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Principles of stochastic dynamic optimization in resource management: the continuous-time case

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ABSTRACT

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A wide range of problems in economics, agriculture, and natural resource management have been analyzed using continuous-time optimal control models, where the state variables change over time in a stochastic manner. Using a firm-level investment model and a model of environmental degradation, this paper provides a concise introduction to continuous-time stochastic control techniques. The process used to derive the differential of a stochastic process is stressed and, in turn, is used to explain Ito's lemma, Bellman's equation, the Hamilton–Jacobi equation, the maximum principle, and the expected dynamics of choice variables. A basic extension of the dynamic duality literature is also provided, where the Hamilton–Jacobi equation is used to derive a stochastic and dynamic analogue of Hotelling's lemma.

1. INTRODUCTION

A range of dynamic optimization techniques have been used to analyze firm and consumer behavior. For example, Kennedy (1988) provides a thorough introduction to discrete-time dynamic programming techniques, with a focus on the management of natural resource and agricultural systems. In general, the decisionmaker is interested in maximizing some

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objective function, often assumed to be the discounted value of profits from exploiting a natural resource, subject to constraints imposed by biological growth functions and existing resource stocks. As Kennedy (1988) notes, the extension of discrete-time methods to stochastic problems is straight forward.

Numerous analyses have used continuous-time optimal control techniques to analyze the management of natural resource and agricultural systems as well as capital investment problems. Dorfman (1969) provides an economic interpretation of basic optimal control results, while Clark (1976) provides a general introduction to control models in natural resource economics (1976). Smith (1975) analyses the extinction of species, Cooper and McClaren (1980) extend static duality concepts to the dynamic case, and Epstein (1981) applies dynamic duality theory to firm investment decisions. Vasavada and Ball (1988) use a continuous-time framework to estimate a dynamic adjustment model for U.S. agriculture. Ehui and Hertel (1989) use an optimal control model to analyze and estimate tradeoffs between deforestation and agricultural productivity in Côte d'Ivoire.

There is also a growing literature that uses continuous-time optimal control techniques where the state variables, such as stock prices or animal populations, change over time in a stochastic manner. For example, initial studies in the finance literature studied the demand for risky assets (Merton, 1969, 1971), the demand for index bonds (Fischer, 1975), and the pricing of stock options (Black and Scholes, 1973; Merton, 1973). More recent analyses include the optimal rotation time for a stochastically evolving forest (Clark and Reed, 1984), optimal production under uncertainty with learning (Majd and Pindyck, 1989), and uncertainty in the theory of renewable resource markets (Pindyck, 1980, 1984).

The need for analyzing stochastic and dynamic problems is clear. Firm debt changes with future and uncertain interest rates, expenditures, and revenues. Research and development firms must allocate expenditures to acquire knowledge that may or may not lead to future marketable products. Most problems in environmental economics involve the management of 'assets', such as plants, pests, and wildlife, that grow according to random biological growth functions. Air and groundwater pollution occurs through uncertain fate and transport dynamics.

This paper provides a concise introduction to continuous-time stochastic control techniques. [1] The goals of the paper are to provide the necessary background for students and practitioners to: (1) read and understand basic

[1] While it is assumed that the reader is familiar with control techniques for the deterministic case, no background in stochastic calculus is necessary.

results in the existing literature; and (2) study more complete developments of the topic found, for example, in Brock (1976), Chow (1979), Kushner (1972), or Malliaris and Brock (1982). As Brock (1976, p. 1) suggests, "Economists use ordinary calculus every day without understanding the intricate details of Lebesque or Riemann integrals... There is no reason why the same cannot be done for the stochastic calculus as well."

The paper proceeds as follows. A dynamic investment model of the firm when future prices and capital stocks are uncertain is first used to introduce stochastic differential equations, interpret the differential of a function of stochastic processes, and identify Bellman's equation. As an example of a potential application, Bellman's equation and the envelope theorem are then used to derive a stochastic and dynamic analogue of Hotelling's lemma. Next, an economic model of environmental degradation, where the quality of an environmental resource evolves stochastically over time, is used to derive and interpret the stochastic maximum principle. A method is also outlined for deriving the expected dynamics of the optimal choice functions, which can then be used to investigate the sensitivity of the optimal choices to parameters changes.

2. FIRM INVESTMENT UNDER UNCERTAINTY

The relationship between capital theory and natural resource management is well recognized (Neher, 1990). For example, the study of capital investment decisions has a long history in economics, where the change in the capital stock $K(t)$ over time is often modelled as a deterministic differential equation $\mathrm{d}K(t)/\mathrm{d}t = I(t) - \delta\ K(t)$, with $I(t)$ representing gross investment in time t and $\delta\ K(t)$ is depreciation. The differential equation, known as the state equation or equation of motion, is easily adapted to the case of stock resources such as mineral deposits and flow resources such as animal populations or forests.

For the purposes of this paper, the change in the capital stock over time is modelled as a *stochastic* differential equation:

$$\mathrm{d}K(t) = [I(t) - \delta\ K(t)]\ \mathrm{d}t + \sigma(K(t))\ \mathrm{d}W(t) \tag{1}$$

where $I(t)$ is gross investment, $\delta\ K(t)$ is depreciation, $\sigma(K(t))$ is a function, and $W(t)$ is a random variable. More specifically, it is assumed that $W(t)$ is a *Wiener process* (or *Brownian motion process*).

The Wiener process $W(t)$ is characterized by the following assumptions:
(1) $W(t)$ is distributed normal with zero mean
(2) $E[\mathrm{d}W(t)] = 0$
(3) $E[\mathrm{d}W(t)\ \mathrm{d}W(t)] = \mathrm{d}t$, and
(4) $E[\mathrm{d}W(t)\ \mathrm{d}W(s)] = 0$ for s not equal to t.

Using these four assumptions, the two terms in equation (1) have straightforward interpretations: $[I(t) - \delta\ K(t)]\ \mathrm{d}t$ is the expected change in the capital stock; and $\sigma(K(t))\ \mathrm{d}W(t)$ is the unexpected change. And, since the variance in the change in the capital stock:

$$\mathrm{Var}(\mathrm{d}K(t)) = E\left\{[\mathrm{d}K(t) - E(\mathrm{d}K(t))]^2\right\} = E\left[\sigma(K(t))^2\ \mathrm{d}W(t)^2\right]$$
$$= \sigma(K(t))^2\ E\left[\mathrm{d}W(t)^2\right] = \sigma(K(t))^2\ \mathrm{d}t, \text{ and}$$

the term $\sigma(K(t))^2$ is the variance of $\mathrm{d}K(t)$ over the period $\mathrm{d}t$.

For example, $\sigma(K(t))\ \mathrm{d}W(t)$ could represent unexpected population fluctuations or depreciation in capital that is either less than or greater than expected depreciation. Thus, while $\delta\ K(t)$ would be expected depreciation of the firm, the term $\delta\ K(t) + \sigma(K(t)\ \mathrm{d}W(t)$ would be total depreciation over the period $\mathrm{d}t$. The term $\sigma(K(t))\ \mathrm{d}W(t)$ could also be interpreted as general technological improvements that increase the effective stock of capital. Alternatively, if the variance of $\mathrm{d}K(t)$ was modelled as $\sigma(K(t), I(t))^2\ \mathrm{d}t$, then $I(t)$ could represent nominal investment, while $\sigma(K, I)\ \mathrm{d}W$ could represent unexpected technical change embodied in new investment.

Armed with the basic definition of a Wiener process, consider the following problem for the firm that makes investment decisions to maximize the discounted value of profits when future prices and capital stocks are unknown:

$$J(t, k, p) = \max_{I(\tau)} E_t \int_t^\infty \mathrm{e}^{-r\tau}[f(K(\tau), I(\tau)) - P(\tau)'\ K(\tau)]\ \mathrm{d}\tau \qquad (2)$$
$$\mathrm{d}K(\tau) = [I(\tau) - \delta\ K(\tau)]\mathrm{d}\tau + \sigma(K(\tau))\ \mathrm{d}Z(\tau) \qquad K(t) = k$$
$$\mathrm{d}P(\tau) = g(P(\tau))\ \mathrm{d}\tau + \alpha(P(\tau))\ \mathrm{d}W(\tau) \qquad P(t) = p$$

where $K(\tau) \geqslant 0$ is an N-vector of capital stocks; $I(\tau) \geqslant 0$ is an N-vector of gross investments; δ is an $N \times N$ diagonal matrix of depreciation rates; $P(\tau)$ is an N-vector of capital rental rates normalized by the output price; f is the production technology; r is the discount rate; $g(P(\tau))$ is a N-vector of functions $g^i(P(\tau))$ that represent expected price changes at τ; $\sigma(K(\tau))$ is an $N \times N$ matrix of functions σ^{ij}; $\alpha(P(\tau))$ is an $N \times N$ matrix of functions α^{ij}; E_t is the expectations operator conditional on information at initial time t; and the variables $Z(\tau)$ and $W(\tau)$ are each N-vectors of Wiener processes, with

$$E[\mathrm{d}Z(\tau)] = E[\mathrm{d}W(\tau)] = 0$$
$$E[\mathrm{d}Z(\tau)\ \mathrm{d}Z(\tau)'] = S\ \mathrm{d}t$$
$$E[\mathrm{d}W(\tau)\ \mathrm{d}W(\tau)'] = B\ \mathrm{d}t$$

and

$$E[\mathrm{d}Z(\tau)\ \mathrm{d}W(\tau)'] = 0$$

The random variables $\mathrm{d}Z$ and $\mathrm{d}W$ are independent across different time periods. The $N \times N$ matrix S is the correlation matrix for the vector $\mathrm{d}Z(\tau)$, and the $N \times N$ matrix B is the correlation matrix for $\mathrm{d}W(\tau)$. The model could be easily extended to the case where $\mathrm{d}Z(\tau)$ and $\mathrm{d}W(\tau)$ are correlated, which would imply that changes in the capital stocks and the capital rental prices are correlated.

Problem (2) states that the firm chooses investment to maximize the expected discounted value of firm profits over the period $\tau = t$ to ∞, given unexpected changes in prices and the capital stock. The value function for the firm's problem $J(t, k, p)$, is the expected maximized discounted value of firm profits. The value function $J(t, k, p)$ is analogous to the expected profit function for the static case with uncertainty and risk neutrality.

Bellman's principle of optimality, which is valid for deterministic or stochastic control problems, implies that (see, e.g., Intriligator, 1971):

$$J(t, K(t), P(t)) = \mathrm{Max}_I\{\mathrm{e}^{-rt}[f(k, I) - pk]\ \mathrm{d}t + E_t[J(t + \mathrm{d}t, K(t + \mathrm{d}t), P(t + \mathrm{d}t))]\} \tag{3.1}$$

which can be rearranged to yield:

$$0 = \mathrm{Max}_I\{\mathrm{e}^{-rt}[f(k, I) - pk] + (1/\mathrm{d}t)\ E_t[\mathrm{d}J]\} \tag{3.2}$$

where $\mathrm{d}J$ is the differential of the value function $J(t, k, p)$.

While (3.2) just restates the principle of optimality, the derivation of Bellman's equation involves substituting for the differential of the value function. However, because J is a function of the stochastic processes, i.e. capital stocks K and prices P, the change in the value function depends on uncertain changes in prices and capital stocks. As a result, the ordinary rules of calculus cannot be used to determine the differential $\mathrm{d}J$. While an infinite set of stochastic calculi can be defined (Brock, 1976), *Ito's lemma* provides the rule for stochastic differentials that will be used here and seems to be most commonly used in economics.

To derive the stochastic differential $\mathrm{d}J$ and indirectly provide a justification for Ito's lemma, the first step is to expand $J(t, k, p)$ in a Taylor series of order $\mathrm{d}t$:

$$\begin{aligned}\mathrm{d}J &= J(t + \mathrm{d}t, K(t + \mathrm{d}t), P(t + \mathrm{d}t)) - J(t, K(t), P(t)) \\ &= J_t\ \mathrm{d}t + J_k'\ \mathrm{d}k + \tfrac{1}{2}\ \mathrm{d}k'\ J_{kk}\ \mathrm{d}k + J_p'\ \mathrm{d}p + \tfrac{1}{2}\ \mathrm{d}p'\ J_{pp}\ \mathrm{d}p \\ &\quad + \mathrm{d}k'\ J_{kp}\ \mathrm{d}p + o(\mathrm{d}t)\end{aligned} \tag{4}$$

where subscripts denote partial derivatives, and $o(\mathrm{d}t)$ is the remainder for terms of order higher than $\mathrm{d}t$. The quadratic terms in k and p are included in the Taylor expansion because they are of order $\mathrm{d}t$ or less after taking expected values. It is also assumed that in the limit $E_t[o(\mathrm{d}t)]/\mathrm{d}t = 0$ as $\mathrm{d}t$ goes to 0.

The next step is to find $E_t[\mathrm{d}J]$, which involves evaluating the expected values of the terms in equation (4). Using the definitions of the Wiener processes W and Z, these expected values are:

$$E_t[J_t\ \mathrm{d}t] = J_t\ \mathrm{d}t \tag{5.1}$$

$$E_t[J_k'\ \mathrm{d}k] = J_k'(I - \delta\ k)\ \mathrm{d}t \tag{5.2}$$

$$E_t[J_p'\ \mathrm{d}p] = J_p'\ g(p)\ \mathrm{d}t \tag{5.3}$$

$$\begin{aligned} E_t[\mathrm{d}k'\ J_{kk}\ \mathrm{d}k] &= E_t[((I - \delta\ k)\ \mathrm{d}t + \sigma(k)\ \mathrm{d}Z)'\ J_{kk}((I - \delta\ k)\ \mathrm{d}t \\ &\qquad + \sigma(k)\ \mathrm{d}Z)] \\ &= E_t[\sigma(k)'\ J_{kk}\ \sigma(k)\ \mathrm{d}Z\ \mathrm{d}Z'] \\ &= \mathrm{TRACE}[\sigma(k)'\ J_{kk}\ \sigma(k)\ S]\mathrm{d}t \end{aligned} \tag{5.4}$$

where the derivation in (5.4) uses the assumption that higher order terms than dt enter the remainder function $o(\mathrm{d}t)$.

Following the steps in (5.4):

$$E_t[\mathrm{d}p'\ J_{pp}\ \mathrm{d}p] = \mathrm{TRACE}[\alpha(p)'\ J_{pp}\ \alpha(p)\ B]\mathrm{d}t \tag{5.5}$$

and

$$E_t[\mathrm{d}k'\ J_{kp}\ \mathrm{d}p] = 0 \quad \text{since } W \text{ and } Z \text{ are independent} \tag{5.6}$$

Therefore, equations (4) and (5.1)–(5.6) yield after a little rearranging:

$$\begin{aligned} E_t[\mathrm{d}J] = \{&J_t + J_k'(I - \delta\ k) + J_p'\ g(p) \\ &+ \tfrac{1}{2}\ \mathrm{TRACE}[\sigma' J_{kk} \sigma S] + \tfrac{1}{2}\ \mathrm{TRACE}[\alpha' J_{pp} \alpha B]\}\ \mathrm{d}t + E_t[o(\mathrm{d}t)] \end{aligned} \tag{6.1}$$

and letting $\mathrm{d}t$ approach 0 yields:

$$\begin{aligned} (1/\mathrm{d}t)\ E_t[\mathrm{d}J] = &J_t + J_k'(I - \delta\ k) + J_p'\ g(p) \\ &+ \tfrac{1}{2}\ \mathrm{TRACE}[\sigma' J_{kk} \sigma S] + \tfrac{1}{2}\ \mathrm{TRACE}[\alpha' J_{pp} \alpha B] \end{aligned} \tag{6.2}$$

since by assumption $E_t[o(\mathrm{d}t)]/\mathrm{d}t = 0$ as $\mathrm{d}t$ approaches 0.

Equation (6.2) is defined as the *differential generator* of the function $J(t,\ k,\ p)$ and gives the expected change through time of a function of stochastic processes (Chow, 1979). This differential generator for the stochastic case is analogous to the total time derivative $\mathrm{d}J/\mathrm{d}t$ for the deterministic case.

Substituting the differential generator (6.2) into equation (3.2) yields Bellman's equation for problem (2):

$$-J_t = \mathrm{Max}_I\Big\{\mathrm{e}^{-rt}(f(k, I) - pk) + J_k'(I - \delta\, k) + J_p'\, g(p) + \tfrac{1}{2}\,\mathrm{TRACE}\big[\sigma' J_{kk}\sigma S\big] + \tfrac{1}{2}\,\mathrm{TRACE}\big[\alpha' J_{pp}\alpha B\big]\Big\} \tag{7}$$

Bellman's equation evaluated at the optimal choice of investment, $I^* = I^*(t, k, p)$, is also known as the Hamilton–Jacobi equation (Intrilligator 1971; Kamien and Schwartz, 1981).

Equation (6.1) can now be used to interpret *Ito's lemma*, which states that the differential of the function $J(t, k, p)$ is:

$$\mathrm{d}J = E_t[\mathrm{d}J] + J_k'\sigma\, \mathrm{d}Z + J_p'\alpha\, \mathrm{d}W + o(\mathrm{d}t) \tag{8}$$

Thus, Ito's lemma (8) for the differential of a function of stochastic processes just implies that the change in a function J includes the expected change $E_t[\mathrm{d}J]$, plus the unexpected change about the mean $J_k'\sigma\, \mathrm{d}Z + J_p'\alpha\, \mathrm{d}W$. This unexpected change is directly related to the variance of $\mathrm{d}J$, since:

$$\mathrm{Var}(\mathrm{d}J) = E\Big[(\mathrm{d}J - E(\mathrm{d}J))^2\Big] = E\Big[\big(J_k'\sigma\, \mathrm{d}z + J_p'\alpha\, \mathrm{d}w\big)^2\Big]$$
$$= \mathrm{TRACE}\big(\sigma J_k J_k'\sigma S\big)\, \mathrm{d}t + \mathrm{TRACE}\big(\alpha J_p J_p'\alpha B\big)\, \mathrm{d}t$$

The main objectives of this section are complete: stochastic differential equations and Wiener processes have been introduced; Bellman's equation and the Hamilton–Jacobi equation have been derived; and Ito's lemma has been defined and interpreted.

A further use of the above results is considered here. Since the Hamilton–Jacobi equation plays a central in dynamic duality theory for the deterministic case (e.g., see Epstein, 1981), it is not surprising that Bellman's equation (7) can be used to derive the dynamic and stochastic analogue of Holtelling's lemma. First, note that since problem (2) is autonomous, the value function $J(t, k, p) = \mathrm{e}^{-rt}V(k, p)$, where V is the firm's intertemporal value of profit discounted to the initial time t (Kamien and Schwartz, 1981). As a result, $-J_t = r\, \mathrm{e}^{-rt}V$, and Bellman's equation (7) can be written as:

$$rV(k, p) = \mathrm{Max}_I\Big\{f - pk + V_k'(I - \delta\, k) + V_p'\, g(p) + \tfrac{1}{2}\,\mathrm{TRACE}\big[\sigma' V_{kk}\sigma S\big] + \tfrac{1}{2}\,\mathrm{TRACE}\big[\alpha' V_{pp}\alpha B\big]\Big\} \tag{9}$$

Applying the envelope theorem to the Hamilton–Jacobi equation (9), after rearranging, yields the stochastic and dynamic analogue of Hotelling's lemma:

$$\begin{aligned} I^*(k, p) = V_{kp}^{-1}\Big\{ & rV_p + k - V_{pp}\ g(p) - V_p'\ g_p(p) \\ & - \tfrac{1}{2}\ \text{TRACE}\left[\sigma(k)'\ V_{kkp}\ \sigma(k)\ S\right] \\ & - \tfrac{1}{2}\ \text{TRACE}\left[\alpha(p)'\ V_{ppp}\ \alpha(p)\ B\right] \\ & - \text{TRACE}\left[\alpha(p)'\ V_{pp}\ \alpha_p(p)\ B\right]\Big\} + \delta\ k \end{aligned} \qquad (10.1)$$

While equation (10.1) is rather complicated, it is a straightforward generalization of the dynamic analogue of Hotelling's lemma for the dynamic and deterministic case. For example, if it is assumed that there is no uncertainty, but prices are expected to follow the deterministic differential equations $\mathrm{d}P/\mathrm{d}t = g(p)$, then equation (10.1) becomes:

$$I^*(k, p) = V_{kp}^{-1}\{rV_p + k - V_{pp}\ g(p) - V_p'\ g_p(p)\} + \delta\ k \qquad (10.2)$$

and if there is no uncertainty and prices are assumed to be constant over time, as in Epstein (1981), then equation (10.1) becomes:

$$I^*(k, p) = V_{kp}^{-1}\{rV_p + k\} + \delta\ k \qquad (10.3)$$

Thus, equation (10.1) shows that stochastic dynamic duality can be used to model investment behavior when capital stock dynamics are uncertain and prices evolve according to a Markovian process. As with the static and deterministic case, the matrix of second derivatives of the value function $V(k, p)$ is symmetric, and the value function is homogenous of degree zero in normalized prices. Unlike the static and deterministic case, and as Stefanou (1987) notes, fourth-order derivatives are in general necessary to characterize the concavity/convexity of the value function and to investigate the response of the optimal choices functions $I^*(k, p)$ to price changes.

For empirical analyses, where degrees of freedom and multicollinearity problems will probably restrict analyses to quadratic-like value functions, equation (10.1) along with simple forms for $g(p)$, $\sigma(k)$, and $\alpha(p)$ may be tractable. For example, consider a one-output/one-quasifixed input investment model where the capital stock evolves in a deterministic manner and price expectations are:

$$\mathrm{d}p = (a + bp)\ \mathrm{d}t + \alpha p\ \mathrm{d}W \qquad (10.4)$$

and the value function is:

$$V(k, p) = a_0 + a_1 p + a_2 k + \tfrac{1}{2}a_{11} p^2 + \tfrac{1}{2}a_{22} k^2 + a_{12} pk \qquad (10.5)$$

The value function $V(k, p)$ is a simplified form of the value function used in Vasavada and Ball (1988), while the price expectations model is similar to those found in Stefanou (1987) and Epstein and Denny (1983).

Since the above formulation of expectations only depends on prices, price expectations can be estimated in a discrete-time model using generalized least squares as:

$$p_t - p_{t-1} = a + bp_{t-1} + e_t \tag{10.6}$$

where e_t are independent and identically distributed errors with $E(e_t) = 0$ and $E(e_t^2) = (\alpha p_{t-1})^2$.

Based on the parameter estimates a^*, b^* and α^*, the quadratic value function, and the assumption of non-stochastic investment dynamics, an empirical net-investment equation based on (10.1) can be estimated using ordinary least squares as: [2]

$$\mathrm{d}k_t = m_0 + m_1 X_{t-1} + m_2 k_{t-1} + v_t \tag{10.7}$$

where

$$\mathrm{d}k_t = k_t - k_{t-1} - (r - b^*)k_{t-1}$$

$$m_0 = (r - b^*)a_1/a_{12}$$

$$m_1 = a_{11}/a_{12}$$

$$X_{t-1} = (r - 2b^* - \alpha^{*2})p_{t-1} - a^*,$$

and

$$m_2 = 1/a_{12}$$

The above framework can be generalized to models with many variable and quasi-fixed inputs and many outputs. Howard and Shumway (1988), Epstein and Denny (1983), Taylor and Monson (1985) and Vasavada and Ball (1988) are examples of empirical applications in agriculture and manufacturing based on deterministic models (i.e. equation 10.3). [3]

3. AN ECONOMIC MODEL OF ENVIRONMENTAL DEGRADATION

A simple economic model of environmental degradation is used in this section to identify the stochastic maximum principle and to derive the

[2] Note that the Hamilton–Jacobi equation (9) could also be used to specify a supply equation, which could be estimated jointly with the quasi-fixed input equation. This two-equation model would allow cross-equation restrictions to be tested or imposed on the model and all parameters of the value function would be estimated.

[3] These studies also indicate the types of data that are necessary to estimate a more complete model for specific investigations.

expected changes in the optimal choice functions (e.g. degradation) over time. Specifically, it is assumed that an environmental resource, such as the productivity of the soil, evolves according to a stochastic differential equation. Since the change in the soil stock over time is influenced by farmer choices, such as inputs, crops grown, tillage practices, as well as weather, the evolution of the soil stock over time is a stochastic process.

Consider the following intertemporal profit-maximization problem: [4]

$$J(x, t) = \underset{s,z}{\text{Max}} E_t\left\{ \int_t^T \mathrm{e}^{-r\tau}[p\, f(s(\tau), x(\tau), z(\tau))\, g(\tau) - c\, z(\tau)]\, \mathrm{d}t + R[x(T)]\, \mathrm{e}^{-rT} \right\} \tag{11}$$

$$\mathrm{d}x(\tau) = (k - s(\tau))\, \mathrm{d}\tau + \sigma(s(\tau), x(\tau), z(\tau))\, \mathrm{d}w(\tau) \qquad x(\tau) = x$$

where all the variables are scalars and p is the output price; $s(\tau)$ is soil erosion; $x(\tau)$ is soil depth with initial value x; $z(\tau)$ is a variable input with price c; $g(\tau)$ is an index of neutral technical change and $f\,g(\tau)$ is the technology which is increasing and concave in s, x and z; r is the discount rate; R is the resale value of the land at the terminal time T given a terminal soil stock $x(T)$; k is the natural regeneration of the soil stock; and $w(\tau)$ is a Wiener process, where $\sigma(s(\tau), x(\tau), z(\tau))^2$ is the variance of $\mathrm{d}x(\tau)$ over the period $\mathrm{d}\tau$; and $J(x, t)$ is the value function. For the purposes of this section, the value function J is not written as a function of the other parameters of the problem (p, c, r, t, k) for notation convenience.

Using the process outlined in the previous section, Bellman's equation for problem (11) is:

$$-J_t = \underset{s,z}{\text{Max}}\{[p\, g(t)\, f(s, x, z) - cz]\, \mathrm{e}^{-rt} + J_x(k - s) + \tfrac{1}{2}\sigma^2 J_{xx}\} \tag{12}$$

The technology parameter $g(\tau)$ in the production function implies that problem (11) in not autonomous. Therefore, following Kamien and Schwartz (1981), equation (12) is written in current value terms, using $J(x, t) = \mathrm{e}^{-rt}\, V(x, t)$, as:

$$r\, V(x, t) - V_t(x, t) = \underset{s,z}{\text{Max}}\{p\, g(t)\, f(s, x, z) - cz + V_x(k - s) + \tfrac{1}{2}\sigma^2 V_{xx}\} \tag{13}$$

[4] This model is a simple generalization of the models in McConnell (1983) and Barbier (1988). Specifically, $f_s > 0$ so that reduced soil erosion in the short run reduces output.

Defining the expected marginal value of the stock V_x as μ, the change in the expected marginal value of the stock with a change in the initial stock (V_{xx}) as μ_x, and substituting μ and μ_x into equation (13), the current-value Hamiltonian for problem (11) is:

$$H(s, x, z, \mu, \mu_x) = p\, g(t)\, f - cz + \mu(k - s) + \tfrac{1}{2}\sigma^2\mu_x \tag{14}$$

The Hamiltonian for the continuous stochastic case has a similar interpretation as that for the deterministic case, i.e. the expected change in the value function at time t includes current returns $p\, g(t)\, f - cz$, expected capital gains $\mu(k - s)$, plus the new term $\frac{1}{2}\sigma^2\mu_x$ which is the cost of uncertainty to the firm at time t. Notice that the variance of the random variable enters the Hamiltonian even though the firm is assumed to be risk neutral. If $V(x,t)$ is concave in x, $V_{xx} \equiv \mu_x$ is less than or equal to zero, and uncertainty tends to reduce the expected value of changes in the value function. However, $V(x, t)$ is not necessarily concave in x, which implies that uncertainty could increase the value of the Hamiltonian. Stefanou (1987) suggests a process for determining the shape of the value function in x.

The stochastic maximum principle is directly analogous to the deterministic case. Using (13) and (14), the stochastic maximum principle implies the following *optimality conditions*:

$$H_s = pgf_s - \mu + \sigma_s\mu_x = 0 \tag{15}$$

$$H_z = pgf_z - c + \sigma_z\mu_x = 0 \tag{16}$$

From the Hamiltonian, $H_\mu = (k - s)$, and the *state equation* is found by:

$$\mathrm{d}x = H_\mu\, \mathrm{d}t + \sigma(k)\, \mathrm{d}w \qquad x(t) = x \tag{17}$$

And, since the costate variable $\mu \equiv V_x$ is a function of the stochastic process x, Ito's differentiation rule (8) is used to find the *costate equation*:

$$\mathrm{d}\mu = (r\mu - H_x)\, \mathrm{d}t + \sigma\mu_x\, \mathrm{d}w + o(\mathrm{d}t) = [r\mu - pgf_x - \sigma\sigma_x\mu_x]\mathrm{d}t + \sigma\mu_x\, \mathrm{d}w + o(\mathrm{d}t) \tag{18}$$

with terminal condition $\mu(T) = R_x[x(T)]$.

Thus, the maximum principle for continuous stochastic processes implies that the optimal paths of z, s, x and μ satisfy equations (15)–(18). In contrast to the deterministic case, the marginal cost of environmental degradation in equation (15) includes the marginal current value of the stock μ plus the extra term $\sigma_x\mu_x$, which takes account of the marginal effect of soil erosion on the standard deviation of $\mathrm{d}x$ over the period $\mathrm{d}t$. This additional term begins to look like an adjustment for risk preferences found in a static expected utility-maximization framework. If it is assumed that $\sigma_s > 0$ and $\mu_x < 0$, so that addition erosion increases uncertainty as measured by the variance of dx, then the total term $\sigma_s \mu_x < 0$ and farmers would choose less erosion under uncertainty than for the deterministic case. On the other hand, if $\mu_x > 0$, the total term $\sigma_s \mu_x > 0$ and farmers would choose more erosion under uncertainty than for the deterministic case. Similar implications follow from equation (16).

When terminal time T is a choice variable, maximizing (11) with respect to T yields:

$$\left\{p\, g(T)\, f(T) - c\, z(T) + \mu(T)\, (k - s(T)) + \tfrac{1}{2}\, \sigma(T)^2\, \mu_x(T)\right\} = r\, R[x(T)] \tag{19}$$

Thus, from (19), the optimal time to sell occurs when the returns from remaining on the land equal the opportunity cost of remaining on the land. While asset replacement has been analyzed in a deterministic context for many years (see, e.g., Goundrey, 1960; Perrin, 1972; or Samuelson, 1972), equation (19) is also an asset replacement criterion for general stochastic control models.

For analytical purposes, it may also be desirable to consider the expected dynamics of the choice variables and how parameters of the problems influence such dynamics. For the dynamic and uncertain model analyzed here, Ito's lemma implies that the expected change in soil erosion $(1/dt)\ E_t[ds]$ will differ from the observed change due to the realization of the random variable dw and any adaptation on the part of the farmer. For example, observing increasing degradation in any time period does not necessarily imply that a farmer planned on such a change.

Two mathematical steps must be followed to derive the expected change in the optimal choice of erosion $s(x, t)$ over the period dt, $(1/dt)\ E_t[ds]$. First, Ito's is applied to the optimality conditions (15) and (16). And second, the two-equation system is solved for $(1/dt)\ E_t[ds]$ and $(1/dt)\ E_t[dz]$. This process is outlined in Appendix A for a simplified example. The optimality conditions (15) and (16) imply that:

$$(1/dt)\ E_t[d(pgf_s)] - (1/dt)\ E_t[d\mu] + (1/dt)\ E_t[d(\sigma_s \mu_x)] = 0 \tag{20}$$

$$(1/dt)\ E_t[d(pgf_z)] - c_t + (1/dt)\ E_t[d(\sigma_z \mu_x)] = 0 \tag{21}$$

The individual terms in equations (20) and (21) are evaluated in Appendix B, using Ito's lemma and the process outlined in Appendix A. Substituting equations (B1)–(B5) from Appendix B into equation (20) and (21), after some mundane (if tedious) algebra, yields:

$$\left\{\begin{matrix}(1/dt)\ E_t[ds] \\ (1/dt)\ E_t[dz]\end{matrix}\right\} = A[B + C - D] \tag{22}$$

where

$$A = \begin{bmatrix} pgf_{ss} + \mu_x \sigma_{ss}, & pgf_{sz} + \mu_x \sigma_{sz} \\ pgf_{zs} + \mu_x \sigma_{zs}, & pgf_{zz} + \mu_x \sigma_{zz} \end{bmatrix}^{-1}$$

$$B = \begin{bmatrix} pgf_s[r - (f_x/f_s) - (g_t/g) - (p_t/p) - (k-s)(f_{sx}/f_s)] \\ pgf_z[(c_t/(c - \sigma_z \mu_x)) - (g_t/g) - (p_t/p) - (k-s)(f_{zx}/f_z)] \end{bmatrix}$$

$$C = \begin{bmatrix} r\mu_x \sigma_s - \sigma\sigma_x \mu_x - \sigma_s[\mu_{xt} + \mu_{xx}(k-s) + \frac{1}{2}\mu_{xxx}\sigma^2] \\ r\mu_x \sigma_z - \sigma\sigma_x \mu_x - \sigma_z[\mu_{xt} + \mu_{xx}(k-s) + \frac{1}{2}\mu_{xxx}\sigma^2] \end{bmatrix}$$

$$D = \begin{bmatrix} \mu_x \sigma_{sx}(k-s) + \sigma^2[\frac{1}{2}(pgf_{sxx} + \mu_x \sigma_{sxx}) + \frac{1}{2}(pgf_{sss} + \mu_x \sigma_{sss}) s_x^2 \\ \quad + \frac{1}{2}(pgf_{szz} + \mu_x \sigma_{szz}) z_x^2 + (pgf_{ssz} + \mu_x \sigma_{ssz}) s_x z_x \\ \quad + (pgf_{ssx} + \mu_x \sigma_{ssx}) s_x + (pgf_{szx} + \mu_x \sigma_{szx}) z_x] \\ \mu_x \sigma_{zx}(k-s) + \sigma^2[\frac{1}{2}(pgf_{zxx} + \mu_x \sigma_{zxx}) + \frac{1}{2}(pgf_{zss} + \mu_x \sigma_{zss}) s_x^2 \\ \quad + \frac{1}{2}(pgf_{zzz} + \mu_x \sigma_{zzz}) z_x^2 + (pgf_{zsz} + \mu_x \sigma_{zsz}) s_x z_x \\ \quad + (pgf_{zsx} + \mu_x \sigma_{zsx}) s_x + (pgf_{zzx} + \mu_x \sigma_{zzx}) z_x] \end{bmatrix}$$

While equations (22) are rather complicated in total, the individual components can be readily interpreted and compared to the analogous conditions for the deterministic case. And, as found in Stefanou (1987) and the investment model in the previous section, the implications of the stochastic and dynamic model include that of the deterministic model as a special case. There are three main sets of terms in equations (22). First, the matrix A is the inverse of the Hessian matrix of the current-value Hamiltonian (14). The analogous term for the deterministic case is the also just the appropriate Hessian matrix. Second, the vector B is identical to the deterministic case (see, e.g., McConnell, 1983), except that $c_t/(c - \sigma_z \mu_x)$ rather than c_t/c is included to take account of the full marginal cost of the variable input z.

The third main sets of terms in equation (22), the vectors C and D, include increasingly complicated effects that are not found in the deterministic model. The vector C takes into account third- and fourth-order changes in the value function due to a change in the state x. The vector D takes into account the third-order effects of firm choices on the production function and the variance function.

Equation (22) provides the expected dynamics of the optimal choice functions $s(x, t)$ and $z(x, t)$ for general functional forms and generalizes the results in Stefanou. However, more specific assumptions can greatly simplify this general result. For example, if it is assumed that firm choices only influence the expected change in the stock but not the unexpected change, as in Stefanou, then $\sigma_s = \sigma_z = 0$ and the vector C reduces to $C' = (-\sigma\sigma_x\mu_x, -\sigma\sigma_x\mu_x)$, the vector D reduces to the production functions terms, and the vector B and matrix A become identical to the deterministic case. If it is assumed that the production function f and the variance function σ are quadratic in s, z and x, then the vector D collapses to $D' = (\mu_x\sigma_{sx}(k-s), \mu_x\sigma_{zx}(k-s))$.

4. CONCLUSION

This paper introduces the basic techniques for the optimum control of stochastic processes. An investment model of the firm is used to introduce stochastic differential equations, interpret Ito's lemma, and derive Bellman's equation. A stochastic and dynamic analogue of Hotelling's lemma is derived. An economic model of environmental degradation is then used to derive and interpret the stochastic maximum principle. Since deterministic models are special cases of their stochastic counterparts, the differences between stochastic and deterministic models can always be readily compared. Thus, at a minimum, analyzing a problem at the theoretical level in a dynamic and stochastic framework highlights the assumptions underlying deterministic models (or static models with uncertainty).

APPENDIX A

The process for deriving the expected soil erosion dynamics $(1/\mathrm{d}t)$ $E_t[\mathrm{d}s]$ from the optimality conditions of the stochastic maximum principle is outlined in this appendix. The basic problem involves using Ito's lemma to determine the differential of a function $y = F(x, t, s(x, t)) = 0$, where $s(x, t)$ is the optimal choice of s at time t given x. For the purposes of this appendix, the simplifying assumptions are made that the state x evolves according to the stochastic differential equation $\mathrm{d}x = (k-s)\,\mathrm{d}t + \sigma\,\mathrm{d}w$, where σ is a scaler. Using the process outlined in the text, equations (4)–(6.1), a second-order Taylor approximation of order dt implies that:

$$\mathrm{d}y = F_t\,\mathrm{d}t + F_x\,\mathrm{d}x + \tfrac{1}{2}F_{xx}\,\mathrm{d}x^2 + F_s\,\mathrm{d}s + \tfrac{1}{2}F_{ss}\,\mathrm{d}s^2 + F_{xs}\,\mathrm{d}s\,\mathrm{d}x + o(\mathrm{d}t) \tag{A.1}$$

where

$$\mathrm{d}x = (k-s)\,\mathrm{d}t + \sigma\,\mathrm{d}w \tag{A2.1}$$

$$\mathrm{d}x^2 = [(k-s)\,\mathrm{d}t + \sigma\,\mathrm{d}w]^2 = \sigma^2\,\mathrm{d}w^2 + o(\mathrm{d}t) \tag{A2.2}$$

$$\mathrm{d}s = s_t\,\mathrm{d}t + s_x\,\mathrm{d}x + \tfrac{1}{2}s_{xx}\,\mathrm{d}x^2 + o(\mathrm{d}t) \tag{A2.3}$$

$$ds^2 = s_x^2\sigma^2\ dw^2 + o(dt) \tag{A2.4}$$

$$ds\ dx = s_x\sigma^2\ dw^2 + o(dt) \tag{A2.5}$$

Using equations (A2.1)–(A2.5), the differential generator of the function $y = F(x, t, s(x, t))$ is:

$$(1/dt)\ E_t[dy] = F_t + F_x(k-s) + \tfrac{1}{2}F_{xx}\sigma^2 + \tfrac{1}{2}F_{ss}s_x^2\sigma^2 + F_{sx}s_x\sigma^2 + F_s(1/dt)\ E_t[ds] \tag{A4}$$

which when set equal to zero can be rearranged to yield:

$$(1/dt)\ E_t[ds] = -(1/F_s)\big[F_t + F_x(k-s) + \tfrac{1}{2}F_{xx}\sigma^2 + \tfrac{1}{2}F_{ss}s_x^2\sigma^2 + F_{sx}s_x\sigma^2\big] \tag{A5}$$

APPENDIX B

Following the process outlined in Appendix A, the terms in equations (21) and (22) can be evaluated as:

$$(1/dt)\ E_t[d(pgf_s)] = (p_t g + pg_t)f_s + pg\big[f_{sx}(k-s) + \sigma^2\big(\tfrac{1}{2}f_{sxx} + \tfrac{1}{2}f_{sss}s_x^2 + \tfrac{1}{2}f_{szz}z_x^2 + f_{ssz}s_x z_x + f_{ssx}s_x + f_{szx}z_x\big)\big] + pgf_{ss}(1/dt)\ E_t[ds] + pgf_{sz}(1/dt)\ E_t[dz] \tag{B1}$$

$$(1/dt)\ E_t[d\mu] = r\mu - pgf_x - \sigma\sigma_x\mu_x \tag{B2}$$

$$(1/dt)\ E_t[d(\sigma_s\mu_x)] = \mu_x(1/dt)\ E_t[d(\sigma_s)] + \sigma_s(1/dt)\ E_t[\mu_x] \tag{B3}$$

where

$$(1/dt)\ E_t[d(\sigma_s)] = \sigma_{ss}(1/dt)\ E_t[ds] + \sigma_{sz}(1/dt)\ E_t[dz] + \sigma_{sx}(k-s) + \sigma^2\big[\tfrac{1}{2}\sigma_{sss}s_x^2 + \tfrac{1}{2}\sigma_{szz}z_x^2 + \tfrac{1}{2}\sigma_{sxx} + \sigma_{ssz}s_x z_x + \sigma_{ssx}s_x + \sigma_{szx}z_x\big]$$

and

$$(1/dt)\ E_t[\mu_x] = \mu_{xt} + \mu_{xx}(k-s) + \tfrac{1}{2}\mu_{xxx}\sigma^2$$

$$(1/dt)\ E_t[d(pgf_z)] = (p_t g + pg_t)f_z + pg\big[f_{zx}(k-s) + \sigma^2\big(\tfrac{1}{2}f_{zxx} + \tfrac{1}{2}f_{zss}s_x^2 + \tfrac{1}{2}f_{zzz}z_x^2 + f_{zsz}s_x z_x + f_{zsx}s_x + f_{zzz}z_x\big)\big] + pgf_{zs}(1/dt)\ E_t[ds] + pgf_{zz}(1/dt)\ E_t[dz] \tag{B4}$$

$$(1/dt)\ E_t[d(\sigma_z\mu_x)] = \mu_x(1/dt)\ E_t[d(\sigma_z)] + \sigma_z(1/dt)\ E_t[\mu_x] \tag{B5}$$

where

$$(1/\mathrm{d}t)\ E_t[\mathrm{d}(\sigma_z)] = \sigma_{zs}(1/\mathrm{d}t)\ E_t[\mathrm{d}s] + \sigma_{zz}(1/\mathrm{d}t)\ E_t[\mathrm{d}z] + \sigma_{zx}(k-s)$$
$$+ \sigma^2\left[\tfrac{1}{2}\sigma_{zss}s_x^2 + \tfrac{1}{2}\sigma_{zzz}z_x^2 + \tfrac{1}{2}\sigma_{zxx} + \sigma_{zsz}s_x z_x + \sigma_{zsx}s_x + \sigma_{zzx}z_x\right]$$

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