absolute risk aversion, U''' must be positive. For these agents, therefore, $I_A > I_B > (1-\alpha)$.

[6] Consider first a rise in μ that leaves $V(Y)$ unchanged, e.g. an addition of the same scalar to all outcomes. The resulting change in I_A would be given by:

$$\frac{dI_A}{d\mu}\bigg|V(Y) - \text{constant} = \frac{(\mu \cdot R'(\mu) - 2R(\mu)) \cdot \text{cv}^2(Y)}{2\mu}$$

and this expression is negative if $\mu R(\mu) < 2R(\mu)$, i.e. if the coefficient of relative risk aversion rises with μ but at decreasing rates – as implied by assumptions (iii) and (iv) on the utility function. To derive the change in I_B, consider the equation:

$$2I_B + I_B^2 \cdot R(\mu) - \frac{V^2(Y)}{\mu^2}R(\mu) = 0$$

from which I_B has been determined in (8). By taking the first-order derivatives of the latter equation with respect to μ, leaving $V(Y)$ unchanged, we get:

$$\frac{dI_B}{d\mu}\bigg|V(Y) - \text{constant} = \frac{(\mu \cdot R'(\mu) - 2R(\mu)) \cdot \text{cv}^2(Y)}{2(1 + I_B \cdot R(\mu))\mu}$$

Hence, if $\mu \cdot R'(\mu) < 2R(\mu)$ then this expression is negative.

A change in $V(Y)$ that leaves the mean unchanged will have the following effects:

$$\frac{dI_A}{dV(Y)}\bigg|\mu - \text{constant} = \frac{R(\mu)}{2\mu^2} > 0$$

$$\frac{dI_B}{dV(Y)}\bigg|\mu - \text{constant} = \frac{R(\mu)}{2\mu^2[1 + I_B \cdot R(\mu)]} > 0$$

[7] See Bigman, Barak and Becker (1989).

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