A Comparison of Parametric and Nonparametric Estimation Methods for Cost Frontiers and Economic Measures

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1. Introduction

The study of producer theory uses several tools for exploring the structure of cost. Estimates of frontier functions, and the distances that firms are from the frontier provides insight into how firms with similar technological access and marketing opportunities achieve different levels of production efficiency. Frontier estimation also provides insight for both managers and economists regarding where cost savings exist for multi-product output firms. Parman et al. illustrate that it is possible to calculate multi-product scale economies and product-specific economies of scale that measure the potential for cost savings through the adjustment of output mix using Data Envelopment Analysis (DEA). Calculations of economies of scope from frontier estimation estimates illustrate how savings are achieved through producing multiple outputs in the same firm versus each output in a separate firm.

Traditionally, cost functions have been estimated using parametric methods with two-sided errors (i.e. OLS) where more efficient firms lie below the "average" frontier and less efficient firms lie above the "average" frontier (Christenson et. al. 1973, Diewert et. al. 1988). The result of such an estimation from a two-sided error model is thus an average cost function for the firms and not truly an estimation of the best practices (Greene 2005). Farrell (1957) used piece-wise linearization to envelope production data. In his analysis, all firms were either on or below the production frontier. In this way, the firms that reside on the frontier are relatively efficient, while those who resided below the frontier experience some amount of inefficiency. The distance from inefficient firms to the estimated frontier is calculated as a ratio of estimated minimum production inputs for a given output to actual production inputs for a given output was

then used as a metric to determine relative efficiency among firms. Later works by Farrell and Fieldhouse (1962), and Afriat (1972) eliminated the restriction of constant returns to scale technology using the nonparametric approach. Charnes, Cooper and Rhodes (1978), while evaluating the technical efficiency of decision making units coined the name Data Envelopment Analysis (DEA) used today to describe the evolved method developed by Farrell.

The DEA method was later augmented using the works of Samuelson (1938) and Shephard (1953) to highlight the dual relationship between costs and production to provide an envelope method to estimate relative cost efficiency among firms. Färe, Grosskopf, and Lovell (1985) provided a method using the dual cost approach with DEA to estimate cost efficiency. In this case, a cost frontier (minimum) is calculated rather than a production frontier (maximum) and thus efficient firms lie on the frontier, but inefficient firms lie above the frontier.

Aigner, Lovell and Schmidt and Meeusen and Van den Broeck; and Battese and Coelli suggest a method of estimation known as the stochastic frontier estimation based on maximum likelihood. They argue that the stochastic frontier conforms more closely to economic theory building a frontier where the observations of cost lie either on, or above a cost frontier. Like traditional parametric estimation methods, the stochastic frontier method requires the specification of a functional form, and all the assumptions that traditional parametric estimation methods must satisfy remain for the function to be consistent with economic theory. Battese and Coelli have expanded this method to include panel estimation of a stochastic frontier using the software program Frontier V4.1[1].

[1] Frontier V4.1 written by Tim Coelli are available online at : http://www.uq.edu.au/economics/cepa/frontier.php

Regression based methods with two-sided errors have been used to envelope the data such as the Conditional Ordinary Least Squares method (COLS) (Greene 2005), and Modified Ordinary Least Squares Method (MOLS) (Afriat 1972). These methods involve either altering the intercept (COLS) or shifting the production/cost function up/down based upon an expected value of the inefficiency distribution (MOLS). These methods are not without challenges and restrictions since the COLS method requires a homoscedastic distribution and the frontier function may not be the same as the minimized sum of squared errors. Also, the MOLS method cannot guarantee that the data is enveloped. A shift or intercept change only affects the calculation of the distance from the frontier calculations but does not affect calculations of marginal costs or incremental costs.

A less investigated parametric method uses OLS, restricting the errors to take on only positive values in the case of a cost function. This method does not require any prior assumptions of distribution and envelopes the data. Further, since it is not a shift, it allows for the marginal cost calculations to be based off of a parametric curve fitted to frontier firms.

The nonparametric approach to frontier estimation has as a few advantages to parametric methods. The most important is that it envelopes the data such that it conforms to economic theory. That is, the cost function is the minimum cost to produce an output bundle (Mas-Colell et al. 1995). As mentioned above, this is a disadvantage to the traditional parametric methods. Another cited advantage is that it does not require the specification of a function and thus is not technologically restrictive. In addition, the nonparametric method does not require the imposition of curvature required for a cost function (Featherstone and Moss 1994).

Recently, studies by Chavas and Aliber (using the dual DEA method shown by Färe et al. 1995) and Chavas et al. (2012) discuss methods for calculating economies of scope. These

articles developed nonparametric frontier estimation and associated incremental cost calculations to determine cost savings from producing multiple outputs simultaneously. However, the methods for calculating multi-product and product-specific scale economies nonparametrically are relatively new (Parman et. al.) and have not been compared to other methods. Such a comparison will evaluate the relative efficiency of the nonparametric approach to estimate the economies of scale measures.

This research examines the robustness of four different estimation approaches to evaluate their ability to estimate a "true" cost frontier and associated economic measures. The manuscript will evaluate three parametric methods including a two-sided error system, OLS with only positive errors, and the stochastic frontier method. The fourth method will be the DEA method (Färe et. al.) augmented to calculate multi-product and product-specific economies of scale (Parman et. al.). The robustness of the four estimation methods is examined using simulated data sets from two different distributions and two different observation quantity levels.

2. Data

The data for the analysis were generated using a modified Monte Carlo procedure found in Gao and Featherstone (2008) run on the SHAZAM software platform with the code found in Appendix A at the end of this document. A normalized quadratic cost function involving 3 inputs (x_1, x_2, x_3) with corresponding prices (w_1, w_2, w_3), and 2 outputs (y_1, y_2) with corresponding prices (p_1, p_2) was used. The normalized quadratic cost/ profit function is used since it is a self-dual cost function and a flexible functional form (Lusk et al.). The input and output prices (w_i, p_i) are simulated randomly following a normal distribution. The assumed distributions for the output prices and input prices shown below were set to provide observed prices strictly greater than zero

with different means and standard deviations to ensure some variability in input/output quantity demands and relative prices. They are:

$$
\begin{aligned}
&w_1 \sim N\,(9,\ 0.99)\\
&w_2 \sim N\,(18,\ 1.98)\\
&w_3 \sim N\,(7,\ 0.77)\\
&p_1 \sim N\,(325,\ 99)\\
&p_2 \sim N\,(800,\ 99)
\end{aligned}
\tag{1}
$$

The input price variability was set proportionate to its mean while the output prices have different relative variability to represent products in markets with different volatilities.

The outputs (y_i) and inputs (x_j) are determined as a function of input and output prices using an assumed underlying production technology. All prices are normalized on the input price w_3 and cost is scaled by w_3 to impose homogeneity. To ensure curvature holds, the "true" cost function is concave in input prices and convex in output quantities. The assumed parameters also satisfy symmetry ($b_{ij}=b_{ji}$). The assumed parameters (Table 1) are used to determine the output quantities y_1 *and* y_2[2]. The general form of the normalized quadratic cost function is:

$$
\begin{aligned}
&C(W,Y) = b_0 + \begin{bmatrix} b_1 & b_2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \\
&\frac{1}{2}\left\{\begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right\} + \begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\end{aligned}
\tag{2}
$$

Output quantities (shown below) are calculated using the assumed parameters of the cost function (Table 1) and the output prices generated in equation 1.

$$
\begin{aligned}
y_1 &= \frac{c_{22}p_1 - c_{12}p_2 + \left(a_{12}c_{12} - a_{11}c_{22}\right)w_1 + \left(a_{22}c_{12} - a_{21}c_{22}\right)w_2 + \left(a_2c_{12} - a_1c_{22}\right)}{\left(c_{22}c_{11} - c_{12}c_{12}\right)}\\
y_2 &= \frac{c_{12}p_1 - c_{11}p_2 + \left(a_{12}c_{11} - a_{11}c_{12}\right)w_1 + \left(a_{22}c_{11} - a_{21}c_{12}\right)w_2 + \left(a_2c_{11} - a_1c_{12}\right)}{-\left(c_{22}c_{11} - c_{12}c_{12}\right)}
\end{aligned}
\tag{3}
$$

[2] The analysis also was completed for alternative assumptions on input.

Using Equation 2, a positive random cost deviation term is added to the cost function following a half-normal distribution that alters the cost efficiency where the absolute value of e is distributed $e \sim N(0,1000)$[3]. The inclusion of this term adds cost inefficiency to the data such that firms are off the frontier effectively increasing their production cost while keeping the output quantities the same. The level of inefficiency is half-normally distributed.

An additional data set[4] is generated assuming a uniform distribution. The uniform deviation ranged from zero to 900. The normal distribution standard deviation of 1,000 generates a mean and standard deviation for cost efficiency roughly equivalent to a uniform distribution with a range from zero to 900.

From equation 2, and using Shephard's Lemma where $(\partial C(W,Y)/\partial w_i)=x_i$, the factor demands for inputs x_1 and x_2 are recovered. Factor demand for x_3 is found by subtracting the product of quantities and prices for x_2 and x_3 from the total cost.

$$\begin{aligned} x_1 &= b_1 + b_{11}w_1 + b_{12}w_2 + a_{11}y_1 + a_{12}y_2 \\ x_2 &= b_2 + b_{12}w_1 + b_{22}w_2 + a_{21}y_1 + a_{22}y_2 \\ x_3 &= \left(C\left(W,Y\right) - \left|e\right|\right) - x_1w_1 - x_2w_2) \end{aligned} \tag{4}$$

The input quantities (x_i's) are then adjusted (x_i^a) by the cost efficiency (CE) effectively increasing the input demands proportionate to the costs generated for each firm (equation 5).

$$x_i^a = \frac{x_i}{CE} \tag{5}$$

Using the above method, 400 observations were simulated where firms produce a combination of both outputs. Fifty firms were generated producing only y_1 with another 50 firms producing only y_2 which is accomplished by restricting either y_1 or y_2 to equal zero and re-

[3] The analysis also examined alternative normal standard deviations.

[4] The analysis was run using 2500 observations. The results were robust for 500 and 2500 observations

running the simulation for 50 separate observations each. Thus, a total of 500 observations were generated with summary statistics shown in Table 2. In Table 2, x_i^n represents inefficient input quantities for the normal error distribution and x_i^u represent the inefficient input quantities for the uniform distribution. The summary statistics for the multi-product scale, product-specific scale, scope, and cost efficiencies for each data point from the "true" cost function are shown in Table 3. Summary statistics for the economic measures are independent of the distribution of cost "inefficiency". Figures 1 through 4 provide a visual representation of the multi-product scale and scope economies as well as cost efficiencies and product-specific scale economies calculated from the "true" cost function.

While the cost efficiency for each firm is altered under a uniform versus a half-normal distribution (Figure 2), the MPSE, PSE's, and economies of scope are identical for each data point (Table 3) for the "true" cost function due to the input prices (w_i's) and output prices (p_i's) being the same. Thus, the output quantities (y_i's) remain unchanged (Equation 3). The input quantities (x_i's) are adjusted such that the deviation in input quantity used by each firm is uniformly distributed. In effect, the uniform data evenly distributes the quantity of firms at each relative distance from the frontier, rather than most firms being clustered around the mean distance as in the half-normal case.

A third data set is simulated using the half-normal distribution. This set uses the same data points as the half-normal case but excludes the single output firms. In this set, there are 400 firms each producing both y_1 and y_2. This data is used to evaluate each method's ability to estimate incremental costs accurately when no zero output firms are observed in the data.

The difference between the "true" estimates and each of the four methods are evaluated. This is done by subtracting each model's estimate from the "true" measure calculated with

Monte Carlo simulation. Since an approximation of the "true" measure is key, the statistics reported are the difference between the "true" measures and what was estimated by each method. Using this approach, any possible bias from each approach can be determined. A positive difference implies that the model underestimates the measure being evaluated and conversely, a negative difference indicates the model overestimates the measure being evaluated. The mean absolute deviation is also reported for all four methods allowing for the comparison of average absolute deviation from zero.

Cumulative density functions are presented for the differences between the true measures and the estimated measures to produce visual representation of both bias and deviation. If there is no difference between the estimated measure and the true measure, the cumulative density function is a vertical line at zero.

3. Estimation Methods

3.1 The Two-Sided Error System Equation

The traditional two-sided error system involves specification of a cost function and single frontier of input quantities and costs from observed prices and outputs. This method fits a curve with observations residing both above and below the fitted curve. The two-sided error method for this study was estimated using the SHAZAM software package using a normalized quadratic cost function with input prices normalized on w_3 (equation 6).

$$\begin{aligned} C(W,Y) &= b_0 + \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \\ &\frac{1}{2}\left\{ \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\} + \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + e_1 \end{aligned} \tag{6}$$

$$x_1 = b_1 + b_{11}w_1 + b_{12}w_2 + a_{11}y_1 + a_{12}y_2 + e_2$$
$$x_2 = b_2 + b_{12}w_1 + b_{22}w_2 + a_{21}y_1 + a_{22}y_2 + e_3 \tag{7}$$

Once the parameters shown in Equations 6 and 7 are estimated, the marginal costs are calculated by:

$$mcy_1 = a_1 + c_{11}y_1 + c_{12}y_2 + a_{11}w_1 + a_{21}w_2; and$$
$$mcy_2 = a_2 + c_{22}y_2 + c_{12}y_1 + a_{12}w_1 + a_{22}w_2 \tag{8}$$

For the normalized quadratic function with two outputs, the incremental costs for each output are:

$$ICy_1 = a_1y_1 + \frac{1}{2}c_{11}y_1^2 + c_{12}y_1y_2 + a_{11}w_1y_1 + a_{21}w_2y_1; and$$
$$ICy_2 = a_2y_2 + \frac{1}{2}c_{22}y_2^2 + c_{12}y_1y_2 + a_{12}w_1y_2 + a_{22}w_2y_2 \tag{9}$$

The costs of producing a single output are:

$$CY_1 = C(W,Y) - ICy_2; and$$
$$CY_2 = C(W,Y) - ICy_1 \tag{10}$$

Once the marginal costs, incremental costs, and single output costs have been estimated, the multi-product scale economies (MPSE), economies of scope (SC), and product-specific scale economies ($PSEy_i$) can be calculated:

$$MPSE = \frac{C(W,Y)}{\left[\sum_{i=1}^{2} mcy_i * y_i\right]} \tag{11}$$

$$SC_i = \frac{\left[\sum_{i=1}^{2} CY_i - C(W,Y)\right]}{C(W,Y)} \tag{12}$$

$$PSEy_i = \frac{ICy_i}{y_i * mcy_i} \tag{13}$$

Cost efficiency is not calculated for the two-sided error system since the deviations from the "frontier" are two sided.

3.2 The OLS Estimator with Positive Errors

A one-sided error model is estimated similar to the two-sided error model with the difference being the error term is one sided and input demand equations (7) are not estimated. Equation 6 is estimated with the restriction that $e_i \geq 0$ for all i using the General Algebraic Modeling Software (GAMS) program. The objective function minimizes the sum of squared errors subject to constraints that define the error. Firms on the frontier exhibit errors equal to zero while those with inefficiency exhibit positive errors. The calculations of MPSE, PSE, and *SC* are identical to the two-sided error model using the coefficient estimates from the one-sided error model.

3.3 The Stochastic Frontier Cost Function Estimator

The stochastic frontier estimation method uses FRONTIER Version 4.1 by Coelli. It is based off the stochastic frontier methods of Battese and Coelli (1992, 1995) and Schmidt and Lovell (1979). One of the primary differences between the stochastic frontier method and the OLS two-sided error method is the error term. Specifically, the error term consists of two elements, V_{it} which are random variables assumed to be iid N(0,$\boldsymbol{\sigma}^2$), and U_{it} which is a non-negative random variable capturing inefficiency. U_{it} is assumed to be half-normal for this analysis and defines how far above the frontier a firm operates. The resulting cost function is:

$$C(W,Y) = b_0 + \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \frac{1}{2}\left\{ \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\} + \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + (V_i + U_i) \tag{14}$$

For simplicity 14 can be rewritten as follows:

$$C(W,Y)_i = X_i B + \left(V_i + U_i\right) \tag{15}$$

The cost efficiency (CE) from the stochastic frontier method takes on a value between one and infinity since $U_i \geq 0$. The cost efficiency from the nonparametric method and the one-sided error model is evaluated by taking the minimized total costs estimate dividing it by the actual total costs resulting in cost efficiency estimates between 0 and 1.

$$CE = \frac{X_i B}{X_i B + U_i} \tag{16}$$

The calculations of marginal costs, incremental costs, the MPSEs, the PSEs, and the economies of scope are the same as those shown in the two-sided error model above using the estimated parameters.

Each of the methods used to estimate the dual cost function are parametric. Symmetry and homogeneity are imposed in the estimation process. Curvature and monotonicity are not and in an empirical estimation they would need to be examined to ensure the cost function estimated is consistent with economic theory.

3.4 The Nonparametric Approach

The nonparametric approach for estimating multi-product scale, product-specific scale and scope economies follow Parman et al. (2013). The cost (C_i) is determined for each firm where costs are minimized for a given vector of input prices (w_i) and outputs (y_i) with the choice being the optimal input bundle (x_i^*).

$$
\begin{aligned}
&\min C_i = w_i^{'} x_i^{*} \\
&s.t \\
&Xz \le x_i^{*} \\
&y^{'} z \ge y_i \\
&z_1 + z_2 + \ldots + z_n = 1 \\
&z_i \in \mathbb{R}^{+}
\end{aligned} \tag{17}
$$

where there are "n" producers. The vector Z represents the weight of a particular firm with the sum of Z_i's equal to 1 for variable returns to scale. From the above model, the costs and output quantities can be estimated. The output quantities (y_i) constrain the cost minimizing input bundle to be at or above that observed in the data. Total cost from the model (C_i) is the solution to the cost minimization problem including the production of all outputs for the ith firm. The cost of producing all outputs except one ($C_{i,all\text{-}p}$) where p is the dropped output and is determined by either forcing one of the outputs to equal zero or by dropping the pth output constraint.

Cost efficiency identifies a firm's proximity to the cost frontier for a given output bundle. It is the quotient of the estimated frontier cost (C_i) and the actual cost (ATC_i) the firm incurred producing their output bundle.

$$
CE_i = \left[\frac{C_i}{ATC_i} \right] \tag{18}
$$

The calculation for economies of scope are:

$$
SC_i = \left[\frac{(\sum_p C_{i,p}) - C_i}{C_i} \right] \tag{19}
$$

The calculation of multi-product economies of scale uses the shadow prices on the output constraints (17) to calculate marginal cost. MPSE is then:

$$MPSE_i = \left[\frac{C_{i,all}}{\sum_p MC_{i,p} Y_{i,p}} \right] \tag{20}$$

Product specific economies of scale (PSE) require the calculation of the incremental costs ($IC_{i,p}$):

$$IC_{i,p} = C_i - \sum_j C_{i,j \neq p} \forall j \tag{21}$$

Average incremental costs ($AIC_{i,p}$) are determined by dividing incremental costs by individual output:

$$AIC_{i,p} = \frac{IC_{i,p}}{y_{i,p}} \tag{22}$$

Using the average incremental cost and the marginal cost calculation above, the PSEs are:

$$PSE_{i,p} = \frac{AIC_{i,p}}{MC_{i,p}} \tag{23}$$

When estimating the frontier nonparametrically using a data set with no single output firms, it is not possible to estimate the incremental costs by forcing one of the output constraints to zero (Equation 17). Thus, the only alternative is to drop one of the constraints. However, when an output constraint is dropped, the program may allow some of the output for the dropped constraint to be produced resulting in an overstatement of the cost of that one output ($C_{i,p}$) which will cause an over statement of economies of scope (equation 19) and an understatement of product specific scale economies (23).

Thus, the additional product-specific production costs from an output being produced when it shouldn't must be. The procedure for adjusting in a two goods case is as follows: the cost of producing y_1 only ($C_{i,1}$) assumes that only (y_1^1) is being produced. However, the optimization program allows some $y_{i,2}^1$ to be produced in this situation overstating the cost of producing y_1

only ($C_{i,1}$). To remove the additional cost, the percentage contribution of $y_{i,1}^{1}$ to cost is multiplied by the cost of producing y_1 only, yielding an adjusted cost ($C^{a}_{i,1}$). This new adjusted cost is then used in the calculation of incremental costs and associated economic measures:

$$C_{i,1}^{a} = C_{i,1}\left(\frac{y_{i,1}^{1}}{y_{i,1}^{1} + y_{i,2}^{2}}\right) \tag{24}$$

The analysis evaluates the difference between the "true" measures of cost efficiency, economies of scope (scope), multi-product scale economies (MPSE), and product-specific economies of scale (PSE) from the four modeling approaches. The statistics and results presented are not the economic measure calculations but the difference between the model estimates and the "true" measure.

The parametric estimators are specified knowing the "true" functional form: the normalized quadratic cost function. Therefore, the differences may represent a "best case scenario" for each parametric method in that the true functional form is known with only the parameter estimates being unknown.

4. Results

Table 4 shows the parameter estimates and standard errors for the parametric methods for all three data sets. The parameter estimates from each method were different under the same distributional assumptions, and different for the same method under different distributional assumptions with the exception of the OLS positive errors model which yielded the same parameter estimates for the uniform and half-normal distributions. For both the two-sided error system, and the stochastic frontier estimation, different distributional assumptions yielded changes in magnitude as well as sign changes for various parameter estimates. Also, when comparing the 500 observation half-normal case to the 400 observation half-normal with no zero

outputs case, there were changes for all three estimation methods as well as changes in magnitude for the estimated parameters. The calculation for the standard errors using GAMS was conducted using the method from Odeh et. al. (2010).

Curvature was checked for each estimation method and each simulation to ensure that it was not violated (Table 5). A curvature violation implies that the shape of the cost frontier estimation does not conform to the "true" cost function which is known in this case, and that it violates the economic theory of the cost function. To check these conditions, the eigenvalues are calculated for the "*b*" (price) and "*c*" (output) matrices where the eigenvalues for "*b*" should be negative (concave in prices) and "*c*" values should be positive (convex in outputs). Each parametric model violated curvature in every simulation for either the "*b*" or "*c*" matrices or both. The one-sided error model and the two-sided system violated curvature of both the "*b*" and "*c*" matrices for the 400 observations simulation.

4.1 Cost Efficiency

Cost efficiency differences evaluate each model's ability to estimate the frontier since it is the ratio of estimated minimum cost to actual total cost. The two-sided error model was not examined because it does not estimate a frontier. The OLS Positive Errors and Nonparametric models performed well for all three data sets in estimating the frontier with average differences below 0.03 in absolute value and standard deviations below 0.04 (Table 6). The most accurate estimation of cost efficiency was the nonparametric model under the uniform distribution simulation with the average, standard deviation, and mean absolute deviation close to zero.

The stochastic frontier method performed almost as well under the half-normal simulation with the average closest to zero, and under the 400 observation simulation with an

average difference of -0.028 but much worse under the Uniform simulation (Figure 5) with an average difference of -0.198, mean absolute deviation of 0.198, and standard deviation of 0.118. This implies that estimating efficiency measures with the stochastic frontier method may be dependent on the correct assumption of the error distribution.

In all cases, the average differences were below zero implying that the OLS positive errors, Stochastic Frontier, and Nonparametric models slightly over estimated the cost efficiencies for most of the firms. This is confirmed by examining the mean absolute deviation in the uniform and 400 observations cases being the same the absolute value of the mean. This is expected given the simulation procedure. Frontier methods envelope the observed data, thus cost efficiencies are overestimated unless there are a significant number of firms where the simulated error is zero. However, the averages were close to zero in most cases with low standard deviations.

4.2 Economies of Scope

Differences in estimates of economies of scope for the four different methods raised more issues than the cost efficiency estimates. For the half-normal and uniform simulations, the two-sided error system had an average furthest from zero at -0.30 in with a standard deviation similar to the other methods (Table 7). For the 400 observation simulation, the Stochastic Frontier Method was furthest from zero at -2.32. Due to scaling, the stochastic frontier method cumulative density is not visible in Figure 6 for the 400 observations case.

The OLS Positive Errors Model and Nonparametric Model estimated economies of scope closely with averages for the half-normal distribution of -0.08 and -0.09 respectively and standard deviations around 0.07 and 0.03 respectively (Table 7). The estimates of scope for the

uniform distribution from the OLS Positive Errors Model and Nonparametric Model were less than 0.02 in absolute value with low standard deviations. The average and standard deviation for the Nonparametric method under the uniform distribution were affected by a few observations being significantly off (Figure 6). For the 400 observation data set, the Nonparametric method had the lowest standard deviation (0.04) and an average closest to zero in absolute value (0.07) (Table 7).

The three parametric estimation methods over estimated economies of scope in all simulations except for the case of a normal distribution where the OLS Positive Errors Model under estimated economies of scope slightly. In many cases, the parametric methods strictly over estimated scope where the absolute values of the means were the same as the mean absolute deviations (Table 7). The Nonparametric Model slightly over estimated scope in both the half-normal and uniform simulations but slightly underestimated scope in the 400 observations data set.

The most robust estimator of economies of scope appears to be the Nonparametric approach with averages close to zero in all three simulations and low standard deviation. The OLS Positive Errors Model does not perform as well in the case of 400 observations simulation, nor does the Stochastic Frontier Model and the standard Two-sided Error System under the half-normal and uniform simulations. Measures of economies of scope are suspect using any of the methods when there are no zero output observations in the data sample.

4.3 Multi-product Economies of Scale

An accurate estimation of MPSE requires both a close approximation of the true frontier and marginal costs. It is possible to have a very good approximation of MPSE but be off on

economies of scope and PSEs due to the necessary estimation of incremental costs for scope and the PSEs.

The nonparametric approach appears to be the most robust estimator of MPSE (Figure 7). It has an average difference closest to zero in all three simulations and the lowest standard deviation in both the half-normal case and 400 observation cases (Table 8). Its mean absolute deviation is also lowest except compared to the OLS Positive Errors model under the uniform distribution. The standard deviation was only slightly higher for the nonparametric approach compared to the OLS Positive Errors model in the uniform case with a standard deviation of 0.05 for the Nonparametric model and 0.04 for the OLS Positive Errors model (Table 8). All average differences except OLS Positive Errors in the uniform case were negative implying that MPSE was, for the most part, over estimated by the models.

Of the four modeling methods in all three simulations, the two-sided error system had the largest average differences from zero and the highest standard deviations (Table 8). No observations were correctly estimated for MPSE (Figure 7) in any of the three simulations. The standard two-sided system approach never approaches the zero difference.

The Stochastic Frontier method results were mixed. While it was out performed by the nonparametric approach in all simulations, it was close to the "true MPSE' in the case of the 400 observations. However, in the uniform distribution simulation, it did not perform well with an average difference of -0.21 and standard deviation of 0.26 (Table 8).

4.4 Product-Specific Economies of Scale

The estimation of the PSEs for both y_1 and y_2 for the half-normal and uniform simulations yielded similar results for all three parametric type estimations (Table 9). The parametric

approaches appear to slightly outperform the nonparametric approach in the estimation of PSE_1 (Figure 8 Panel A) in the half-normal simulation but performed similarly in the estimation of PSE_2 (Figure 8 Panel B) under the same distribution in terms of absolute distance from zero. For the uniform simulation, the PSE_1 and PSE_2 estimates from the Nonparametric Model were similar to both the Stochastic Frontier Method and the two-sided error systems with the OLS Positive Errors Model being the closest to zero under the uniform simulation (Table 9).

Under the half normal and the uniform simulations, the two-sided error system and the stochastic frontier underestimated PSE's for y_1 and y_2. OLS Positive Errors under estimated PSEs under both distributions except for the half-normal PSE_1. In the 400 observation simulation, OLS overestimated both PSEs where that was not the case for the Nonparametric Model and OLS Positive Errors Model.

The average difference and standard deviation for the PSEs from the Stochastic Frontier Method in the 400 observation simulation are off significantly (Table 9). Of the parametric methods, it appears that two-sided error system performed best when there were no single output firms having the lowest standard deviations and averages fairly close to zero, especially for PSE_2 (Figure 9 Panel B).

In the 400 observations simulation, while the standard deviation was higher for the nonparametric method than OLS and OLS Positive Errors, the average for PSE_1 was closest to zero using the nonparametric method and closer than OLS Positive Errors and the Stochastic Frontier Method for PSE_2 (Table 9). None of the methods accurately predict the PSEs when there were no zero observations (Figures 8 and 9, panel C).

The challenge for each model in the 400 observations simulation is that there are no firms producing only a single output. This requires each method to extrapolate estimates out of sample for the purpose of calculating incremental costs. If the smallest firms are not efficient, a linear projection will be inaccurate depending on the amount of inefficiency of the smaller firms.

5. Implications of the Results

Results suggest the two-sided error system is least accurate for estimating a frontier function and associated cost measures. This method lacks consistency with the economic definition of a cost function. This is apparent in that it does not, in any simulation, robustly estimate the MPSE or the economies of scope.

The stochastic frontier method appears susceptible to incorrect distributional assumptions on the one sided error as it estimates the frontier much closer to the "true" frontier under a half-normal distribution rather than the uniform distribution. Results also suggest that the stochastic frontier method has difficulty extrapolating when there are no zero output firms as shown by its inability to accurately estimate economies of scope or PSEs for the 400 observations simulation. However, in the case of a normal distribution it accurately estimates the frontier and, with the existence of zero output observations, accurately estimates economies of scope and PSEs.

The OLS positive errors model appears to accurately project the cost frontier regardless of the distributional assumption and whether there are no single output firms. However, like the stochastic frontier method, the OLS positive errors method has difficulty extrapolating when there are no single output firms. Thus, under no single output cases, the economies of scope estimations from the positive errors model may be incorrect, as may be PSE estimates.

The nonparametric method in all three simulations is fairly robust in estimating the "true" cost frontier and associated economic measures. It is also the model most capable of handling data with no single output firms as shown by its proximity to zero in estimating economies of scope and PSEs. It does not appear to be particularly susceptible to distributional assumptions.

It is important to remember that all of the parametric methods used the correct function form (normalized quadratic). These results may be different should the data not be consistent with that functional form. Functional form and statistical assumptions are not necessary in the case of the nonparametric method, thus, the results will likely be more robust. Therefore, if a researcher is unsure of model specification or the data generation process, the nonparametric approach may be a good alternative to parametric estimation.

6. Conclusions

Four methods for estimating a cost frontier and associated economic measures were examined under three different simulations including a half-normal distribution, uniform distribution, and a data set with no single output firms. The first method examined was a traditional two-sided error system regression with costs residing above and below the fitted curve. The second was the stochastic frontier method initially proposed by Aigner, Lovell, and Schmidt where the error term ensures all observations lie on or above the cost frontier. The third method was an OLS regression where the error term was restricted to take on positive values only ensuring that all observations lie on or above the cost frontier. Finally, a nonparametric DEA method proposed by Färe et. al. using a series of linear segments was used to trace out a cost frontier. For each simulation, cost efficiency, economies of scope, multi-product scale

economies, and product specific scale economies were calculated and compared to the known values from the "true" cost frontier.

The results show that the three frontier estimators are capable of estimating the "true" frontier in some simulations however; the stochastic frontier method was not as robust as the nonparametric method or the OLS Positive Errors Model. This result was also observed in the calculation of multi-product scale economies where all three frontier functions estimated the 400 observations data set and the half-normal data set close, whereas the Stochastic Frontier Model was not. The OLS method could not estimate a frontier and corresponding cost efficiency and was the furthest from the "true" calculation of multi-product scale economies indicating it was not close in estimating marginal cost.

The Stochastic Frontier Model appears to be less robust for estimating the "true" measures that require calculating incremental costs such as economies of scope and product-specific scale economies. Though the two-sided error model was less accurate in obtaining the "true" estimates in the half-normal and uniform simulations, the Stochastic Frontier method was less accurate in estimating scope economies or product specific economies of scale when there were no single output firms.

Overall, the nonparametric approach estimated the frontiers and associated economic measures close to the "true" values considering no special assumptions or specifications were required in its estimation. It's estimation of the frontier was about as close, or closer to the "true" values as any of the methods examined and its calculations of MPSE and economies of scope were the closest in several of the scenarios presented. The nonparametric approach did not significantly fail to estimate PSEs compared to any other method. Therefore, it appears that the DEA method is robust for estimating scale and scope measures.

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Tables

Table 1 Assumed coefficients used in cost function for data simulation for half-normal and uniform distributions**.**

Coefficient	Value
A_1	30.0
A_2	80.0
A_{11}	0.50
A_{12}	1.00
A_{21}	0.6
A_{22}	0.50
B_0	20.0
B_1	10.0
B_2	35.0
B_{11}	-0.09
B_{12}	-0.15
B_{22}	-0.47
C_{11}	1.44
C_{12}	-0.24
C_{22}	2.29

Table 2 The average, standard deviation, minimum and maximum for the input/output quantities and input prices in half-normal (x_i^n) and uniform (x_i^u) cases.

N=500	Average	Standard Deviation	Minimum	Maximum
x_1^n	42.29	11.95	13.35	88.33
x_2^n	69.85	23.29	38.44	268.76
x_3^n	2602.60	1154.75	152.95	8083.87
x_1^u	36.93	8.644	14.06	68.89
x_2^u	60.16	10.25	38.43	136.13
x_3^u	2302.06	1027.79	147.92	6585.05
w_1	9.05	0.98	5.42	11.98
w_2	17.95	1.88	13.15	24.70
w_3	6.98	0.78	4.85	9.75
y_1	11.67	5.90	0.00	30.19
y_2	14.31	7.53	0.00	37.92

Table 3 Summary statistics for efficiency calculations from generated data including half-normal and uniform distributions.

Economic Measure	Average	Standard Deviation	Minimum	Maximum
------*Half-normal Distribution*------				
Multi-product Scale Economies	0.931	0.108	0.772	1.989
Cost Efficiency	0.721	0.177	0.129	1.000
Scope	0.096	0.051	0.037	0.513
Product-specific Scale Economies for y1	0.728	0.246	0.000	0.957
Product-specific Scale Economies for y2	0.763	0.257	0.000	0.995
------*Uniform Distribution*------				
Cost Efficiency	0.799	0.133	0.268	1.000
------*400 Observations*------				
Multi-product Scale Economies	0.918	0.082	0.773	1.989
Cost Efficiency	0.751	0.159	0.129	1.000
Scope	0.085	0.028	0.062	0.514
Product-specific Scale Economies for y1	0.808	0.047	0.678	0.957
Product-specific Scale Economies for y2	0.848	0.041	0.733	0.996

Note: Economies of Scope, Multi-product Scale Economies, and Product-specific Scale economies are identical for the half-normal and uniform distributions

Table 4 Parameter estimates and standard errors for three simulations of each parametric model

	N= 500 Half-Normal Distribution			N=500 Uniform Distribution			N=400 No Zero Outputs		
	Two-sided Errors	One-sided Errors	Stochastic Frontier	Two-sided Errors	One-sided Errors	Stochastic Frontier	Two-sided Errors	One-sided Errors	Stochastic Frontier
A_1	28.83	56.05	32.00	29.77	56.05	28.90	60.92	76.82	302.28
	(4.67)	(3.61)	(27.32)	(2.05)	(1.88)	(3.74)	(23.12)	(5.24)	(12.03)
A_2	79.33	88.39	46.05	80.18	88.39	78.24	52.38	54.74	-221.34
	(3.83)	(2.75)	(21.10)	(1.62)	(1.43)	(4.41)	(17.58)	(4.40)	(10.46)
A_{11}	0.49	-9.73	2.91	0.47	-9.73	3.04	1.17	24.21	-45.54
	(0.05)	(2.42)	(19.55)	(0.04)	(1.26)	(1.85)	(0.39)	(3.51)	(22.64)
A_{12}	0.67	-3.83	-7.58	0.56	-3.83	1.31	1.91	-16.74	-88.03
	(0.19)	(1.39)	(11.78)	(0.08)	(0.72)	(1.57)	(0.75)	(1.78)	(21.90)
A_{21}	0.54	-5.00	1.69	0.76	-5.00	4.45	0.12	-34.24	65.82
	(0.69)	(1.71)	(14.72)	(0.03)	(0.89)	(1.59)	(0.29)	(2.71)	(18.24)
A_{22}	-1.16	0.44	5.61	-0.39	0.44	1.66	-1.67	-0.99	78.41
	(0.15)	(1.13)	(9.84)	(0.07)	(0.58)	(1.26)	(0.59)	(1.50)	(18.03)
B_0	684.95	360.20	-2011.12	401.01	360.20	460.56	689.81	-194.91	-4270.54
	(60.23)	(100.29)	(42.22)	(26.66)	(52.15)	(1.34)	(80.79)	(42.64)	(1.82)
B_1	26.18	-267.67	1833.84	20.20	-267.67	12.97	25.65	103.79	3225.37
	(26.17)	(104.00)	(107.89)	(0.75)	(54.07)	(1.17)	(1.74)	(46.62)	(8.11)
B_2	70.13	-200.80	825.29	57.29	-200.80	48.74	68.61	56.44	2109.47
	(5.25)	(66.17)	(104.11)	(2.34)	(34.40)	(1.71)	(4.74)	(26.04)	(6.91)
B_{11}	0.34	46.59	-834.08	-0.09	46.59	-65.45	0.12	-210.43	-529.61
	(0.18)	(89.08)	(74.46)	(0.08)	(46.36)	(1.00)	(0.17)	(34.17)	(8.11)
B_{12}	0.83	143.01	-297.13	0.20	143.01	-28.37	0.22	195.80	-1076.81
	(0.51)	(34.06)	(62.21)	(0.23)	(20.21)	(1.05)	(0.50)	(14.26)	(10.09)
B_{22}	2.85	27.48	-135.50	0.64	27.48	-8.34	0.25	2.74	-283.21
	(1.88)	(26.88)	(114.14)	(0.87)	(14.90)	(1.17)	(1.80)	(10.34)	(13.61)
C_{11}	2.29	1.64	2.20	1.76	1.64	2.05	-0.35	-1.41	22.91
	(0.15)	(0.18)	(1.25)	(0.06)	(0.09)	(0.50)	(1.67)	(0.64)	(13.05)
C_{12}	-0.78	-0.25	0.10	-0.48	-0.25	-0.75	1.40	0.013	-15.66
	(0.53)	(0.07)	(0.35)	(0.02)	(0.04)	(0.19)	(1.25)	(0.53)	(8.36)
C_{22}	3.13	2.41	3.16	2.66	2.41	2.38	1.32	6.40	14.66
	(0.68)	(0.12)	(0.78)	(0.04)	(0.06)	(0.32)	(0.97)	(0.43)	(5.63)

Table 5 Eigenvalues for "B" (prices) and "C" (outputs) matrices for each model and simulation

Half-Normal		Uniform		No Zero Outputs	
B	C	B	C	B	C
------*Two-sided Error System*------					
3.09	3.50	0.70	2.80	0.41	2.10
0.09	1.80	-0.10	1.50	-0.40	-1.14
X	√	**X**	√	**X**	**X**
------*Stochastic Frontier*-------					
-37.0	3.20	3.10	2.90	677	34.8
-931	-2.20	-76.0	1.40	-1489	2.60
√	**X**	**X**	√	**X**	√
------*OLS Positive Errors*------					
180	2.40	180	2.40	118	6.00
-106	1.50	-106	1.50	-325	-1.50
X	√	**X**	√	**X**	**X**

Note: The known cost function is concave in prices (B matrix) and convex in outputs(C matrix). For concavity, the matrix must yield negative eigenvalues and for convexity the matrix must yield positive eigenvalues. A "√" implies correct curvature while "X" implies a curvature violation.

Table 6 Statistics for simulated cost efficiency differences for the OLS positive errors, stochastic frontier, and nonparametric estimations

	Average	Standard Deviation	Minimum	Maximum	Mean Absolute Deviation
		------Half-normal Distribution------			
OLS Positive Errors	-0.020	0.039	-0.277	0.063	0.023
Stochastic Frontier	-0.015	0.043	-0.304	0.155	0.024
Nonparametric	-0.025	0.041	-0.530	-0.003	0.026
		------Uniform Distribution------			
OLS Positive Errors	-0.011	0.013	-0.062	0.122	0.013
Stochastic Frontier	-0.198	0.118	-1.478	-0.058	0.198
Nonparametric	-0.004	0.007	-0.079	0.000	0.004
		------400 Observations------			
OLS Positive Errors	-0.017	0.023	-0.173	0.015	0.017
Stochastic Frontier	-0.028	0.049	-0.351	0.139	0.039
Nonparametric	-0.022	0.032	-0.386	-0.003	0.022

Table 7 Statistics for economies of scope differences from all four methods from all three data sets.

	Average	Standard Deviation	Minimum	Maximum	Mean Absolute Deviation
		------Half-normal Distribution------			
Two-sided Error System	-0.300	0.057	-0.474	-0.194	0.300
OLS Positive Errors	-0.082	0.067	-0.291	0.093	0.088
Stochastic Frontier	-0.101	0.056	-0.266	0.052	0.103
Nonparametric	-0.089	0.030	-0.249	0.029	0.089
		------Uniform Distribution------			
Two-sided Error System	-0.300	0.058	-0.489	-0.191	0.300
OLS Positive Errors	0.010	0.023	-0.044	0.190	0.018
Stochastic Frontier	-0.158	0.041	-0.312	-0.091	0.158
Nonparametric	-0.019	0.079	-0.904	0.017	0.020
		------400 Observations------			
Two-sided Error System	-0.148	0.115	-0.437	0.152	0.175
OLS Positive Errors	-0.187	0.053	-0.376	-0.048	0.187
Stochastic Frontier	-2.324	0.607	-4.109	-0.142	2.324
Nonparametric	0.070	0.036	0.025	0.514	0.070

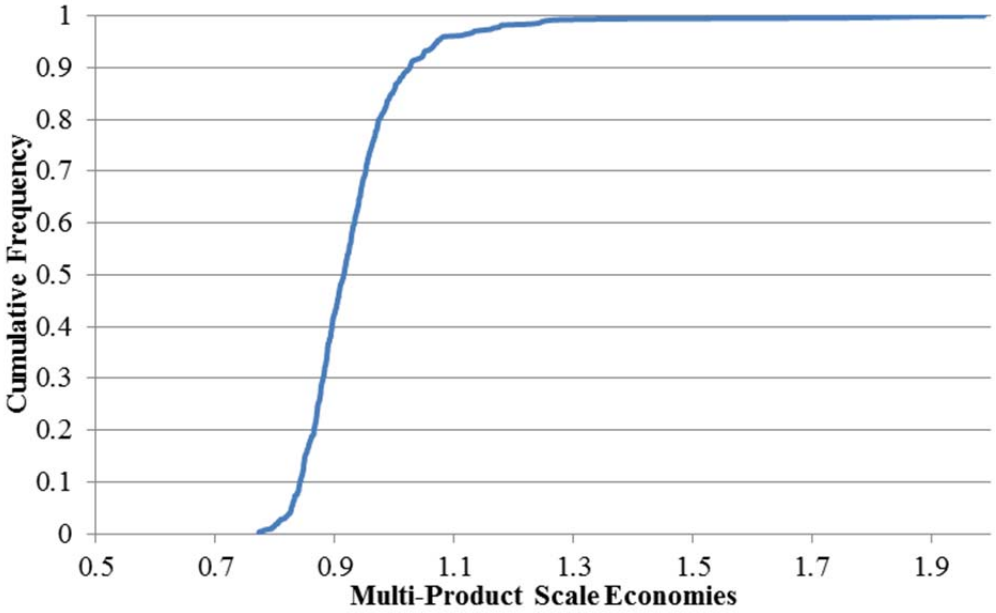
Table 8 Statistics for Multi-product Scale Economies differences from all four methods from all three data sets.

	Average	Standard Deviation	Minimum	Maximum	Mean Absolute Deviation
		------Half-normal Distribution------			
Two-sided Error System	-0.443	0.361	-2.917	-0.067	0.443
OLS Positive Errors	-0.257	0.658	-5.995	0.104	0.272
Stochastic Frontier	-0.084	0.183	-1.577	0.107	0.107
Nonparametric	-0.002	0.096	-0.678	0.739	0.049
		------Uniform Distribution------			
Two-sided Error System	-0.482	0.609	-7.857	-0.068	0.482
OLS Positive Errors	0.023	0.044	-0.124	0.515	0.027
Stochastic Frontier	-0.210	0.258	-3.384	-0.030	0.210
Nonparametric	-0.012	0.054	-0.277	0.114	0.029
		------400 Observations------			
Two-sided Error System	-0.371	0.300	-4.642	-0.065	0.371
OLS Positive Errors	-0.087	0.131	-1.331	0.105	0.103
Stochastic Frontier	-0.074	0.152	-0.727	0.177	0.118
Nonparametric	-0.009	0.137	-0.821	0.743	0.058

Table 9 Statistics for Product-specific Scale Economies differences for y_1 and y_2 from all four methods and all three data sets

		Average	Standard Deviation	Minimum	Maximum	Mean Absolute Deviation
		------Half-normal Distribution------				
	Two-sided Error System	0.099	0.024	0.047	0.147	0.099
y1	OLS Positive Errors	-0.056	0.039	-0.130	0.140	0.064
	Stochastic Frontier	0.111	0.018	0.081	0.176	0.111
	Nonparametric	0.128	0.202	-0.259	0.573	0.202
	Two-sided Error System	0.053	0.011	0.022	0.081	0.053
y2	OLS Positive Errors	0.075	0.016	0.037	0.148	0.075
	Stochastic Frontier	0.060	0.006	0.004	0.072	0.060
	Nonparametric	0.027	0.099	-0.344	0.303	0.085
		------Uniform Distribution------				
	Two-sided Error System	0.098	0.025	0.024	0.147	0.098
y1	OLS Positive Errors	0.012	0.013	-0.012	0.049	0.013
	Stochastic Frontier	0.056	0.018	0.012	0.093	0.056
	Nonparametric	0.020	0.123	-0.241	0.299	0.097
	Two-sided Error System	0.052	0.011	0.002	0.075	0.052
y2	OLS Positive Errors	0.005	0.002	0.000	0.011	0.005
	Stochastic Frontier	0.004	0.004	-0.008	0.012	0.005
	Nonparametric	-0.004	0.053	-0.186	0.126	0.039
		------400 Observations------				
	Two-sided Error System	-0.218	0.054	-0.368	-0.048	0.218
y1	OLS Positive Errors	-0.421	0.296	-4.110	-0.066	0.421
	Stochastic Frontier	1.696	32.94	-619.9	120.5	6.480
	Nonparametric	0.187	0.310	-0.311	0.957	0.266
	Two-sided Error System	-0.036	0.015	-0.086	0.000	0.036
y2	OLS Positive Errors	0.302	0.028	0.042	0.373	0.302
	Stochastic Frontier	-9.224	248.9	-4952	444.3	18.15
	Nonparametric	0.132	0.209	-0.248	0.937	0.183

Figures



Note: the MPSE calculations for both the half-normal and uniform error distribution is identical.

Figure 1 Frontier Multi-Product Scale Economies Cumulative Frequency for Simulated Data.

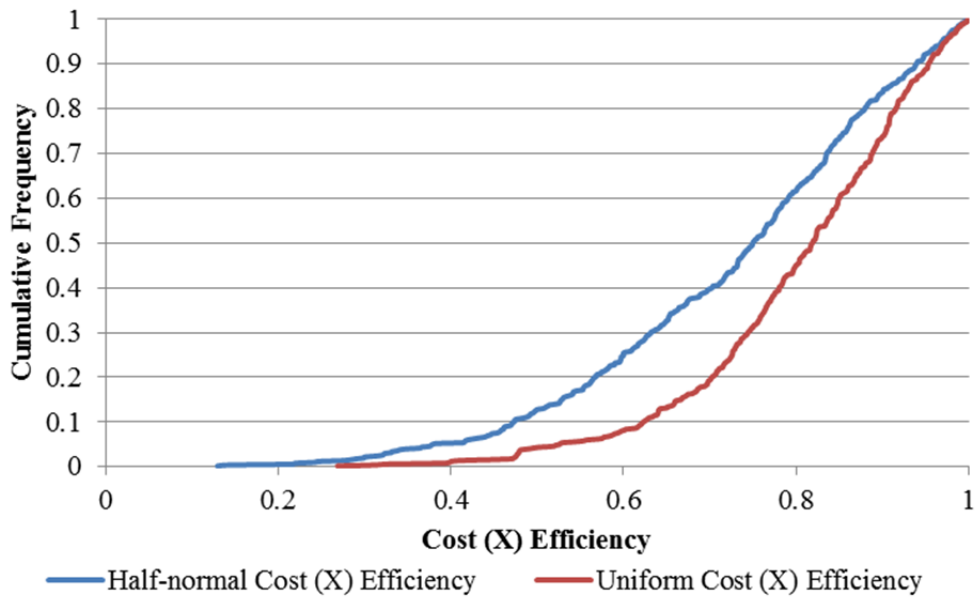


Figure 2 Frontier Cost Efficiencies Cumulative Frequency for both Half-normal and Uniform Distributions.

Note: The Economies of Scope calculations for both the half-normal and uniform error distribution is identical.

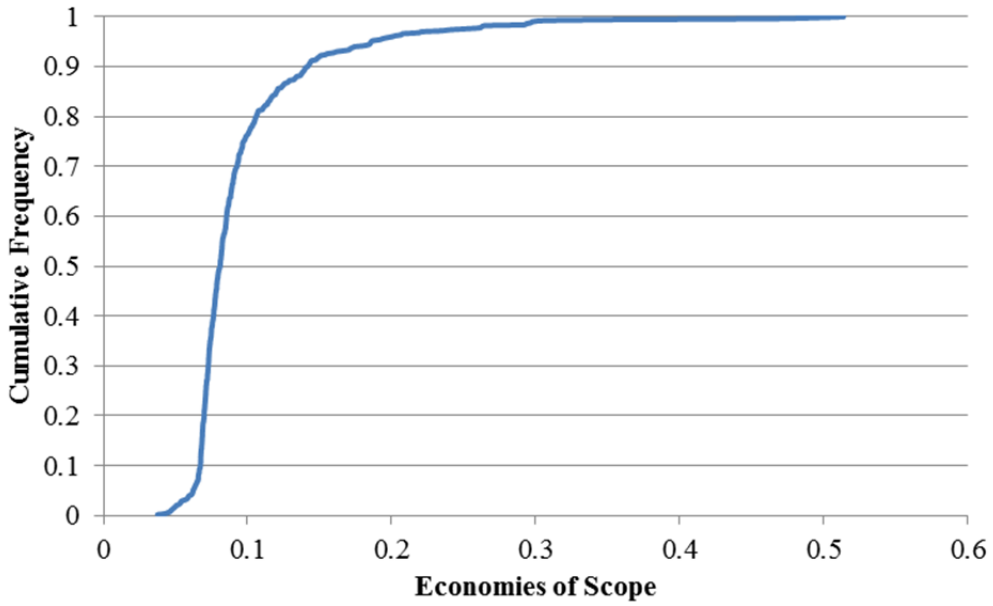


Figure 3 Frontier Economies of Scope Cumulative Frequency

Note: The PSE calculations for Y1 and Y2 for both the half-normal and uniform error distribution are identical

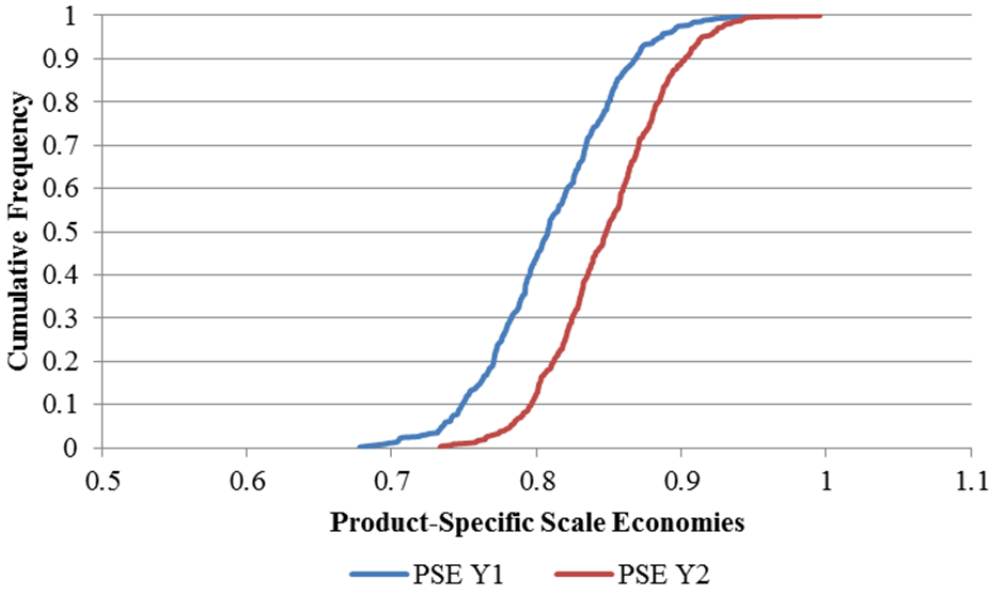


Figure 4 Frontier Product-Specific Scale Economies

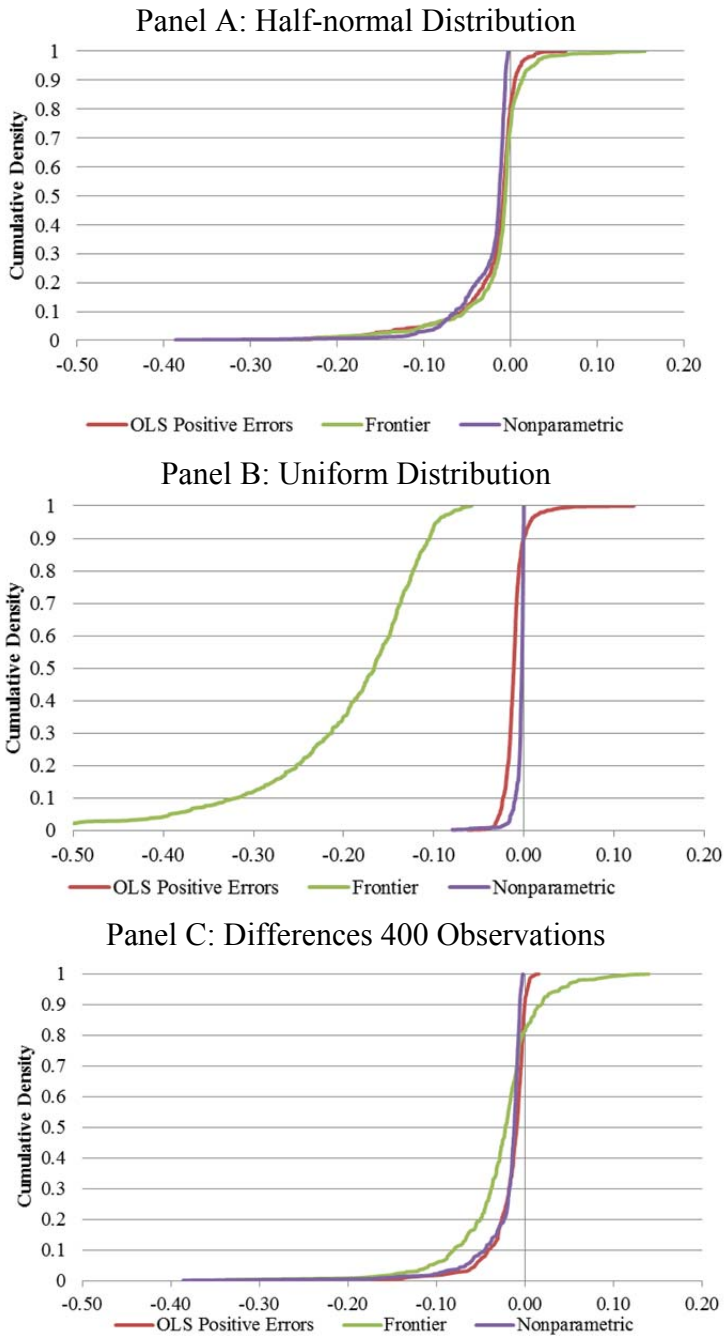


Figure 5 Differences between frontier cost efficiency and estimated cost efficiency for the nonparametric, frontier, and OLS positive errors models

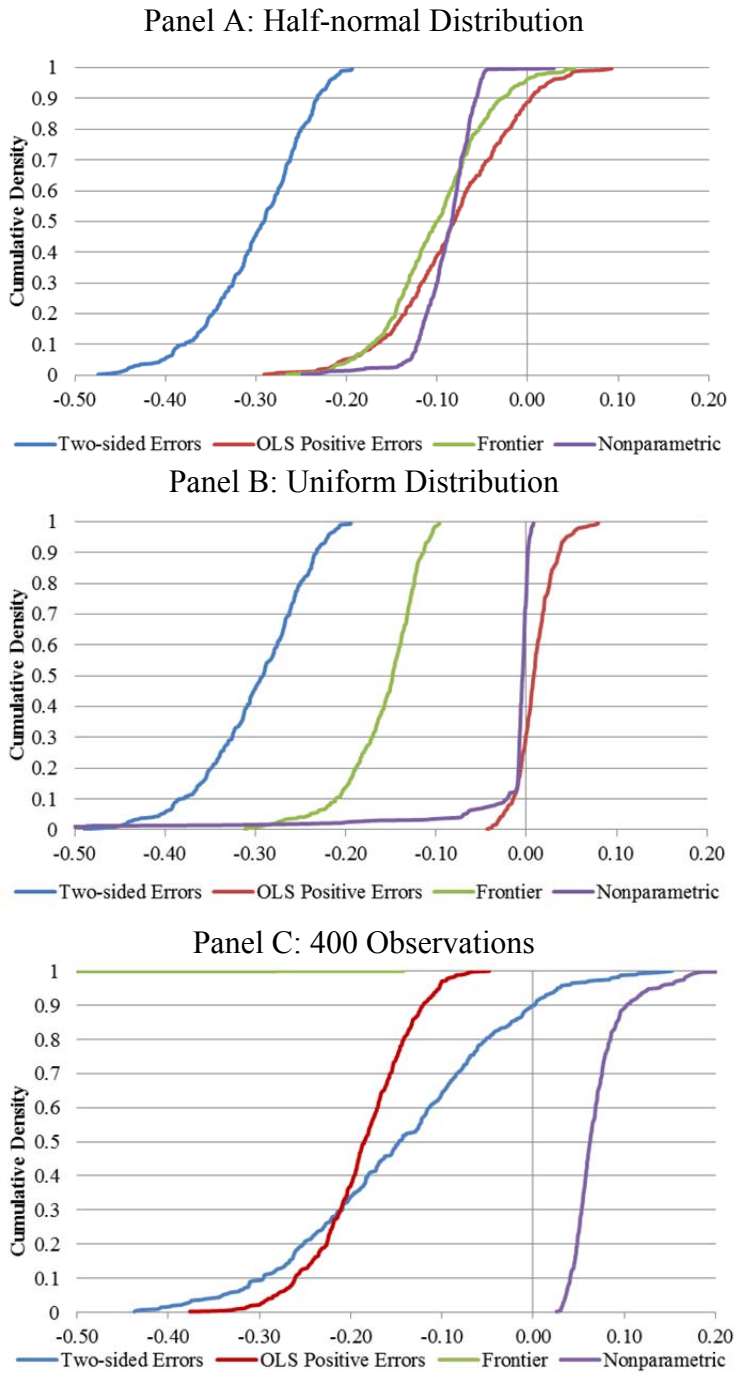


Figure 6 Differences between frontier Economies of Scope and estimated Economies of Scope from Two-sided Errors, OLS Positive Errors, Frontier, and Nonparametric models.

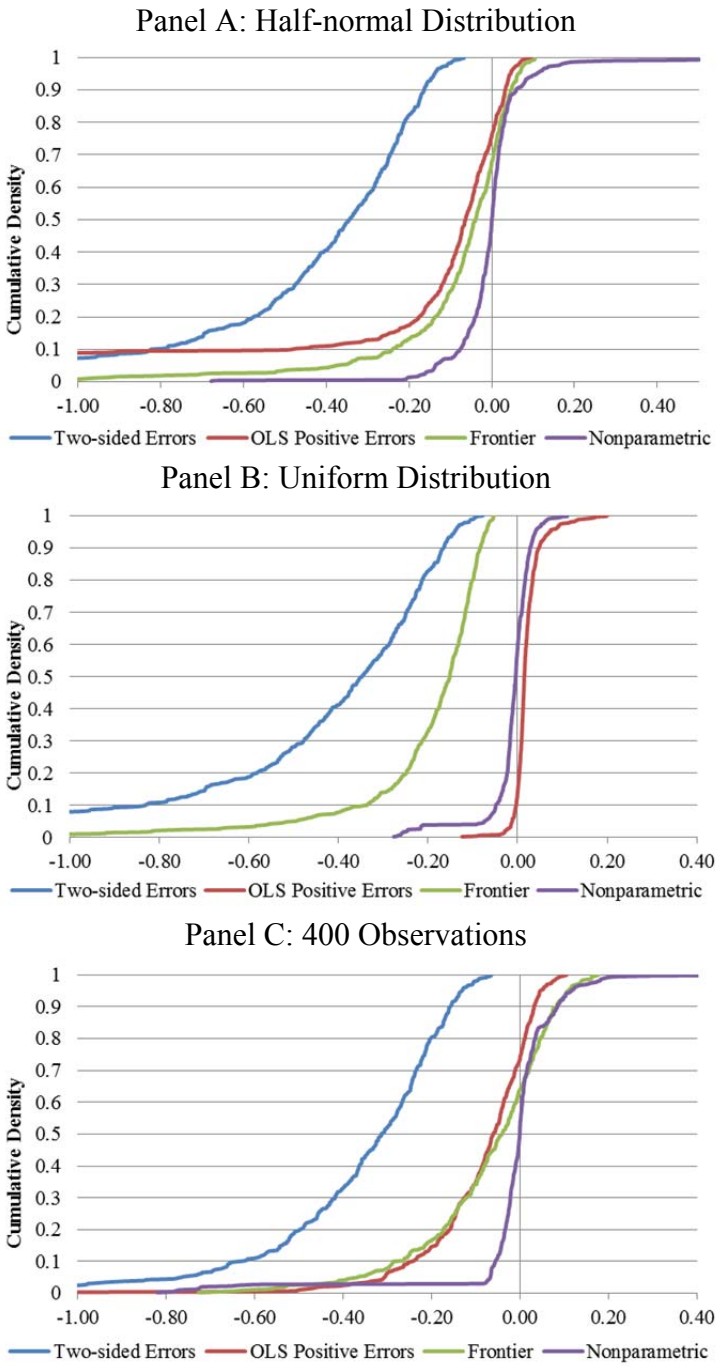


Figure 7 Differences between frontier Multi-product Scale Economies and estimated Multi-product Scale Economies from the Two-sided Errors, OLS Positive Errors, Frontier, and Nonparametric models.

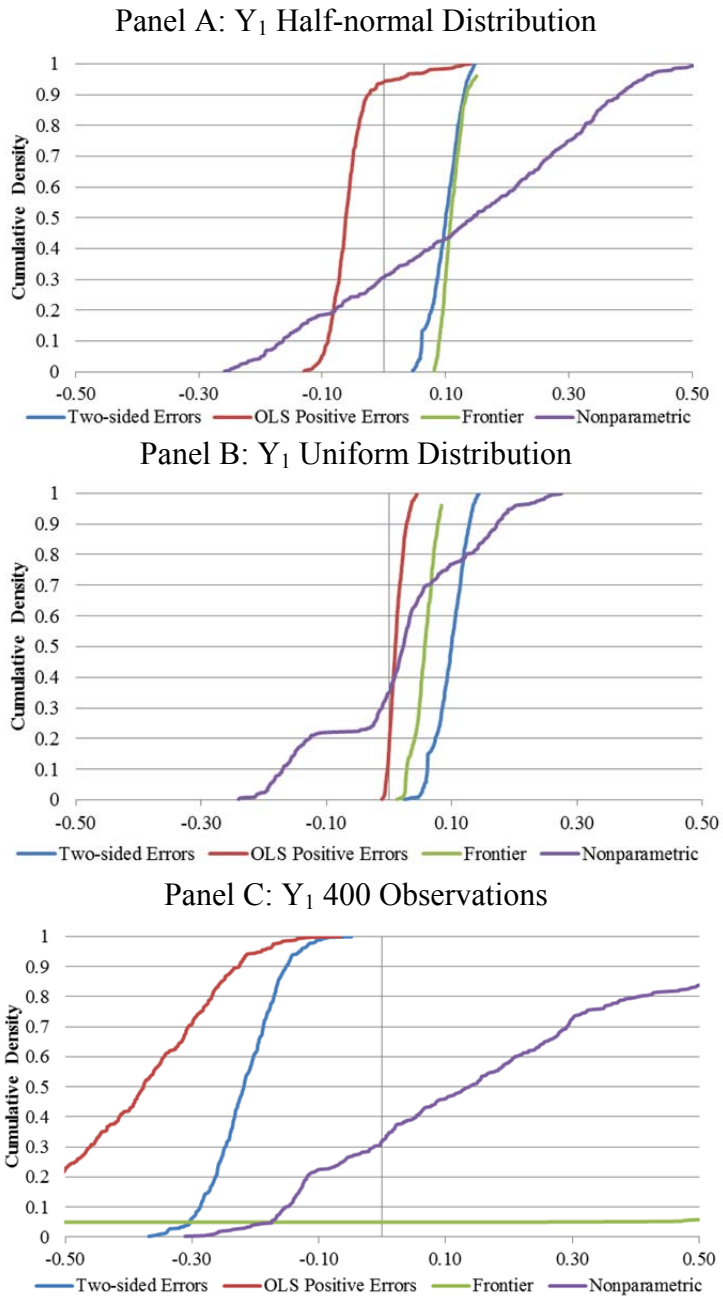


Figure 8 Differences between frontier Product-specific Scale Economies for Y_1 and estimated Product-specific Scale Economies for Y_1 from the Two-sided Errors, OLS Positive Errors, Frontier, and Nonparametric models.

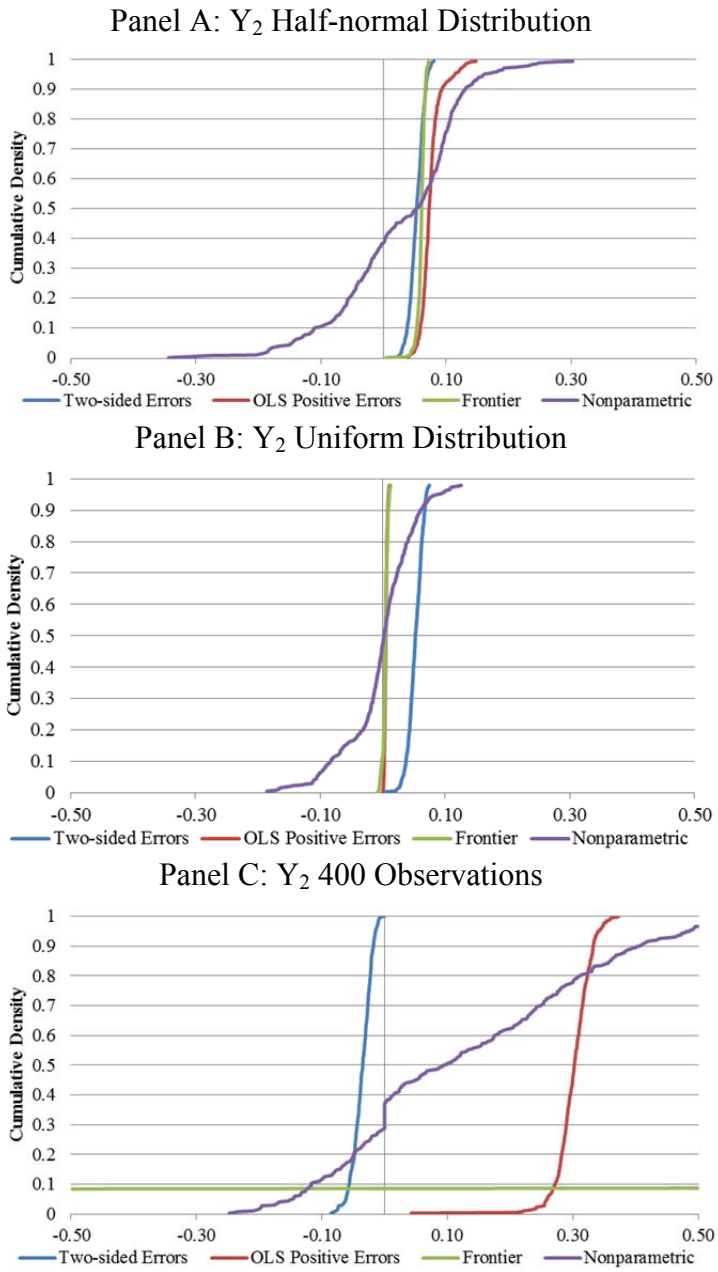


Figure 9 Differences between frontier Product-specific Scale Economies for Y_2 and estimated Product-specific Scale Economies for Y_2 from the Two-sided Errors, OLS Positive Errors, Frontier, and Nonparametric models.