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# Principles of Dynamic Optimization in Resource Management

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## Abstract

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The type of resource problem amenable to static analysis is distinguished from that requiring dynamic analysis. Possibly due to the apparent complexity of optimal control theory methods, often dynamic models have not been applied where they would be appropriate. In this article dynamic programming arguments are used to derive optimality conditions directly and simply. They are derived for a renewable resource such as a fishery, but they have application to resource management in general. The approach is illustrated by examples to the extraction of a depletable resource, to feeding for weight gain, and to applying fertilizer when some fertilizer carriers over from one crop season to another. Conditions for the optimal replacement of biological units are also considered.

## 1. Introduction

The main aim of this article is to explain and illustrate the principles of dynamic optimization applied to the management of agricultural and natural resource systems. McInerney (1976, 1978) does this for two-period problems using diagrammatic analysis for non-renewable resources, and for renewable resources with simple growth functions. The scope of the analysis is extended here with the minimum of mathematics to $n$-period and infinite-period problems and to the use of general growth functions.

A subsidiary aim of this article is to distinguish the types of problem which can be satisfactorily solved with static models from those requiring more complex dynamic models. Whilst many management problems are most appropriately formulated as dynamic problems, they are too often solved as static problems. An optimal control theory approach may be required. Although this point has been made before (see e.g. Rausser and Hochman, 1979; Kennedy, 1981a; Chavas et al., 1985) it is still worth emphasizing.

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It is one thing to formulate an optimal control problem, but another to solve it. Efficient solution methods often make use of the necessary conditions for optimality, or the maximum principle. For example, Chavas et al. (1985) use the optimality conditions in solving a swine production problem. However, references to the Hamiltonian, costate or adjoint variables, and transversality conditions are likely to discourage some students, if not some potential practitioners.

The novel contribution of this article is to show how the optimality conditions can be derived very directly for multistage decision problems using the logic of dynamic programming. Dynamic programming arguments are frequently employed to derive the maximum principle for continuous time problems (see, e.g., Dorfman, 1969; Intriligator, 1971; Koo, 1977), but the mathematics is fairly involved. In practical management settings, however, decisions are made at discrete intervals, so that the relevance of the continuous time problem is only as a limiting case. The maximum principle for multistage problems with discrete time is invariably derived by differentiating a Lagrangean expression (see, e.g., Dorfman, 1969; Benavie, 1970). In contrast, the dynamic programming approach adopted here starts by defining the capital value of an asset such as a resource stock, and derives values of the marginal value of the asset at all decision stages as a by-product of optimization. The directness of the approach should make it a useful pedagogic device in teaching.

The approach is applied to three finite-stage problems to show how it aids the understanding of intertemporal tradeoffs. The applications are to mining a depletable resource, feeding livestock and fertilizer provision when there are carryover effects. A simple but little-known rule is derived for the last situation.

For many resource problems there is no natural planning horizon. Examples are the harvesting of a renewable resource stock such as fish, and the continual replacement of biological units such as trees or livestock. Optimal rules are derived for problems with infinite decision stages, but with return functions and stock dynamics which are stage-invariant.

## 2. Limited scope of static models

The farm production process has often been characterised by production functions relating farm output $(y)$ to dependent variables $(u)$, categorised as controllable or uncontrollable, measurable or unmeasurable. Let the production function be written as $y\{u\}$, and for the moment let $u$ be a scalar representing a controllable input. Profit from the production process is:

$$a = p^y y\{u\} - p^u u \tag{1}$$

where $p^y$ and $p^u$ are the prices of the output and input, respectively, independent of $y$ and $u$. The decision problem is how to set $u$ so as to maximize profit.

If $y$ is a concave function of $u$ and there are no constraints on $u$, the condition for optimal $u$ is:

$$p^y \, dy/du = p^u \qquad (2)$$

This is the requirement that for optimal $u$ the value of the marginal product of the input equal the price of the input.

Over what period of time is the decision maker interested in maximizing profit? For how long does (1) apply? A process may consist of applying all of the input at one point in time, followed by harvesting all of the output at another point in time, as in the case of applying fertilizer for crop production. If so, (1) applies for the period set by these points in time. Alternatively, the input may be applied continually and the output harvested continually, as in the case of feeding cows for milk production. Then the longest that (1) can apply is the period over which there is no change in $p^y/p^u$ or $y\{u\}$. The function $y\{u\}$ may change because of changes in uncontrollable inputs not included in $u$, or just through the ageing of the biological unit.

The decision maker is usually interested in maximizing profit over a longer period than that for which (1) holds. If so, the optimal value of $u$ changes at different stages over the planning period. For $n$ decision stages the goal is:

$$\max \sum_{i=1}^{n} \alpha^{i-1} a_i\{u_i\} \qquad \text{with respect to } u_1, \ldots, u_n \qquad (3)$$

where $a_i\{u_i\}$ is the profit at stage $i$ for $u_i$, and $\alpha$ is the discount factor. A crucial question is under what circumstances can this goal be met by the independent stage-by-stage application of rule (2). It is only possible if there is separability between decision periods, in the sense that $u_i$ does not affect $p_j^y, p_j^u$ nor $y_j\{u_j\}$ for $j > i$. In the traditional theory of the firm, and in much applied work with agricultural production functions and linear programming, it is assumed that separability holds, at least to a reasonable approximation.

Often in biological management the assumption is not reasonable. For example, $u_i$ may be a two-element vector consisting of a control variable and a state variable which is not controllable at stage $i$, but which is a function of the control variable in earlier periods. In resource management problems the state variable will often be a stock level such as biomass, or the level of feed, fertilizer or water available for allocation in all remaining decision periods. These problems are ones of optimal control.

If $y$ represents a vector of outputs which are linearly related to a vector of controllable inputs, $u$, in limited supply, maximizing (1) is a single-period linear programming problem. If there are no state variables, such as opening stocks of cash, livestock or soil status, the more realistic multistage problem (3) is separable. The solution is obtained by solving the sequence of single-period linear programming problems. If state variables do have to be taken into account, the problem can be solved for input levels in all periods simultaneously using multiperiod linear programming. Thus, in the case of linear technology, linear programming can be used to solve both a sequence of single-period problems and a multiperiod problem.

Where input–output relations are non-linear, the marginal analysis underlying rule (2) does not extend in such a straightforward way to the rule for optimal $u_i$ in the multiperiod case. Perhaps because the rule is more complex, or because there are no perceived market incentives to plan with a horizon longer than one period, non-separable problems are sometimes treated as if they were separable. However, ignoring carryover effects can lead to serious policy problems at the aggregate level, such as applying too much fertilizer, mining soil nutrients as if they had zero value and the over-exploitation of forests and fisheries. We next consider a simple derivation of the multiperiod rule.

## 3. Optimality conditions for the dynamic resource problem

Consider the following general resource problem, which applies to harvesting fish but which can be readily reinterpreted to apply to crop, livestock and forestry production. The overall problem is to set the level of harvesting, $u_i$, in each period $i$ so as to maximize the present value of net returns. The state variable is the biomass of the fish stock, $x_i$. The net return from fishing in each period is $a_i\{x_i, u_i\}$. After the final period return, $a_n\{x_n, u_n\}$, allow for the possible realization of the value of the terminal stock, $F\{x_{n+1}\}$.

The period net return is:

$$a_i\{x_i, u_i\} = p_i u_i - c_i\{x_i, u_i\} \tag{4}$$

where $p_i$ is the price of fish and $c_i\{x_i, u_i\}$ is the cost of harvesting, which depends on $x_i$, a measure of the concentration of fish in the sea. The biomass of the fish stock at the beginning of period $i+1$ is given by the first-order difference equation:

$$x_{i+1} = x_i + g_i\{x_i\} - u_i \qquad (i=1, ..., n) \tag{5}$$

where $g_i\{x_i\}$ represents autonomous growth in stock as a result of reproduction, natural mortality and fish growth.

The problem is expressed as:

$$\max_{u_i,...,u_n} v_1\{x_1, u_1, ..., u_n\} = \sum_{i=1}^{n} \alpha^{i-1} a_i\{x_i, u_i\} + \alpha^n F\{x_{n+1}\} \tag{6}$$

subject to (5). As a constrained optimization problem the optimality conditions could be obtained by differentiating the appropriate Lagrangean expression (see e.g. Benavie, 1970). They can be obtained more directly and simply using dynamic programming arguments.

Equation (6) defines the value of the fish stock as $v_1$ for any harvesting sequence $u_1, ..., u_n$. The usual economic capital valuation which would be given to the fish stock is $v_1$ for the optimal harvesting sequence $u_1^*, ..., u_n^*$ defined as:

$$V_1\{x_1\} \equiv v_1\{x_1, u_1^*, ..., u_n^*\} \tag{7}$$

Equation (7) can be used to rewrite (6) as:

$$\begin{aligned} V_1\{x_1\} &= \max_{u_1} [a_1\{x_1, u_1\}] + \sum_{i=2}^{n} \alpha^{i-1} a_i\{x_i, u_i^*\} + \alpha^n F\{x_{n+1}\} \\ &= \max_{u_1} [a_1\{x_1, u_1\}] + \alpha v_2\{x_2, u_2^*, ..., u_n^*\} \\ &= \max_{u_1} [a_1\{x_1, u_1\}] + \alpha V_2\{x_2\} \end{aligned} \tag{8}$$

In this way the problem of setting $n$ harvesting levels is reduced to that of setting one harvesting level. Note that the level $x_2$ is related to $x_1$ and $u_1$ by (5) for $i=1$. Of course, $V_2\{x_2\}$ is initially unknown, but it can be determined at least in principle by a process of backward induction.

By similar reasoning, (8) generalizes to:

$$\begin{aligned} V_i\{x_i\} &= \max_{u_i} [a_i\{x_i, u_i\} + \alpha V_{i+1}\{x_{i+1}\}] \\ &= \max_{u_i} z_i\{x_i, u_i\} \\ &= a_i\{x_i, u_i^*\} + \alpha V_{i+1}\{x_i + g_i\{x_i\} - u_i^*\} \qquad (i=1, ..., n) \end{aligned} \tag{9}$$

with

$$V_{n+1}\{x_{n+1}\} = F\{x_{n+1}\} \tag{10}$$

Equation (9) is the fundamental equation of dynamic programming. Together with (10) and (5), it may be used for finding $u_i^*$ for all $i$ by backward recursion. The solution process is initiated by solving (9) for the last decision stage with $i=n$. Equation (10) gives the required value of $V_{n+1}\{x_{n+1}\}$ in (9). Values of $V_n$ and $u_n^*$ are found for all feasible values of $x_n$. With $V_n\{x_n\}$ known, (9) may be used to determine $u_{n-1}^*$ and $V_{n-1}$ for all $x_{n-1}^*$, and so on until $u_1^*$ is found. Solutions may be derived numerically for a finite range of values of the state variable in each period. Alternatively, they may be derived analytically, either exactly for problems with special structures, or by a process of successive approximation.

If the functions in (5), (9) and (10) are differentiable it is possible to find optimality conditions, analogous to (2) for the separable case. It may be possible to use these conditions for solving problems with differentiable functions.

A necessary condition for $u_i = u_i^*$ to be an interior solution to (9) is:

$$\partial z_i/\partial u_i = \partial a_i/\partial u_i + \alpha(\mathrm{d}V_{i+1}/\mathrm{d}x_{i+1})(\partial x_{i+1}/\partial u_i)$$
$$= \partial a_i/\partial u_i - \alpha(\mathrm{d}V_{i+1}/\mathrm{d}x_{i+1}) = 0 \tag{11}$$

The term $\mathrm{d}V_{i+1}/\mathrm{d}x_{i+1}$ in (11) plays a key role in conditions for dynamic optimization. It is the present value of a marginal unit of the fish stock, evaluated at the beginning of period $i+1$ when the stock level is $x_{i+1}$. It corresponds to the current value Lagrange multiplier in the Lagrangean derivation, and to the costate variable used in the discrete maximum principle. Writing for notational convenience $\mathrm{d}V_{i+1}/\mathrm{d}x_{i+1}$ evaluated at $x_{i+1}$ as $\lambda_{i+1}$, (11) can be expressed as:

$$\partial a_i/\partial u_i - \alpha\lambda_{i+1} = 0 \tag{12}$$

The first and second terms in (12) represent the immediate and future contributions respectively to the present value of a marginal change in $u_i$. Note that if the future contribution ($\alpha\lambda_{i+1}$) is zero, (12) is the same as (2). The second term can also be interpreted as the marginal user cost of harvesting the marginal fish at stage $i$. The cost is the present value of reduced net returns flowing from the lower fish stock at the start of period $i+1$.

If for any stage $i$, $x_i$ and $\lambda_{i+1}$ were known, it would be possible to find $\partial a_i/\partial u_i$ and hence $u_i^*$ from (12). However, values of $\lambda_{i+1}$ are in general not directly observable, except when $i=n$. For $i=n$:

$$\lambda_{n+1} = \mathrm{d}V_{n+1}/\mathrm{d}x_{n+1} = \mathrm{d}F/\mathrm{d}x_{n+1} \tag{13}$$

The relationship between $\lambda_i$ and $\lambda_{i+1}$ is found by differentiating (9) with respect to $x_i$:

$$\lambda_i = \partial a_i/\partial x_i + \alpha\lambda_{i+1}(1+\mathrm{d}g_i/\mathrm{d}x_i) \qquad (i=1, ..., n) \tag{14}$$

Any change in $x_i$ affects $u_i^*$. However, it can be shown that if (11) holds, $\mathrm{d}V_i/\mathrm{d}x_i$ can be specified without partial derivatives with respect to $u_i$. This can also be interpreted as a result of the envelope theorem.

Equation (14) states that for optimality the increase in the optimal value of the fish stock resulting from an additional fish must equal the increase in the immediate return, plus the value of the increased stock one period later discounted one period. As before, backward recursion may be used in principle to find $\lambda_i$ for $i=n, ..., 1$, using (13), (14) and (5).

## 4. Example applications

### 4.1 Mining

An important result, which Solow (1974) refers to as a "fundamental principle of the economics of exhaustible resources", can be obtained from (14). If $x_i$ represents the available amount of a depletable resource, $u_i$ the quantity mined and $a\{\cdot\}$ the concave function of net return from mining after deducting costs not dependent on $x_i$, then $g\{x\}=0$ and $\partial a/\partial x_i=0$. Equation (14) after rearrangement yields:

$$(\lambda_{i+1}-\lambda_i)/\lambda_i=r \tag{15}$$

This states that under an optimal mining policy (with $u_i^*>0$), the value of the marginal unit of the resource rises through time at the rate of interest. The marginal user cost of the resource at stage $i$, $\alpha\lambda_{i+1}$, also rises through time at the rate of interest. Consequently, quantity mined decreases from stage to stage.

### 4.2 Livestock feeding

Traditional production function approaches to finding optimal rations for producing weight gains have not been entirely satisfactory. Allowance has to be made for the changing productivity of feeds fed to an animal as it grows. One approach has been to divide total weight gain over the feeding period into a number of weight gain intervals. For each interval an isoquant function is specified relating the different levels of feeds which produce the same weight gain. The time taken to achieve the weight gain is also related to the combination of feeds. Information on these functions for each weight gain interval, together with prices and costs, enables optimal feeding regimes to be found using heuristic processes (see e.g. Heady et al., 1976; Melton et al., 1978).

A more direct approach is to view the problem as one of optimal control. Weight gain resulting from feed inputs can be specified as the control variable, and weight and possibly age as state variables. Suppose initially that the date of sale and the returns per unit liveweight of the finished animal are known. Let the fattening time be divided into $n$ feeding periods.

The cost of feeding in the $i$th feeding period is given by $a_i\{x_i, u_i\}$, dependent on liveweight $x_i$ at the beginning of the $i$th period, and $u_i$, the weight gain over the $i$th period. In this example the use of subscript $i$ on the cost function allows for weight gain to be a function of age as well as liveweight.

The basic recursive functional equation is:

$$V_i\{x_i\}=\max_{u_i}[-a_i\{x_i, u_i\}+\alpha V_{i+1}\{x_i+u_i\}] \qquad (i=1, .., n) \tag{16}$$

with

$$V_{n+1}\{x_{n+1}\}=px_{n+1}$$

where $V_i\{x_i\}$ is the present value of net returns from feeding an animal of liveweight $x_i$ an optimal sequence of rations over the remaining $n-i+1$ feeding periods, and $p$ is the price per unit liveweight at the end of the $n$th feeding period.

The rule for optimal weight gain in any feeding period $i$ may be determined by inductive reasoning. For period $i$ the rule is to find $u_i^*$ such that:

$$-\partial a_i/\partial u_i+\alpha(\mathrm{d}V_{i+1}/\mathrm{d}x_{i+1})=0 \qquad (i=1, ..., n) \tag{17}$$

assuming an interior solution. For $i=n$, we have:

$$-\partial a_n/\partial u_n+\alpha p=0$$

Statement of the rule for $i<n$ requires knowledge of $\mathrm{d}V_{i+1}/\mathrm{d}x_{i+1}$. Differentiating (16) for $i=n$ with respect to $x_n$ gives:

$$\mathrm{d}V_n/\mathrm{d}x_n=-\partial a_n/\partial x_n+\alpha p \tag{18}$$

where the partial derivative on the RHS is evaluated at $u_n=u_n^*$. Substituting for $\mathrm{d}V_n/\mathrm{d}x_n$ in (17) gives the rule for optimal weight gain for period $n-1$:

$$-\partial a_{n-1}/\partial u_{n-1}+\alpha[-\partial a_n/\partial x_n+\alpha p]=0 \tag{19}$$

The rule generalizes to:

$$\partial a_i/\partial u_i=\alpha^{n-i+1}p-\sum_{j=1}^{n-i}\alpha^j\ \partial a_{i+j}/\partial x_{i+j} \tag{20}$$

The result shows that the question of whether to feed for an extra unit of weight gain in period $i$ cannot be settled solely by comparing the extra feed cost with the present value of the weight gain by the sale date. The future changes in ration costs resulting from the extra unit of weight gain have to be taken into account also. Assume, as seems reasonable, that ration cost is a positive convex function of both weight gain and liveweight. The rule implies that the optimal weight gain in period $i$ is less the lower is $i$ (i.e. the greater the number of feeding periods before sale), the lower is $p$, and the greater is the increase in ration cost with liveweight.

Equation (16) can also be used to obtain numerical solutions to livestock feeding problems. Although the method has not been widely adopted, applications have been reported by Glen (1980, 1983), Hochman and Lee (1972), Kennedy (1982), Kennedy et al. (1976), Meyer and Newett (1970), Nelson and Eisgruber (1970) and Yager et al. (1980). These applications are reviewed by Kennedy (1986a). Chavas et al. (1985) have also argued persuasively the need for a control theory approach to livestock production, and obtain a numerical solution for a swine production problem. Unlike the other applications just mentioned, they obtain a solution using a gradient method to solve equations based on (12) and (14).

### 4.3 Fertilizer carryover

The determination of the optimal level of fertilizer application is usually defined as a static problem of maximizing net returns, $a$, in (1), where $y\{u\}$

specifies crop level as a function of fertilizer application $u$. Equation (2) then describes the condition for optimal fertilizer application. It is implicitly assumed that applying fertilizer in period $i$ has no significant consequences for period $i+1$. In fact there is always some carryover effect, the extent depending on the fertilizer, soil characteristics, crop type, crop yield, and the weather. Heady and Dillon (1951, pp. 524–525), Fuller (1965) and Anderson (1967, pp. 53–54) mentioned carryover and its relevance to calculating optimal application rates but gave no general rules. A general rule was given by Kennedy et al. (1973) and Dillon (1977) for the case where fertilizer carryover from period $i$ to period $i+1$ is proportional to the fertilizer carried over to period $i$ plus the fertilizer applied in period $i$. The rule was derived by backward induction. Rules were subsequently derived in the same way for more complex carryover functions (Kennedy, 1981b) and for the general stochastic case (Taylor, 1983). Recently Kennedy (1986b) has shown how these rules can be derived more directly. The argument is outlined below as another illustration of the application of the optimality conditions (12) and (14). Suppose there are to be $n$ successive applications of fertilizer to $n$ successive crops. The basic recursive functional equation is:

$$V_i\{x_i\}=\max_{u_i}[p_i^y y_i\{x_i+u_i\}-p_i^u u_i+\alpha V_{i+1}\{b(x_i+u_i)\}] \quad (i=1, ..., n) \qquad (21)$$

where $x_i$ is the level of fertilizer carried over to the start of period $i$, and $b$ is the proportion of fertilizer available in period $i$ $(x_i+u_i)$ carried over to period $i+1$. Differentiating the RHS of (21) with respect to $u_i$ gives (for an interior solution):

$$p_i^y\ \partial y_i/\partial u_i-p_i^u+\alpha b(\mathrm{d}V_{i+1}/\mathrm{d}x_{i+1})=0 \qquad (22)$$

Differentiating (21) with respect to $x_i$ gives:

$$\mathrm{d}V_i/\mathrm{d}x_i=p_i^y\ \partial y_i/\partial x_i+\alpha b(\mathrm{d}V_{i+1}/\mathrm{d}x_{i+1}) \qquad (23)$$

Combining (22) and (23) gives:

$$\mathrm{d}V_i/\mathrm{d}x_i=p_i^u$$

a result which is easily rationalized. If applications are optimal, the value of an additional unit of fertilizer carried over to period $i$ must equal the cost of providing an additional unit of fertilizer in period $i$. This means that (22) can be rewritten as:

$$p_i^y\ \partial y_i/\partial u_i=p_i^u-\alpha b p_{i+1}^u \qquad (24)$$

which is the same as the rule when there is no carryover except for the subtraction of the second term on the RHS. Assuming that crop yield is a positive concave function of available fertilizer, recognition of carryover reduces the optimal marginal product of available fertilizer, and hence increases the optimal level of available fertilizer. The value of the marginal product of fertilizer is equated with the cost of provision, only with carryover this is the price of fertilizer less the present value of the fertilizer that will not need to be applied in the next period.

Note that (24) shows that the optimal level of fertilizer to apply in period $i$ does not depend on fertilizer prices beyond period $i+1$, nor on crop prices or crop production functions beyond period $i$. In other words, in this particular case, the multiperiod problem *can* be solved as a sequence of separable one-period problems with parameters relating to the current and next period only. If the price of fertilizer is the same in all periods, (24) becomes:

$$p^y \, \partial y/\partial u = (1-\alpha b)p^u \tag{25}$$

which is the same as (2) with the price of fertilizer discounted by the factor $(1-\alpha b)$.

As an example, consider the determination of the optimal level of phosphate fertilizer to apply each year for 6 years in growing sorghum in the Northern Territory of Australia. The example is investigated in more detail in Kennedy (1986a). The sorghum response function is:

$$y = 2129(1-\exp(-0.04(x+u+8.8))) \tag{26}$$

where $y$ is yield of sorghum in kg/ha, $u$ is the rate of phosphate application per year in kg/ha, and $x$ is phosphate carried over from the previous year in kg/ha. It is assumed that 8.8 kg/ha of phosphate are always accessible in the soil whether fertilizer is applied or not, and that no fertilizer has been applied prior

TABLE 1

A comparison of outcomes for each rule when there is carryover

| Year $i$ | No carryover rule | | | Carryover rule | | |
|---|---|---|---|---|---|---|
| | $u_i$ (kg/ha) | $x_i$ (kg/ha) | $y_i\{x_i+u_i\}$ (kg/ha) | $u_i$ (kg/ha) | $x_i$ (kg/ha) | $y_i\{x_i+u_i\}$ (kg/ha) |
| 1 | 41 | 0 | 1844 | 60 | 0 | 1992 |
| 2 | 41 | 24 | 2018 | 26 | 34 | 1992 |
| 3 | 41 | 37 | 2064 | 26 | 34 | 1992 |
| 4 | 41 | 45 | 2081 | 26 | 34 | 1992 |
| 5 | 41 | 49 | 2089 | 26 | 34 | 1992 |
| 6 | 41 | 52 | 2092 | 7 | 34 | 1884 |
| Totals | 246 | | 12188 | 171 | | 11844 |
| Net revenue (kg/ha) | 539 | | | 566 | | |

to year 1. Prices are $p^y = \$0.08$ per kg and $p^u = \$0.83$ per kg. The rate of discount is 10% per annum which makes $\alpha = 0.9091$. The proportion of fertilizer carried over is $b = 0.57$.

Table 1 shows the results of applying the rule for no carryover when there is in fact carryover. This is the rule given by (2). Each year, 41 kg/ha of phosphate are applied. The amount *available* to the sorghum plants increases each year as fertilizer accumulates and is carried over. Yields therefore rise each year. Also shown are the results of applying the optimal rule given by (25). In year 1, 60 kg/ha of phosphate are applied, and 26 kg/ha thereafter until year 6. Nevertheless, in each of the years 1–5, 60 kg/ha of phosphate are available. In the final year, year 6, the application drops to 7 kg/ha. Available phosphate is 41 kg/ha, the optimal rate when no value is attached to phosphate carried over. Overall, less fertilizer is applied, yields are lower, but net revenue is higher by about 5%.

## 5. Optimality conditions for renewable resources

At some point in time the productivity of a biological unit declines with age and the unit eventually dies. In the case of natural living resources, such as fish and wildlife, the biological units are replaced without human intervention. Management of the stock of biological units primarily concerns the optimal rate of harvesting in each season.

In the case of other living resources which are farmed, such as livestock and forests, there is not only the problem of how much input to inject or output to extract at each stage, but also the question of whether to replace the biological unit.

In the following two sections optimality conditions are derived for both of these problems in turn, using dynamic programming arguments. In both cases it is assumed that the planning horizon is infinite, that period net returns are discounted, and that net return and growth functions are the same in all periods. This means that the optimal value functions, $V\{x\}$, are finite and the same for all periods. Subscripts for periods can be dropped.

### 5.1 Optimal sustainable yield

A yield from a resource stock is said to be sustainable if the yield is just balanced by the natural growth of the stock. If sustainable yields are taken from a stock, then the stock level remains constant. In many cases the optimal policy for a renewable resource is to harvest stocks until some optimal stock level is reached and from then on to harvest a sustainable yield. It is simple to show the condition which must hold for the optimal steady-state stock, $\boldsymbol{x}$, and harvest level $\boldsymbol{u}$. For any steady-state stock, autonomous growth $g\{\boldsymbol{x}\}$ equals sustainable yield $\boldsymbol{u}$, so that the basic recursive equation for the general resource problem (8) can be rewritten:

$$\begin{aligned} V\{\boldsymbol{x}\} &= a\{\boldsymbol{x}, \boldsymbol{u}^*\} + \alpha V\{\boldsymbol{x}\} \\ &= (1+r)a\{\boldsymbol{x}, \boldsymbol{u}^*\}/r \end{aligned} \tag{27}$$

where $a\{\boldsymbol{x}, \boldsymbol{u}^*\}$ is the optimal annual sustainable rent and $r$ is the rate of discount. The result is the usual formula for the present value of an annuity. Differentiating both sides of (27) with respect to $x$ gives:

$$\lambda r = (1+r)\ \mathrm{d}a/\mathrm{d}x \tag{28}$$

which is equivalent to Munro's Golden Rule of Resource Conservation (1981, p. 134). The increase in annual sustainable rent to be obtained from a one unit increase in stock must be equal to the interest which could be obtained by investing the worth of the unit elsewhere.

## 5.2 Optimal replacement

In the livestock feeding problem considered earlier, the possibility in period $i$ of replacing the current animal with another of lower liveweight and feeding it to period $n$ was ignored. This is now allowed for, taking the simplest case of stationary functions and an infinite planning horizon. For ease of exposition, only the number of feeding periods over which the animal has been kept for fattening, $t$, is made a state variable. The liveweight state variable $x$ and change-in-liveweight variable $u$ still apply but are suppressed. Replacement animals are all of the same weight and age.

The recursive equation is:

$$V\{t\} = \begin{cases} -a\{t\} + \alpha V\{t+1\} & t=0 \\ \max[-a\{t\} + \alpha V\{t+1\}, s\{t\} + V\{0\}] & \text{for} \quad 0<t<T \\ s\{t\} + V\{0\} & t=T \end{cases} \tag{29}$$

where $V\{t\}$ is the present value of following an optimal feeding and replacement regime for an animal which has been fed over $t$ feeding periods, $a\{t\}$ is the optimal feeding cost if the animal is kept for further feeding, $s\{t\}$ is the revenue received from selling the animal, and $T$ is the maximum number of periods an animal can be fed before replacement. Thus for $t=0$ and $t=T$ there is only one option. For all other $t$, if $-a\{t\} + \alpha\ V\{t+1\} > s\{t\} + V\{0\}$, continued feeding is optimal; otherwise, replacement is optimal.

Typically, for any particular replacement age and liveweight, and optimal feeding regime, there is an optimal duration of fattening, $t^*$, such that:

$$-a\{t\} + \alpha V\{t+1\} \gtreqless s\{t\} + V\{0\} \qquad \text{for} \quad t \lesseqgtr t^* \tag{30}$$

It is of interest to know what condition must hold for optimal replacement as a possible means of identifying $t^*$. The condition can be derived analytically using calculus if the relevant functions are differentiable. It is derived here more simply but less formally.

If the length of each feeding period is made sufficiently small, there is an insignificant difference between $V\{t\}$ and $V'\{t\}, t=1, ..., T$, where $V\{t\}$ is based on the optimal total fattening duration of $t^*$ feeding periods, and $V'\{t\}$ is based on a total fattening duration of $t^*+1$ feeding periods. This means in particular that there is no significant difference between $V\{0\}$ and $V'\{0\}$, and between $V\{t^*\}$ and $V'\{t^*\}$. Given:

$$V\{t^*\}=s\{t^*\}+V\{0\} \tag{31}$$

and

$$\begin{aligned} V'\{t^*\} &= -a\{t^*\}+\alpha V'\{t^*+1\} \\ &= -a\{t^*\}+\alpha(s\{t^*+1\}+V'\{0\}) \end{aligned} \tag{32}$$

the following equation holds as a reasonable approximation:

$$s\{t^*\}+V\{0\}=-a\{t^*\}+\alpha(s\{t^*+1\}+V\{0\}) \tag{33}$$

This can be rewritten as:

$$s\{t^*+1\}-s\{t^*\}-a\{t^*\}=r(a\{t^*\}+s\{t^*\}+V\{0\}) \tag{34}$$

which states that $t$ is the optimal number of feeding periods when the gain through postponing replacement by one more period (the LHS) equals the loss in forgone interest through not investing the proceeds which could be realized immediately (the RHS).

It remains to give a value to $V\{0\}$. Given the nature of the optimal policy:

$$\begin{aligned} V\{0\} &= -\sum_{j=0}^{t^*-1} \alpha^j a\{j\} + \alpha^{t^*} V\{t^*\} \\ &= -\sum_{j=0}^{t^*-1} \alpha^j a\{j\} + \alpha^{t^*}(s\{t^*\}+V\{0\}) \\ &= -(\sum_{j=0}^{t^*-1} \alpha^j a\{j\} + \alpha^{t^*} s\{t^*\})/(1-\alpha^{t^*}) \end{aligned} \tag{35}$$

Substituting for $V\{0\}$ in (34), the condition for the optimal time of replacement becomes:

$$s\{t^*+1\}-s\{t^*\}-a\{t^*\}=r(a\{t^*\}+s\{t^*\}-\sum_{j=0}^{t^*} \alpha^j a\{j\})/(1-\alpha^{t^*}) \tag{36}$$

This is a discrete generalization of the Faustmann formula for identifying the optimal rotation of a forest, with $s\{t\}$ equal to stumpage value, and $a\{t\}$ equal to replanting costs for $t=0$, and equal to zero for $t>0$.

## 6. Conclusion

Other writers have noted the need for modelling the dynamics of agricultural resource management. Dillon (1977, p. 97) comments that some early work in livestock production functions "did not really comprehend the problem of profit maximization over time". Johnson and Rausser (1977, p. 164) state that "many agricultural economists for a number of years have been busily applying static neoclassical theory to intrinsically dynamic systems". Heady (1981, p. 38) has suggested "perhaps conventional optimizing theory has been used more widely in recent decades because theory related to time and stochastic phenomena was not yet sufficiently operational". Hanf and Schiefer (1983, p. 16) note that "in most operational decision models the time dimension of managerial decisions is not considered adequately".

The determination of optimal injections into and harvests from agricultural and natural resources is often best specified as an optimal control problem. Yield at stage $i$ is commonly a function of both the injection level and a state variable such as biomass or age. Values of the state variable at subsequent stages depend on the current levels of the state and control variables. It is argued that crop and livestock management problems could advantageously be more routinely formulated as optimal control problems. Indeed, this argument could be extended to other areas of farm management which entail what are really dynamic problems, such as advertizing, extension, information acquisition and entering futures contracts.

Simple example problems have been considered in this article for ease of exposition. They involved single state and decision variables, and were deterministic. However, it is straightforward to extend the approach to more general problems. For example, the fertilizer carryover problem has been reformulated and solved for situations in which fertilizer applications in any number of previous periods must be specified as state variables (Kennedy, 1986b).

If the state variable $x_{i+1}$ is a stochastic function of $x_i$ and the control variable $u_i$, the methods of stochastic dynamic programming can be applied. The basic recursive equation (9) is modified by preceding $V_{i+1}\{x_{i+1}\}$ by the expectation operator, and interpreting $V_i\{x_i\}$ as the *expected* value of managing the resource under an optimal policy. Interest in analytically treating fisheries management problems as stochastic has been increasing recently, and most approaches have used stochastic dynamic programming (see, e.g., Andersen and Sutinen, 1984; Smith, 1986).

Since the initial recognition of the scope for dynamic programming as a farm management tool by practitioners such as Candler and Musgrave (1960), Throsby (1964) and Burt (1965), the technique's limitations and power have been tested over a wide range of applications (see, e.g., Kennedy, 1981a, 1986a; Burt, 1982). One message from this article is that it has much to offer to resource economists, not just as a numerical solution technique, but also as an analytical device.

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