Environmental Sustainability
with a Pollution Tax

by

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Abstract

This paper examines environmentally sustainable growth with reference to climate change assuming two final outputs and two factors of production, accounting for both pollution flow and stock effects. If the elasticity of marginal utility of consumption is greater than one, an optimal pollution tax ensures sustainable growth without any further government intervention. Otherwise, either a high temporal elasticity of substitution in production or consumption is required for sustainability. Even a suboptimal pollution tax may allow sustainable development provided the tax time profile meets certain conditions that are developed and described in this paper.

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1. Introduction

This paper examines the feasibility of environmentally sustainable economic growth in a dynamic general equilibrium framework of a closed economy with two final outputs and two factors of production. It explicitly accounts for both pollution flow effects and the existence of irreversible thresholds affecting the stock of renewable natural resources (i.e., the stock of clean air in the upper atmosphere). The paper highlights the important role played by two key facets of consumer preferences, namely the temporal substitution among final goods of diverse environmental impacts (represented by their elasticity of substitution) and the inter-temporal substitution of consumption (represented by the elasticity of marginal utility of consumption, *EMU*). If the *EMU* is greater than one, an optimal pollution tax ensures sustainable growth even if the elasticity of substitution in production between clean and dirty inputs and in consumption between clean and dirty consumer goods are well below one without requiring any further government intervention. If the *EMU* is less than one, sustainable growth is still feasible but requires much more demanding conditions: either temporal elasticity of substitution must be substantially greater than one.

This paper finds further that even a suboptimal pollution tax may allow sustainable development as long as the tax time profile meets certain plausible conditions that are developed below. Finally, numerical simulation results in section 8 demonstrates that if the pollution tax used as the sole policy instrument to prevent climatic disaster is well designed, it may only modestly affect the rate of economic growth.

The paper assumes that there exists a threshold level of the stock of the renewable natural resource which, if crossed, may drastically and irreversibly harm human health with the utility of the representative consumer falling to minus infinity (Cropper, 1976; Keller et al., 2004; Nævdal, 2006; Nævdal and Oppenheimer, 2007; Leizarowitz and Tsur, 2012). However, as long as such stock is above this threshold, human welfare is only affected by the flow levels of pollution emissions gradually. [1] The paper explores the properties of a pollution tax for sustainable

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[1] Consider the case of climate change gases; the emission flows consist not of one gas but of a cocktail of pollutants, including pollutants of mostly local effects (i.e., carbon monoxide), of local and global climatic effects (i.e., soot), and mainly global effects (i.e., carbon dioxide). The latter two pollutants accumulate in the upper atmosphere, thus affecting the stock of "clean air" over time. The effect of these flows is to cause health and other detrimental effects gradually over time while the stock accumulation effect is of little immediate effect as long as certain threshold stock levels are not surpassed. If the stock of CO2 in the atmosphere increased marginally from 250 parts per

development under the resource stock constraint and identifies a family of growth paths (including suboptimal paths as well as an optimal one) each of which guaranteeing environmentally sustainable economic growth.

The theoretical literature on growth and the environment over the last few decades has provided significant insights regarding the role of institutional and policy conditions in supporting environmentally sustainable economic growth (i.e., Bovenberg and Smulders, 1995; Bovenberg and de Mooij, 1997; Stokey, 1998; Brock and Taylor, 2010; Golosov et al., 2011; Acemoglu et al., 2012). Despite substantial progress in modeling, the existing growth theoretic literature still relies on certain restrictive models and assumptions that often fail to persuade many (especially environmentalists and ecologists) of the idea that persistent positive economic growth over the long run may eventually be consistent with an improving environment, thus preventing environmental catastrophe. The present paper is mainly inspired by and related to the landmark studies by Stokey (1998) and Acemoglu et al. (2012). It generalizes their findings in several respects by highlighting the role of a variety of features of consumer preferences and producer technologies, demonstrating that, contrary to the conclusion of most studies, elastic production and/or consumer choices are not necessary conditions for sustainable economic growth.

Most existing growth models assume one final good, which precludes the existence of an output composition effect, often considered important by empirical analyses (i.e., Grossman and Krueger, 1995; Cole and Elliot, 2003). A model with two final goods and two factors of production, as the one we developed below, may be considered isomorphic to existing models which assume one final good produced using two inputs one of which is a composite input in turn produced with another clean input and a dirty one (as in Acemoglu et al., 2012). However, this is not necessarily the case; a model that explicitly recognizes more than one final good where both endogenous savings and technological change are sources of economic growth, as the one developed below, brings to the forefront peculiarities of consumer conditions, in particular the role of the *EMU* vis-à-vis the temporal elasticity of substitution either among consumption goods or factors of production. We show that the relationship between *EMU* and the temporal elasticity of substitution in consumption or production plays a key role in sustainable

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million (*ppm*), there is little consequence for human life. However, if it surpasses, for example, 650 *ppm*, the potential catastrophic effects of the stock accumulation may be felt.

development, an insight lost in models that assume a single final good with technological change as the primary source of economic growth.[2]

Standard growth models that allow for savings as a source of growth often assume that the value of *EMU* is greater than one, an assumption that has been criticized by prominent authors on conceptual grounds (i.e., Aghion and Howitt, 1997; Ogaki and Reinhart 1998). Additionally, the empirical evidence regarding the size of *EMU* is mixed; some recent studies tend to contradict this assumption (i.e., Ogaki and Reinhart 1998; Vissing-Jørgensen and Attanasio, 2003; Layard et al., 2008). We thus relax this assumption and consider sustainable development alternatively considering levels of *EMU* above or below one.

Most existing models assume either unitary or highly elastic substitution between man-made and environmental factors of production (i.e., Stokey, 1998; Acemoglu et al., 2012).[3] This assumption has been challenged by environmentalists claiming that natural capital (i.e., the environment) and man-made capital are complements rather than substitutes (Daly, 1992). Moreover, to some degree, a number of empirical studies seem to support the claims made by environmentalists, concluding that factor input substitution is indeed substantially less than one (i.e., Field and Grebenstein, 1980; Kemfert and Welsch, 2000; van der Werf, 2008; Hassler et al., 2012).

Empirical studies report stronger substitution between clean and dirty consumer goods than among factors of production, often obtaining elasticity of substitution estimates well above 3 for consumer goods. Consequently, it appears that the scope for substitution between clean and dirty goods by consumers is greater than the substitution potential among inputs by producers, a feature that we explicitly consider in this study (i.e., Lin et al., 2008; Galarraga et al., 2011).[4]

Clean input-augmenting exogenous technological change is often assumed (i.e., Stokey, 1998; Brock and Taylor, 2010). However, recent studies have emphasized the endogenous nature of

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[2] See Baylis, Fullerton and Karney (2013) for the importance of considering at least two final goods and two productive inputs in examining the effects of unilateral carbon policy in a static equilibrium model.

[3] Recent growth theoretic studies do allow for factor input complementarities but in the context of non-renewable resources; their depletion is assumed to induce endogenous innovation (Bretschger and Smulders, 2012; Peretto, 2009). The focus on non-renewable resources, however, prevent consideration of the possibility of catastrophic and irreversible losses of renewable natural resources such as the atmospheric stock of clean air, a central focus of the present paper.

[4] Moreover, studies have shown that the consumers' flexibility with regards to clean goods is highly responsive to increased information and public education on the pollution content of the various consumer goods, as well as to eco-labeling (Kotchen and Moore, 2007). This is in sharp contrast with the reported lack of responsiveness to these interventions by manufacturing firms (Banerjee and Solomon, 2003).

technological change; for example, Acemoglu et al. (2012) allows for endogenous technological change, showing that targeted research subsidies may transform pollution-augmenting technological change into clean input-augmenting technological change as long as the elasticity of substitution between the clean and dirty inputs is much greater than one. Otherwise, targeted research subsidies are impotent to affect the structure of technological change. We consider exogenous technological change allowing alternatively for various types of it (neutral, pollution-augmenting and/or clean input-augmenting), an assumption that simplifies the analysis considerably. In view of the point made by Acemoglu et al. (2012) regarding the impotency of research subsidies when the elasticity of substitution between clean and dirty inputs is less than one, the assumption of exogenous technological change is innocuous, given that we focus mostly on cases where this elasticity is in fact less than one. Moreover, as Golosov et al. (2011) show, whether technological change is endogenous or exogenous is irrelevant in deriving an optimal disaster-avoiding pollution tax.

The standard neoclassical growth model of sustainable development has been criticized by environmentalists mainly on the grounds that man-made and natural capital are not likely to be strong substitutes in production, as assumed by most neoclassical growth models (i.e., Daly, 1992) and that there is excessive optimism regarding the role of technological change (i.e., Vollebergh and Kemfert, 2005).

The fact that we show that environmental sustainability accompanied with positive and persistent economic growth can be achieved in economies where natural and man-made capital have low elasticity of substitution, and that it may proceed under any type of technological change, constitutes an important response to the above critiques.

2. Framework of the analysis

The economy produces two goods: a clean good and a dirty one. The dirty good sector includes traditional manufacturing industries and primary industries that generate air and/or water pollution as a byproduct of their production processes. The clean good sector includes services and other goods that generate little or no pollution.

2.1. Production

Let k denote the total man-made composite input available at time t in the economy. This composite input includes human capital as well as other more tangible forms of capital.

Henceforth, we refer to k as "capital", which is momentarily distributed between the clean industry and the dirty industry. Let k_d denote the amount of capital employed in the dirty industry. The flow of pollution from the dirty sector is represented by x. Following Cropper and Oates (1992), López (1994), and Copeland and Taylor (2004), we consider pollution as a factor of production directly. The output of the dirty good is:

$$y_d = A_d F(k_d, bx). \tag{1}$$

The parameter A_d denotes total factor productivity with proportional growth rate, $\dot{A}_d / A_d \equiv g_d \geq 0$ and $b > 1$ is a factor-augmenting technological factor with $\dot{b}/b \equiv \zeta \geq 0$.

The dirty sector produces only a final consumer good. F is a Constant Elasticity of Substitution (CES) function, and it is given as follows:

$$F(k_d, bx) = \left[\alpha k_d^{-\frac{1-\omega}{\omega}} + (1-\alpha)(bx)^{-\frac{1-\omega}{\omega}} \right]^{-\frac{\omega}{1-\omega}},$$

where ω is the elasticity of substitution between capital and pollution and α is a fixed distribution coefficient.

The output of the clean good is assumed to depend only on the capital input and is governed by the linear production technology, as follows:

$$y_c = A_c(k - k_d).$$

where the parameter A_C is the return to capital in the clean sector and k is the total stock of capital in the economy at a point in time. The clean sector produces a final consumer good as well as new capital (or investment). Mostly for the sake of reducing notational clutter, we focus primarily on pollution-augmenting and neutral technological change. Later in the paper, however, we show that the results remain mostly unchanged by considering capital-augmenting technological change.

We consider two sources of economic growth, technological change and capital accumulation. We specify the various types of technological change below. Here we focus on capital accumulation using the budget constraint of the economy. If we normalize the price of the clean good to unity (i.e., $p_c = 1$), the economy's budget constraint can be written as:

$$\dot{k} = A_c(k - k_d) + pA_d F(k_d, bx) - c - \delta k, \tag{3}$$

where $p \equiv p_d / p_c$ is the relative price of dirty goods, $c \equiv c_c + pc_d$ is the total consumption expenditure expressed in units of the clean good, δ is the rate of capital depreciation, and $\dot{k} \equiv dk / dt$ is the net capital accumulation. The sum of the first two terms on the right-hand side of Equation (3) represents the income of the economy expressed in units of clean goods. The gross capital accumulation, $\dot{k} + \delta k$, is equal to net savings (income less consumption), which is also expressed in units of the clean good.[5]

2.2. Stock of clean air

Economic activity releases pollution flows into the atmosphere. A portion of the pollution emissions are removed by nature's revitalization processes but another portion of them remains as a stock that accumulates in the upper atmosphere. Pollution emissions (whether they accumulate in the atmosphere or rapidly dissipate) have instantaneous direct negative effects on welfare. In addition, the fact that a portion of the emissions accumulates in the upper atmosphere causes very gradual and subtle changes in climate, which may have negligible direct effects on welfare unless such accumulations reach a threshold level at which point catastrophic events may be triggered, causing massive welfare losses.

Thus, pollution reduces the stock of clean air, so that the changes in the stock of clean air are the net result of two forces, the natural purification rate of pollution and the flow emission of pollution. Following most of the literature we assume a constant rate of environmental regeneration (i.e., Aghion and Howitt, 1997; Acemoglu et al., 2012). Denote the stock of clean air in the upper atmosphere as E, the threshold of minimal stock of clean air below which an environmental catastrophe occurs as $\underline{E}$, the pristine stock level by $\overline{E}$, and let $0 < \psi < 1$ be the constant rate of natural atmospheric purification. Then we have:

$$\dot{E} = \psi E - x \quad \text{for} \quad \underline{E} \le E < \overline{E}. \tag{4}$$

$$= -x \qquad \text{for} \quad E < \underline{E}.$$

For future reference we note that by integrating (4) within the specified boundaries we obtain:

$$E(t) = \exp(\psi t)\left(E_0 - \int_0^t x(\upsilon)\exp(-\psi\upsilon)d\upsilon \right) \tag{4'}$$

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[5] We assume that the investment in capital is irreversible. Once the economy builds capital, it cannot be transformed back into consumption goods; capital can be reduced over time only by allowing it to depreciate.

for $E(t) \geq \underline{E}$; E_0 is the initial, predetermined level of the stock of clean air.

2.3. Consumption and welfare

The welfare function of the representative consumer is comprised of two parts, a utility derived from the consumption of goods and the disutility generated by pollution. We represent the utility derived from the consumption of goods by an indirect utility function as follows:

$$u = \frac{1}{1-a}\left(\frac{c}{e(1,p)}\right)^{1-a},$$

where c denotes the total consumption expenditure, $e(1,p)$ is the unit (dual) expenditure function or cost-of-living index, and $a > 0$ is a parameter equal to EMU. If $a < 1$, we adopt a positive utility scale such that $0 < u < \infty$, while we scale the utility index to $-\infty < u < 0$ when $a > 1$. Of course, a special case of the above specification occurs when $a = 1$, in which case we obtain the often-used logarithmic specification, $u = \ln[c / e(1,p)]$. The indirect utility function is assumed to be increasing and strictly concave in the real consumption level, $c / e(1,p)$.

We assume that the consumer's underlying preferences for goods are described by a CES utility function, so that the unit expenditure function is:

$$e(1,p) = \left[\gamma_c + \gamma_d p^{1-\sigma}\right]^{\frac{1}{1-\sigma}},$$

where σ is the consumption elasticity of substitution between the dirty and clean goods, and $\gamma_c > 0$ and $\gamma_d > 0$ are fixed parameters. The indirect utility function defined above presumes homothetic preferences. Consumer demand for the clean good c_c and dirty good c_d can be retrieved from the indirect utility function using Roy's identity. The optimal level of c is determined by the inter-temporal optimization (as detailed below).

The second part of the welfare function corresponds to the disutility generated by pollution. Let $v(x;E)$ denote the environmental damage function, which is assumed to be increasing and convex in the level of pollution, x. We assume that the environmental damage function is:

$$v(x;E) = \frac{x^{1+\eta}}{1+\eta} \quad \text{if } E \geq \underline{E},$$

$$= \infty \quad \text{if } E < \underline{E}.$$

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Also, $\eta > 0$ denotes the elasticity of marginal damage caused by pollution and is assumed to be a fixed parameter. Therefore, the consumer's total welfare function is:

$$U(c,x;E) \equiv \frac{1}{1-a}\left(\frac{c}{e(1,p)}\right)^{1-a} - \frac{x^{1+\eta}}{1+\eta} \text{ when } E \geq \underline{E}$$

$$\equiv -\infty \qquad\qquad\qquad \text{when } E < \underline{E} .$$

Assuming a fixed pure time discount rate (ρ) and socially optimal intervention, the competitive economy is modeled "as if" it maximizes the present discounted value of the utility function:

$$\int_0^\infty U(c,x;E)\exp(-\rho t)dt,$$

subject to the budget constraint (i.e., Equation (3)), clean air stock level constraint $E \geq \underline{E}$ (Equation (4)) and the initial conditions $k = k_0$ and $E = E_0$. In other words, the competitive behavior of the representative consumer and producer under optimal pollution tax and lump-sum reimbursement is described by the choices of the optimal levels of c and x at each point in time.

We assume that both goods are always produced, which implies that $k_d(t) < k(t)$ for all t. Thus, the current value Hamiltonian function assuming an interior solution is:

$$H_E = U(c,x,E) + \lambda\left[A_C(k-k_d) + pAF(k_d,bx) - c - \delta k\right] + \mu\left[\psi E - x\right] + \phi\left[E - \underline{E}\right]$$

where λ and μ denote co-state variables each representing the shadow price of man-made capital and natural capital, respectively while $\phi \geq 0$ is a time-varying Lagrange multiplier associated with the stock constraint.

2.4. Analytical strategy

We assume that the economy maximizes H_E subject to the market equilibrium conditions for the final goods to be introduced later in the next section. This means that in addition to the usual endogenous variables of the optimal control problem we need to solve for the endogenous market prices. Using the system of necessary conditions for dynamic optimization (Maximum principle and Kuhn-Tucker conditions) and the said market clearing conditions, we may in principle solve for seven endogenous variables $(c,k_d,x,p,\lambda,\mu,\phi)$ at each point in time. While the analysis of the original problem is extremely complex given the fact that the utility function

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is discontinuous at $E = \underline{E}$, the dynamic optimization process can be examined in a more tractable way if the shadow price of the stock of pollutant, ϕ, is zero (that is, if the stock constraint is not binding).

We therefore use the following strategy: First, we solve the model of dynamic optimization and market equilibrium using as a maintained assumption that $\phi = 0$, that is, that the stock of clean air remains above $\underline{E}$ throughout all time. Next, we analyze the conditions under which, given the solution derived from the first step, the constraint $E(t) \geq \underline{E}$ is satisfied for all t given initial stock levels of the natural and man-made assets, E_0 and K_0. Thus, the first part of the solution is obtained by maximizing H_E (subject to the relevant market clearing conditions) with $\phi = 0$ and the second part examines whether or not this solution satisfies the stock constraint.

Under our stated assumptions on preferences and production technology, H_E is strictly concave with respect to state and control variables, and the necessary conditions become sufficient. In fact there exists a unique solution for the optimal control problem.[6] In the subsequent sections, we also characterize the conditions for the clean air stock to remain above the threshold level. If the optimal path of emissions obtained by maximizing H_E does not permit the stock of clean air to fall below the critical threshold at any point in time then it constitutes an optimal solution for the original problem of dynamic market equilibrium with stock constraint.

It is now necessary to define what we mean by "sustainable economic growth".

Definition 1: *We say that sustainable growth is possible if, at some point along the growth process, the economy is able to continue growing indefinitely while pollution emissions permanently decline and the stock of natural capital never falls below the critical threshold level.* Therefore, sustainability requires that there exists a finite time, $T \geq 0$, such that at any time $t > T$, $\hat{x} < 0$, which implies that $\lim_{t \to \infty} \hat{x} \leq 0$, and that $E(t) \geq \underline{E}$ for all t .[7]

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[6] We note also that the Inada condition is satisfied. In other words, for any $a > 0$ our utility scale guarantees that $\lim_{c \to 0} U_x(c, x, E) = \infty$ for any finite x and $\underline{E} \leq E$.

[7] A similar notion has been adopted by several authors, including Stokey (1998) and Brock and Taylor (2010). This concept of sustainable growth conforms to the concept of sustainable development in Arrow et al. (2010).

2.5. Additional considerations

Here we establish some basic properties of the consumption and factor shares which are essential for the ensuing analysis. The budget share of the dirty final good in the consumption expenditure for the CES utility function is $s(p) = \dfrac{\gamma_d}{\gamma_c p^{\sigma-1} + \gamma_d}$ and the factor share of the clean input in the cost of production of the dirty good for a CES production function is

$$S_k(k_d / bx) = \alpha \left[(1-\alpha) \left(\frac{k_d}{bx} \right)^{\frac{1-\omega}{\omega}} + \alpha \right]^{-1}.$$ Of course, the share of the dirty input in the cost of production of the dirty final good is $1 - S_k$. Then we have the following remark:

Remark 1: *The share $s(p)$ is an increasing (decreasing) function of p if $\sigma < 1(\sigma > 1)$. The share $S_k(k_d / bx)$ is increasing (decreasing) in k_d / bx if $\omega > 1(\omega < 1)$.*

Remark 1 is important for subsequent analysis because it allows us to predict the evolution of $s(p)$ and $S_k(k_d / bx)$ over time if we know the dynamics of p and k_d / bx, on the basis of the size of the elasticity of substitution. As shown below the dynamics of these shares are key factors determining the sustainability (or lack of sustainability) of the economy.

2.6. Assumptions

We make the following assumptions:

Assumption 1: *The clean sector of the economy is sufficiently productive so that the marginal return to capital (A_c) is higher than the marginal opportunity cost of capital ($\rho + \delta$); hence, $M \equiv A_c - \rho - \delta > 0$.*

Assumption 2: *Technological change can be pollution-augmenting occurring at an exogenous rate $\zeta \geq 0$ and/or neutral, raising the total factor productivity of the dirty sector at an exogenous rate $g_d \geq 0$. However, the rate of technological change is bounded from above as follows: $\zeta + g_d \leq min\{M, M / a\}$.*

Assumption 1 is a necessary condition for the economy to be able to accumulate capital over time. Meanwhile, Assumption 2 implies that all exogenous technological changes are concentrated in the dirty industry. The assumption of dirty input (pollution)-augmenting

technological change in the context of endogenous technological change is consistent with the so-called laissez-faire or market solution arising when the government does not intervene to subsidize research and development to increase the productivity of the clean inputs (i.e., Acemoglu et al., 2012). In section 6, we relax this assumption by also allowing for capital-augmenting technological change.

Assumption 2 also places a limit on the speed of technological progress. As we shall show below, this limit is necessary for technical reasons. It assures that the net effect of the two primary sources of growth, namely capital accumulation and technological change, is pollution-increasing while the technique and composition effects are pollution-reducing. If this assumption is not satisfied then we would obtain that the direct effect of economic growth (i.e., the factor accumulation-cum-technological change effect) would be pollution-reducing while the technique and composition effects would be pollution-increasing. This baffling condition would in fact render the analysis of sustainable development meaningless. If the direct effect of economic growth were to lower pollution then we would have sustainable development even in the absence of a pollution tax and, hence, in the absence of technique and composition effects.

3. Optimality and market clearing conditions

3.1. Optimality conditions

The first-order necessary conditions for maximization of the Hamiltonian function imply that the marginal utility of consumption must be equal to the shadow price of capital, λ :

$$e(1,p)^{a-1}c^{-a} = \lambda. \tag{5}$$

Meanwhile, along the optimal path the well-known no arbitrage condition must be satisfied:

$$\frac{\dot{\lambda}}{\lambda} = -\left[A_c - \rho - \delta\right] \equiv -M. \tag{6}$$

There are two additional conditions for optimality as follows: first, the marginal value product of capital should be equal across the two sectors; second, firms equalize the marginal value product of pollution to the optimal pollution tax. Therefore, assuming an interior solution, we have:

$$pA_d \frac{\partial F(k_d, bx)}{\partial k_d} - A_c = 0, \tag{7}$$

$$pA_d \frac{\partial F(k_d, bx)}{\partial x} - v'(x)/\lambda = 0 \ . \tag{8}$$

Equation (7) indicates that in equilibrium the marginal value product of capital should be equalized across the two sectors. Equation (8) says that the optimal pollution tax, which is equal to the marginal rate of substitution between pollution and consumption, $\tau \equiv v'(x)/\lambda$, is equalized to the marginal value product of pollution. Finally, the savings should be equal to the net investment at each moment of time, so that we have Equation (3) as an additional first order condition. Moreover, we have the standard transversality condition, $\lim_{t \to \infty} \lambda k(t) e^{-\rho t} = 0$.

3.2. Market clearing conditions

In Appendix we show that the rate of growth of the consumer demand for dirty goods is:

$$\hat{c}_d = \frac{1}{a} M - \left[\frac{s(p)}{a} + (1 - s(p))\sigma \right] \hat{p}. \tag{9}$$

A circumflex above the symbol reflects its corresponding rate of growth. In addition, the rate of growth of production of the dirty goods is:

$$\hat{y}_d = g_d + \hat{F}(k_d, bx) = g_d + S_k \left(\widehat{\frac{k_d}{bx}} \right) + \left(\widehat{bx} \right). \tag{10}$$

Because the dirty goods are used for consumption only, market equilibrium requires that $y_d = c_d$ at all points in time. Furthermore, once the dirty goods market is cleared, the market for the clean goods automatically clears because the current savings are equal to the current investment, as stipulated in Equation (3). Therefore, the relative price of dirty goods must adjust endogenously over time to allow for such equilibrium to persist. Along the equilibrium path, the growth rate of production and demand for the dirty good must be equal, so that $\hat{y}_d = \hat{c}_d$.

4. Dynamic equilibrium

4.1. The conditions

Using Equation (9) and Equation (10), we obtain:

$$z\hat{p} + S_k \left(\widehat{\frac{k_d}{bx}} \right) + \hat{x} = \frac{M}{a} - g_d - \zeta \tag{11}$$

where $z \equiv s(p)\dfrac{1}{a} + (1 - s(p))\sigma > 0$ (also recall that $\zeta \equiv \dot{b}/b$ and $g_d \equiv \dot{A}_d / A_d$). The function z corresponds to the weighted average of the inter-temporal elasticity of substitution ($1/a$) and the temporal elasticity of substitution, using the budget shares as weighting factors.

From Equation (7), we have $\hat{p} + \hat{A}_D + \hat{F}_1(k_d, bx) = 0$, which given the CES production function implies that:

$$\hat{p} - \frac{1}{\omega}(1 - S_K)\left(\widehat{\frac{k_d}{bx}}\right) = -g_d. \tag{12}$$

Finally, in Appendix we show that using Equation (8) the following expression follows:

$$\hat{p} + \frac{1}{\omega}S_K\left(\widehat{\frac{k_d}{bx}}\right) - \eta\hat{x} = M - g_d - \zeta. \tag{13}$$

This states that the rate of increase of the private marginal revenue of the dirty input,

$\hat{p} + \dfrac{1}{\omega}S_K\left(\widehat{\dfrac{k_d}{bx}}\right) + \zeta + g_D$, is equal to the rate of increase of the input price, which in turn equals

rate of increase of the pollution tax, $\hat{\tau} = \eta\hat{x} + M$.

4.2. Solution of the dynamical system

In Appendix, we show that the dynamical system of Equations (11), (12), and (13) solves for

the equilibrium growth rates of $\hat{p}$, $\left(\widehat{\dfrac{k_d}{bx}}\right)$ and $\hat{x}$ as follows:

$$\hat{p} = \frac{1}{|W|\omega}\left(\left[M(1 - S_k)\left(\frac{\eta}{a} + 1\right)\right] - g_d\left[(1 - S_k)(\eta + 1) + \omega S_k\left(\eta + \frac{1}{\omega}\right)\right] - \zeta\left[(1 - S_k)(\eta + 1)\right]\right), \tag{14}$$

$$\left(\widehat{\frac{k_d}{bx}}\right) = \frac{1}{|W|}\left[M\left(\frac{\eta}{a} + 1\right) + g_d\eta(z - 1) - \zeta(\eta + 1)\right] > 0, \tag{15}$$

$$\hat{x} = \frac{1}{|W|\omega}\left\{M\left(\frac{1}{a} - z(1 - S_k) - \omega S_k\right) + g_d(z - 1) + \zeta\left(z(1 - S_k) + \omega S_k - 1\right)\right\}, \tag{16}$$

where $|W| \equiv \dfrac{1}{\omega}\left[(1 - S_k)(1 + z\eta) + S_k\right] + \eta S_k > 0$.

Using Equation (16) we can decompose the dynamics of pollution flows into four partial effects, as follows:

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$$\hat{x} = \frac{1}{\omega |W|}[\varepsilon_k + \varepsilon_t + \varepsilon_s + \varepsilon_c],\tag{16'}$$

where $\varepsilon_k \equiv M/a > 0$ is the *pure capital increasing effect*; $\varepsilon_t \equiv -(\zeta + g_D) < 0$ is the *pure technological change effect*; $\varepsilon_s \equiv -\omega S_k[M - \zeta] < 0$ is the *technique effect*; and $\varepsilon_c \equiv z\{S_K g_D - (1 - S_k)[M - \zeta - g_D]\}$ is the *output composition effect*.

The pure capital effect and technological change effect constitute the two primary sources of economic growth. Meanwhile, the technique and output composition effects are dependent on the primary sources of growth. The pure capital scale effect, *ceteris paribus*, increases pollution while the pure technological change effect reduces pollution because it reflects the fact that the effective dirty input may rise over time without necessarily increasing pollution. Assumption 2 guarantees that the net direct effect of economic growth, $\varepsilon_k + \varepsilon_t$, is pollution-increasing.

Expanding income due to the two primary sources induces an increase of the pollution tax due to the fact that the marginal utility of consumption, λ, falls as $M > 0$. This means that the relative price of the dirty input (pollution) increases over time which, in turn, triggers a technique or input substitution effect that has a pollution-reducing effect. The tax increase also causes an output composition effect by raising the cost of production and, hence the relative price, of the dirty good which in turn induces consumers to substitute consumption of dirty goods with clean goods and, hence, reduce pollution.

Pollution-augmenting technological change weakens both the technique and composition effects. Assumption 2 assures that although technological change only partially mitigates these effects, it cannot reverse them. The increase of the productivity of pollution due to technological change counters the effect of the increased pollution tax because the relative price of effective pollution increases less, causing the incentives to substitute pollution with clean inputs to weaken. Similarly, the increased productivity associated with technological change attenuates the cost increase of the dirty goods caused by the pollution tax. This, in turn, reduces the price increase of the dirty goods and, hence, weakens the consumers' incentives to substitute dirty goods with clean ones.

4.3. The optimal pollution tax dynamics

Finally, we derive the dynamics of the optimal pollution tax that is consistent with the system (14) to (16). Noting that $\tau = v'(x)/\lambda$ we have that $\hat{\tau} = \eta\hat{x} + M$. Therefore, using Equations (8), (13) and (15) we can derive the rate of change of the pollution tax over time:

$$\hat{\tau} = \frac{\eta}{|W|\omega}\left(\left(\frac{1}{a} + \frac{1}{\eta}\right)M - (\zeta + g_d) + zg_d + ((1 - S_k)z + \omega S_k)\zeta\right) > 0.$$

By Assumption 2, $M/a \geq \zeta + g_d$, which means that the pollution tax increases continuously along the optimal path. While the tax increases over time, the share of the pollution tax costs on the total value of consumption, $\tau x/c$, may eventually decline along the optimal path.

4.4. Suboptimal pollution paths

The fact that we can obtain an explicit and tractable solution for the optimal rates of change of pollution and the other relevant variables show that, with enough information regarding the key parameters considered, this part of the solution is relatively easy to obtain for a government or planner. But this is, of course, not a complete solution; in order to obtain a complete solution we need to solve for the *initial* values of the endogenous variables (p, k_d/bx, x and, therefore, τ) in addition to their optimal rates of change as provided by (14) to (16). In fact, determining such initial values is extremely complex, not only for analysts but also for governments. Fortunately, as can be seen though an inspection of equations (14) to (16), the optimal rates of change of the variables are *not* dependent on the initial values of such variables.

This characteristic of the dynamical solution is very important because, as we shall see below, it allows us to determine the maximal critical initial level of pollution that assures that the stock of clean air will never fall below the catastrophic threshold. An imperfect government that is unable to ascertain the optimal initial values of the endogenous variables could still determine such a critical level and its job would be reduced to ensuring that the initial pollution level is below the critical point and from then on follows the myopic growth rule dictated by equation (16). The result would be a suboptimal rule, implying higher pollution levels than the optimum at all points in time, but one that assures sustainable and positive economic growth thus preventing environmental disaster. section 6 deals with these issues.

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5. Economic growth

An important issue is whether the dynamic path described by Equations (14) to (16) implies a positive rate of consumption growth despite that the pollution tax is continuously increasing. The following proposition shows that this is indeed the case:

Proposition 1: (i) *The growth rate of real consumption expenditure is:* $\left(\dfrac{\hat{c}}{e}\right)=\dfrac{1}{a}\left[M-s(p)\hat{p}\right],$ *where $\hat{p}$ is given by (14).*(ii) *The rate of growth of real consumption remains positive throughout the equilibrium dynamic path for any positive ω and σ.*(iii) *If either input substitution or consumption substitution is elastic (if $\omega>1$ or $\sigma>1$), but not both, the rate of growth of real consumption converges from below towards a rate M/a. If both $\omega>1$ and $\sigma>1$, then the growth rate of real consumption converges to $(1/a)(M+g_d)$.*(iv) *If $\omega<1$ and $\sigma<1$, then the rate of growth of real consumption converges from above towards a rate $\left((1+\eta)/(a+\eta)\right)(\zeta+g_d)<M/a$.*

Proof: See Appendix.

Proposition 1 demonstrates that the dynamic equilibrium path described by Equations (14) to (16) is associated with a positive rate of growth of real consumption regardless of the size of the elasticity of substitution. However, the economy's growth rate is below its potential as a consequence of the fact that the optimal pollution tax forces the relative price of dirty goods to continuously increase over time. This, in turn, increases the cost of living for consumers, implying that economic growth must be partially sacrificed. However, as shown in Remark 1, if $\sigma>1$, the share of the dirty goods in the consumption bundle declines, and if $\omega>1$, the share of the clean input in production increases. In either of these cases the sacrifice of the growth rate vis-à-vis its potential level becomes progressively smaller beyond a certain point in time. That is, the growth rate of the economy approaches in the long run its maximum potential rate, which in this case is equal to M/a in the absence of neutral technological progress in the dirty sector.

The fact that when $\sigma>1$ or $\omega>1$ the convergence (or long run rate of growth) of the economy is not affected by the rate of pollution-augmenting technological change might seem surprising. The reason for this fact is that, in this case, the consumer budget share of pollution

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and/or the share of pollution in the cost of production approaches zero.[8] That is, pollution-augmenting technological change becomes irrelevant for economic growth over the long run because the share of the dirty input in the production of the dirty goods and/or the share of dirty final goods constitute a negligible fraction of the economy.

Furthermore, from Remark 1 it follows that if $\omega < 1$ and $\sigma < 1$, the share of the dirty input (pollution) in the cost of production increases over time and the share of dirty goods in the consumer budget increases over time, both converging to 1. Therefore, in such a case the technological change becomes the key determinant of the convergence rate of economic growth. Conversely, because the share of the clean goods approaches zero, the capacity of the economy to expand such goods becomes increasingly irrelevant for economic growth. This means that in the inelastic case the economy's growth rate declines and becomes increasingly dependent on the rate of technological change and less dependent on the rate of capital accumulation as the shares of the dirty input and dirty final output increase over time. Moreover, Assumption 2 implies that the growth rate of the economy converges to a lower level than in the elastic case.

The following corollary to Proposition 1 summarizes the results discussed in the previous two paragraphs:

Corollary 1: *Economies characterized by elastic producer and/or consumer choices tend to grow more rapidly and converge towards higher secular growth rates than economies exhibiting inelastic producer and consumer choices.*

6. Conditions for sustainable growth (assuming that $\phi = 0$)

We first consider the case when *EMU* is greater than one, as assumed by standard sustainable growth models. In this study we will also consider the case when *EMU* is less than one, in light of the fact that some recent studies have shown that the *EMU* may reach levels below one, contrary to what has previously been assumed to be the case (i.e., Attanasio and Browning, 1995; Vissing-Jørgensen, 2002). Although the analysis is conducted under the assumption of pollution-augmenting and neutral technological progress in the dirty sector, the results hold under the more

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[8] This is true if $\sigma > 1$ but $\omega < 1$ because in this case the consumption share of the dirty goods approaches zero and hence the participation of the dirty goods in the economy becomes negligible in the very long run. Furthermore, if $\sigma < 1$ but $\omega > 1$ the share of pollution in production of the dirty goods approaches zero, meaning that in the very long run the participation of pollution as an input becomes negligible.

general assumptions on technological changes, including capital-augmenting technological progress in the dirty sector. This is shown in Appendix.

6.1. The case when EMU is greater than one

A consequence of allowing $a > 1$ is that the rate of economic growth is slower than in a case where $a < 1$. In other words, the scale effect is less powerful and, hence, *ceteris paribus*, pollution emissions will tend to grow more slowly as the economy grows. This makes the conditions for sustainability much weaker than in a case where $a < 1$. From Proposition 1 and Equation (16), the following proposition emerges:

Proposition 2: *Suppose $a > 1$, technological change is either pollution-augmenting and/or neutral or non-existent, and that Assumptions 1 and 2 hold. Then, if either σ and/or ω is positive, an optimal pollution tax is sufficient to induce sustainable development.*

Proof: See Appendix.

Therefore, the conditions for sustainable development are extraordinarily weak in the case where $a > 1$. In this case, a society's willingness to pay for a marginal reduction of pollution increases rapidly with income. The growth effect then becomes relatively weak vis-à-vis the case where $a < 1$. Even when both consumption and input elasticity of substitution are less than one, sustainable development arises.

The intuition of this important result is as follows: assuming that $\sigma < 1$ and $\omega < 1$, and using Equation (8) (noting that $s \to 1$ in the long run) and the expression for $\left(\dfrac{\hat{c}}{e}\right)$ in Proposition 1, the secular or long run rate of growth of real consumption is found to be equal to the growth rate of dirty consumer goods. The rationale for this result is that in the long run, the clean consumption goods become a negligible fraction of total consumption and, hence, the rate of growth of total consumption is given by the rate of growth of the dirty consumption goods only. This, in turn, implies that the rate of long run growth of the dirty output is also equal to the long run growth rate of real consumption. Therefore, using part *(iii)* of Proposition 1, it follows that:

$$\left(\frac{\hat{c}}{e}\right)^{\infty} = \hat{c}_d^{\infty} = \hat{y}_d^{\infty} = \frac{1+\eta}{a+\eta}(\zeta + g_d) < \zeta + g_d,$$

where the ∞ superscript denotes the long run values (i.e., $\hat{y}_d^\infty \equiv \lim_{t \to \infty} \hat{y}_d$).[9] We note from the above expression that since $a > 1$, the long run rate of growth of the dirty good is less than the growth rate of technological change. On the other hand, from Equation (9) it follows that since over the long run $S_k \to 0$ (because $\omega < 1$) then $\hat{y}_d^\infty = \zeta + g_d + \hat{x}^\infty$. Hence, $\hat{x}^\infty < 0$. If $a > 1$, then the economy's growth rate is low enough to have a smaller impact on pollution. This, in turn, means that an optimal pollution tax is sufficient to cause pollution to decrease over the long run, even if the economy is wholly inelastic.

Of course, while sustainability is in this case attained, the rate of economic growth of the economy remains positive; however, if both $\sigma < 1$ and $\omega < 1$, this rate can be quite low and may be below the rate of technological change. In other words, the sacrifice in terms of economic growth imposed by environmental sustainability is, in this case, large and permanent. However, this is not the case when either producers or consumers exhibit higher rates of flexibility. As shown in the proof of Proposition 2, if $\sigma > 1$ and/or $\omega > 1$, then sustainable growth also arises. Moreover, in such cases part (ii) or (iii) of Proposition 1 apply, meaning that the long run rate of growth of real consumption is M/a, which is of course greater than the long run rate of growth prevailing when both $\sigma < 1$ and $\omega < 1$ ($M/a > \dfrac{1+\eta}{a+\eta}(\zeta + g_d)$). In other words, in such a case, the growth rate sacrifice in terms of environmental sustainability is much smaller and is merely temporary.

The reason why this important result is missed by the standard growth theoretical models is that they drastically limit the consumer's role in the economy by assuming only one final good. Proposition 2 arises because the growth rate of the consumption of the dirty goods dictates the long run rate of growth of real consumption, which is sufficiently slow to permit pollution to eventually start falling within a finite period of time. Therefore, we are able to derive this considerably important new insight by explicitly allowing for more than one type of consumer good. If the *EMU* of consumption is greater than one, then the sustainable economic growth is effectively a natural condition, provided an optimal environmental tax is implemented.

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[9] We note that $\hat{c}_d^\infty = (1/a)[M - \hat{p}^\infty]$, where $\hat{p}^\infty = (a/1+a)[(M/a - \zeta - g_d)\eta + (M - \zeta - g_d)]$. In addition, since the market equilibrium condition implies that $\hat{c}_d^\infty = \hat{y}_d^\infty = \zeta + g_d + \hat{x}^\infty$, we have; $\hat{x}^\infty = (1/a)[M - \hat{p}^\infty(\zeta)] - \zeta - g_d$ $= ((1+\eta)/(a+\eta))(\zeta + g_d) - \zeta - g_d < 0$.

6.2. Capital-augmenting technological progress

We now introduce capital-augmenting technological progress in the dirty sector to demonstrate the robustness of our results in Proposition 2. In the case of capital-augmenting technological change affecting the dirty sector, we simply augment capital by factor, n, with $\dot{n}/n = \theta > 0$.

Corollary 2: *Suppose $a > 1$, technological change in the dirty sector augments any factor of production (and/or is neutral or non-existent), and that Assumptions 1 and 2 hold. Then, if either σ and/or ω is positive, an optimal pollution tax is sufficient to induce sustainable development.*

Proof: See Appendix.

Corollary 2 implies that progressively higher optimal pollution tax along the growth path induces sustainable growth under any type of exogenous technological changes. Corollary 2 also implies that when $\omega < 1$ and $\sigma < 1$, the necessary and sufficient condition for sustainable growth is that EMU is greater than one.

When the technical elasticity of substitution between clean and dirty inputs is greater than one, capital-augmenting technological change decreases the relative price of dirty goods even under the rising pollution tax. Since the expenditure share of dirty goods increases when the consumption elasticity of substitution is greater than one, the flexibility requirement in the production of dirty goods under capital-augmenting technological change becomes more stringent than in its absence. On the other hand, if the consumption elasticity is less than one, the presence of capital-augmenting technological progress makes it easier to achieve sustainable growth than its absence would.

Finally, it can be shown that if the capital-augmenting technological progress takes place not only in the dirty sector but also in the clean sector, together with pollution-augmenting technological progress, sustainable growth occurs under an optimal pollution tax. Thus, as long as EMU is greater than one, sustainable growth occurs under an optimal pollution tax for any type of exogenous technological progress. *6.3. The case when EMU is less than one*

Here, we demonstrate that when $a < 1$ the conditions for sustainable economic growth are more demanding than in the previous case. This section will first characterize the output composition effect and will then look into the input substitution (or technique) effect.

6.3.1. The output composition effect

The composition effect works when consumers substitute dirty goods with clean goods in the face of the rising relative price of the dirty goods. Here we consider the case when the consumption elasticity of substitution is strictly greater than 1, but the production elasticity of substitution is less than 1. In this case, the feasibility of sustainable growth relies exclusively on consumer flexibility. Using Remark 1, it follows that the factor share of the clean input in the output value of the dirty final goods, S_k, converges to zero (and concomitantly, the share of the dirty input converges to 1). The fact that the relative price of dirty goods continuously increases over time means that consumers substitute dirty goods with clean ones.

Therefore, assuming that $\sigma > 1$ and $\omega < 1$, then the limit to Equation (16) is:

$$\lim_{t \to \infty} \hat{x} = \frac{M\left(\frac{1}{a} - \sigma\right) - (\zeta + g_d)(1 - \sigma)}{(1 + \sigma\eta)}. \tag{17}$$

From Equation (17) it follows that $\lim_{t \to \infty} \hat{x} < 0$ if and only if

$$\sigma > \frac{\frac{M}{a} - (\zeta + g_d)}{M - (\zeta + g_d)} \equiv d(M, a; \zeta, g_d) > 1.$$

The threshold level, $d(M, a; \zeta, g_d)$, above which sustainable growth becomes possible, is increasing in ζ and g_d respectively. As a consequence of technological change in the dirty sector, sustainable growth becomes more difficult. The threshold level reduces to $1/a$ in the absence of any form of technological progress. The following lemma summarizes the previous results:

Lemma 1 (on the role of the composition effect): *Suppose that technological progress is pollution- augmenting and/or neutral or non-existent, and that Assumptions 1 and 2 hold. If $a < 1$, then $\omega < 1$ does not preclude sustainable economic growth if and only if σ is greater than a threshold level exceeding one (i.e., $\sigma > d(M, a; \zeta, g_d) > 1$).*

Lemma 1 underlines the importance of the composition effect in circumventing the case of an inelastic production technology. All of the previous analyses have assumed a single final good, and hence have ignored the output composition effect, concluding that a flexible production

technology ($\omega \geq 1$) is a necessary condition to allow for sustainable development. Lemma 1 shows that this is not true as long as consumer preferences are sufficiently flexible ($\sigma > d(M,a;\zeta,g_d) > 1$). Remarkably, sustainable growth under an optimal pollution tax may occur even if the production function of dirty goods is Leontief ($\omega = 0$); that is, even if clean and dirty inputs are complements rather than substitutes. Also, the absence of technological change means that $d(M,a;\zeta,g_d) = 1/a$ and thus the condition for sustainable development is not qualitatively affected.

A sufficient condition for the share of dirty consumption goods to approach zero in the long run is that $\sigma > 1$ when $\omega < 1$, so that the relative price of dirty goods increases over time. It might seem surprising that this condition is not sufficient for sustainable development. This is the case because the share of dirty goods approaching zero does not necessarily imply that the rate of growth of the demand for (and hence supply of) the final dirty goods will become negative. In fact, the growth rate of dirty goods continues to be positive over the long run if the economy's growth rate is sufficiently rapid, and may even surpass the rate of pollution-augmenting technological change, in which case pollution will continue to increase in the long run. Lemma 1 shows that only when the elasticity is sufficiently large ($\sigma > d(M,a;\zeta,g_d) > 1$) will the consumption of dirty goods (and hence the production of dirty goods) grow at a rate that is below the pollution-augmenting technological change, thus leading to a reduction of pollution levels.[10]

6.3.2. The input substitution or technique effect

We will now consider the case when the technical elasticity of substitution between the two inputs is strictly greater than one, while the consumption elasticity of substitution is less than one but still positive. In this case, the cost share of the clean input approaches one, while the share of the dirty good in the consumer budget also approaches one. The feasibility of sustainable growth depends solely on technique effect. From Equation (16) we have:

$$\lim_{t \to \infty} \hat{x} = \frac{\left(\dfrac{M}{a} - \zeta\right) - \omega(M - \zeta) + g_d(\sigma - 1)}{1 + \omega\eta}. \tag{18}$$

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[10] Given that $\omega < 1$, which implies that $\lim_{t \to \infty} S_k = 0$, it follows from (10) that the rate of growth of the dirty good production over the long run is equal to the growth rate of effective pollution, $\hat{y}_d^\infty = \hat{x} + \zeta$. Hence, if $\hat{y}_d^\infty = \hat{c}_d^\infty > \zeta$ then $\hat{x} > 0$, where a superscript ∞ denotes long run levels.

The first term of the numerator of Equation (18) represents the technique effect resulting from a change in the relative factor costs of production. The optimal pollution tax causes the pollution input to become increasingly expensive. In addition, if the elasticity of substitution between the clean and the dirty input is greater than one, the pollution input is gradually substituted with capital, causing its share to converge to zero. The second term of the numerator (which is positive) captures the productivity effect of pollution, an effect that makes it more difficult to achieve sustainable growth over the long-term. The third term represents the effect of growth of total factor productivity in the dirty sector, which reduces pollution growth when $\sigma < 1$. It follows that sustainable growth only becomes possible if the technique or substitution effect outweighs the technological change effect. This condition is satisfied if $\omega > d(M,a;\zeta,0) > 1$ where

$$d(M,a;\zeta,0) = \frac{\left(\dfrac{M}{a} - \zeta\right)}{M - \zeta}.$$

Consequently, if $a < 1$, a Cobb-Douglas production function ($\omega = 1$) is not consistent with sustainable development when $g_d = 0$. As we demonstrate below, the standard growth models have almost always assumed Cobb-Douglas production functions, and are therefore able to conclude that growth is sustainable only because they assume that the *EMU* is greater than one. The following lemma summarizes these findings.

Lemma 2 (on the technique or input composition effect): *Suppose that technological progress is pollution-augmenting and that Assumptions 1 and 2 hold. If $a < 1$, then $\sigma < 1$ does not preclude sustainable economic growth if an optimal pollution tax is implemented and ω is greater than a threshold level, $d(M,a;\zeta,0)$ that exceeds one.*

In our model (unlike, for example, the model in Acemoglu et al., 2012) capital (i.e., the clean input) is expanding in a growing economy and, moreover, the rate of economic growth is endogenous. Hence, even if technological change is only pollution-augmenting and concentrated in the dirty sector (as we assume), the capital-to-effective pollution ratio (k_d / bx) may increase without requiring so rapid an increase of the pollution tax as to smother economic growth. This follows because the technique effect does not rely exclusively on the pollution tax, but is reinforced by the capital growth effect. Therefore, if the elasticity of substitution between capital

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and pollution is greater than the threshold level, then the substitution effect may dominate the expansion effect within the dirty sector and pollution will begin decreasing at some finite time along the growth path. Combining Lemmas 1 and 2, we obtain the following proposition:

Proposition 3: *Suppose that technological change is pollution-augmenting and Assumptions 1 and 2 hold. If $a < 1$, then sustainable growth is feasible if an optimal pollution tax is implemented and either ω or σ is greater than the threshold level, $d(M, a; \zeta, 0)$, which exceeds one.*

Proof: See Appendix.

Proposition 3 demonstrates that even if technological progress benefits only the dirty sector and is biased toward the dirty input in a pollution-augmenting fashion, and if the *EMU* is less than one, then an optimal pollution tax may be sufficient to induce environmental sustainability if either the consumer's preferences or the producer's technologies exhibit sufficient flexibility. From Proposition 1, it follows that this occurs while the economy's growth rate is positive throughout the full adjustment path. Moreover, since environmental sustainability requires that either $\sigma > 1$ or $\omega > 1$, Proposition 1 clearly shows that economic growth is lowered in the short run but that the economy's growth rate gradually recovers towards its potential rate over the long run. Therefore, the optimal pollution tax alone can lead to sustainable growth without requiring further policy interventions (such as subsidies directed at transforming technological change from pollution-augmenting to clean sector or clean input augmenting).

7. Stock effects: Conditions for avoiding an environmental disaster

In this section we analyze the conditions under which the solution for the dynamical system developed in the previous sections is indeed consistent with avoidance of environmental disaster at any point in time. Assuming that the dynamic path of pollution is defined by equation (16), we find that for any given initial level of clean air stock there exists a corresponding critical level of initial emission flow such that if the initial value of pollution emissions is less than such critical level, the clean air stock remains at all times above a minimal threshold level that prevents environmental disaster. Otherwise, if the initial pollution level is above the critical level, then the clean air stock falls below the threshold level and catastrophic environmental disaster will eventually ensue. The intuition behind this result is that since equation (16) gives the (optimal) rate of change of pollution for all times, then the full path of pollution is entirely

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determined by the initial level of pollution. The question is whether along this path the stock of clean air ever reaches the catastrophic level. If we find the initial (critical) level of pollution that in conjunction with (16) causes a pollution path that exactly avoids reaching such a catastrophic stock level, then any other pollution path following the same rate of change established by (16) but starting from a lower pollution level will also avoid catastrophe.

In order to identify such a critical level of initial emissions, we first note that for any given initial level of man-made capital, the system of equations (14) to (16) yields a unique optimal growth path for p, (k_d / x) and x.[11] In particular, we can define the pollution level at a point in time as:

$$x(t) = x_0 \int_0^t \exp(g(\upsilon)\upsilon)d\upsilon,$$

where $g(\upsilon)$ is the rate of change of pollution at time υ, which is a function of all parameters and the predetermined variable, k_0. As we show below, the effect of the initial clean air stock on $x(t)$ occurs entirely through its effect on x_0. In addition, the stock of clean air at any point in time is given by Equation (4'). Hence, we can define the unique path of pollution emission flows and stock of clean air as conditional functions of the (endogenous) initial value of pollution as well as of the (predetermined) initial stocks of clean air and natural capital as follows:

$$x(t) = G(t, x_0; k_0, \chi) \text{ and } E(t) = J(t, x_0; k_0, E_0, \chi),$$

where the function $J(t, x_0; k_0, E_0, \chi)$ is defined in (4') and $\chi = (a, \sigma, \omega)$ denotes a vector of structural parameters. Also, we have that $x(0) = G(0, x_0; k_0, \chi) = x_0$ and $E(0) = J(0, x_0; k_0, E_0, \chi) = E_0$ by the fixed point theorem. From Equation (4') it is clear that unless the pollution emissions $x(t)$ eventually starts falling over time the stock constraint, $E(t) \geq \underline{E}$ for all $t \geq 0$, cannot be satisfied.

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[11] We note that the system of equations (14), (15) and (16) can be represented as a system of autonomous differential equations $\dot{p} = \Theta(S_k, s(p), p)$, $\dot{(k_d / bx)} = \Gamma(S_k, s(p), (k_d / bx))$ and $\dot{x} = \Phi(S_k, s(p), (k_d / bx), x)$. Since $\Theta(\cdot)$, $\Gamma(\cdot)$ and $\Phi(\cdot)$ are all continuously differentiable functions, there exists a unique solution for each set of initial values. We also note that the solution for emission, x, constitutes an optimal control for dynamic optimization in the absence of stock constraints. The initial level of emission is determined endogenously within the system. Likewise, initial values of k_d and therefore p are all endogenously determined within the system.

Let χ^* denote the set of $\chi = (a, \sigma, \omega)$ which guarantees eventual decline of pollution emissions, and are the parameters that satisfy the conditions established by either Propositions 2 or 3. Then for any χ in χ^*, and man-made stock of capital, we can define the *admissible* set, $D(\chi, k_0)$ of initial values of clean air stock and flow level of pollution which assures sustainable growth. Thus,

$$D(\chi, k_0) = \left\{ (x_0, E_0) \big| J(t, x_0; k_0, E_0, \chi) \geq \underline{E}, \text{ for all } t > 0 \right\}.$$

Given the initial level of clean air, E_0, the set $D(\chi, k_0; E_0)$ of initial levels of flow pollution that an economy can emit while maintaining the stock of clean air above the threshold level is bounded above and closed. This is so because the function $J(t, x_0; k_0, E_0, \chi)$ is continuous as shown by (4') and is also bounded from above. There exists the maximal element, $x_0^c(E_0)$ of the set $D(\chi, k_0; E_0)$, above which an environmental disaster occurs. We define $C(\chi, k_0) = \left\{ E_0, x_0^c(E_0) \big| \underline{E} \leq E_0 \right\}$, which constitutes the boundary or envelope of the set $D(\chi, k_0)$.

Alternatively, we note that for any eventually declining pollution path, there exists a time $T \geq 0$ after which pollution decreases in a monotonic way. It follows that there exists a critical turnaround time $t^* > T$ such that

(19) $$x(t^*) = G(t^*, x_0^c; k_0, \chi) = \psi \underline{E},$$

(20) $$E(t^*) = J(t^*, x_0^c; k_0, E_0, \chi) = \underline{E},$$

where x_0^c is the *maximum* initial level of pollution emissions that corresponds to any given $E_0 > \underline{E}$ consistent with avoiding environmental disaster and t^* is the critical turnaround time at which the stock of clean air reaches the minimum level necessary to avoid a catastrophe. The two equations (19) and (20) solve for the two endogenous variable, $x_0^c = x_0^c(E_0; k_0, \underline{E}, \chi, \psi)$ and $t^* = N(E_0; k_0, \underline{E}, \chi, \psi)$. [12]

Figure 1 illustrates the previous analysis. The thick curve, denoted as C, is the envelope of set D as defined above. Therefore, C provides an envelope for all trajectories of x as a function of E_0 that satisfy the constraint $E(t) \geq \underline{E}$ at all times, which is called set D in Figure 1. By

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[12] Section 8 presents an explicit solution of these endogenous variables in a Cobb-Douglas economy.

contrast, any trajectory that is outside (above) the envelope C, denoted as a complement of set D (set D^c) in Figure 1 (which is shaded), reaches an environmental catastrophe. Figure 1 shows the particular case where pollution emissions follow an inverted U-shaped pattern where the envelope C reaches $\underline{E}$ at the turnaround time t^*. The uniqueness property of the adjustment paths guarantees that any two different trajectories starting from different initial positions move in parallel and never cross each other. Hence, any trajectory starting below $x_0^c(E_0)$ never reaches the catastrophic stock level, while any trajectory starting above C is bound to eventually violate the stock constraint.
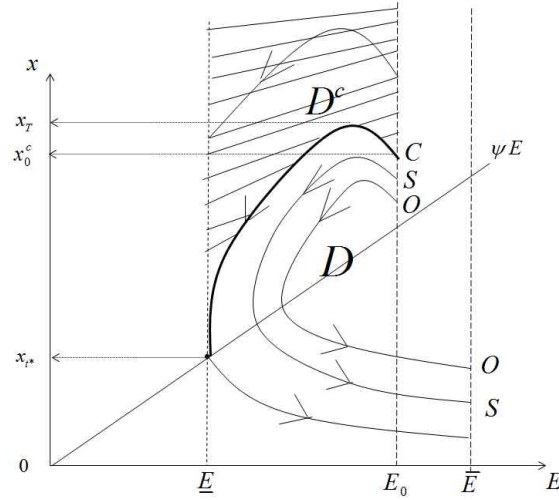

Fig.1. The admissible set D and the Envelop C in E-x space

In Figure 1 the curve labeled OO represents the optimal trajectory while the curve SS shows an arbitrary suboptimal but sustainable trajectory associated with a suboptimal tax. The tax that underlying trajectory SS satisfies two conditions: first, it is sufficiently high to permit the initial pollution level to be below the critical level (x_0^c) as defined earlier and second, it adjusts over time to allow for an optimal rate of change of pollution according to Equation (16). In general, finding x_0^c is easier and demands much less information than determining the optimal initial pollution level. It must be noted that, as expected, pollution levels within trajectory OO are lower than those within trajectory SS at each point in time.[13]

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[13] Figure 1 does not illustrate time profiles of pollution emissions for the two trajectories. It can be shown, however, that each level of E is reached at an earlier time along the trajectory OO than SS. Although it appears in the

8. Numerical calibrations

Here we develop a numerical example to obtain further insights into the propositions of this paper.[14] In order to highlight the role of the consumption composition effect, we assume that the clean and dirty inputs are complements (i.e., $\omega = 0$). For simplicity we focus only on pollution-augmenting technological progress. We first calibrate our model only with flow emissions of pollution using parameters based on data from the US economy and check the sustainability condition for the stock constraint later.

8.1. Parameter choices

In the recent literature the long-run annual growth rate of the US economy is often assumed to be 2 percent (i.e., Nordhaus, 2007; Weitzman, 2007; Acemoglu et al., 2012). As shown in Proposition 1 above, this corresponds to M / a where $a = EMU$. Since the EMU is assumed to be approximately 2 in the literature, the net return to the capital input, M, is approximately 0.04. We thus assume that the net return to capital is four percent, and examine the feasibility of sustainable growth under varying assumptions of the EMU and temporal substitution parameters in consumption, σ.

Based on recent econometric estimates we alternatively consider values of EMU of 2 and 0.8. (i.e., Ogaki and Reinhart, 1998 and Vissing-Jørgensen, 2002). For $M = 0.04$, the long run growth rate of the economy becomes 5 percent when $EMU = 0.8$, which is much greater than the commonly accepted rate of 2 percent. In spite of this, we perform this simulation to highlight the fact that when EMU is low, the scale effect is much larger and therefore makes sustainable growth more difficult to achieve. In addition, in order to highlight the role of the composition effect, we consider three different values for σ, namely, 4, 2 and 0.8. Finally, we assume that the rate of pollution-augmenting technological progress is $\zeta = 0.005$, the parameter for the elasticity of marginal damage is $\eta = 1$, the ratio A_C / A_D is 1 and the ratio γ_c / γ_d in the unit expenditure function is 0.7.

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figure that the level of pollution emissions is higher in OO than SS beyond the turnaround level, this is due to the fact that the visual comparison considers indeed different points in time. At each point of time the level of E is higher within trajectory OO than SS.

[14] Here we provide a succinct description of the simulation methodology. For further detail, please check the online resource.

8.2. The pollution emissions path

Figure 2 provides the growth of pollution emissions over time for various values of EMU.[15]
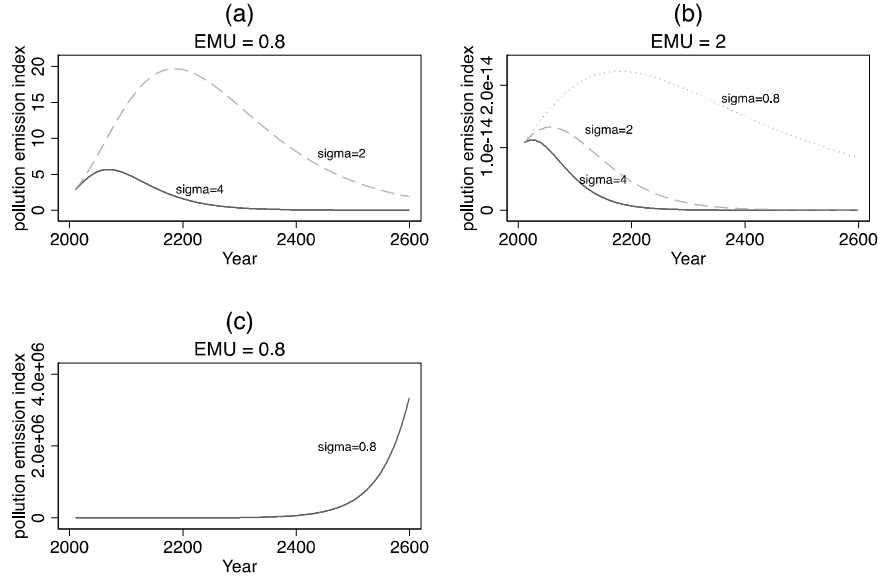
Source: Authors' calculation

Fig.2. Pollution emissions for different values of σ and EMU

Panel (a) shows the case when $EMU = 0.8$. If the elasticity of substitution is greater than the

threshold level, $\left(\dfrac{M/a - \zeta}{M - \zeta} \right) \approx 1.28$, implied by Proposition 3, there exists a critical time until

which pollution increases monotonically and after which declines over time. This turning point

depends on the level of σ. If $\sigma = 4$, the turning point takes place in the year 2069, and if

$\sigma = 2$, in 2185. This is due to the fact that the consumption composition effect becomes more

effective when σ is larger. Panel (b) depicts the case when $EMU = 2$: if $\sigma = 4$ then pollution

begins falling very quickly by the year 2025, but if $\sigma = 2$ or $\sigma = 0.8$ then the turning point

occurs during a much later year (2057 and 2178, respectively). Panel (c) illustrates the pollution

emissions path for the case when both EMU and σ are less than one, in which case pollution

increases in all periods. Given that $\sigma < 1.28$, pollution emissions continue to increase over time

for all periods as indicated by Lemma 1. In summary, if $EMU < 1$, sustainable growth requires

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[15] For illustration purposes, we use a time scale obtained by calibrating the changes in the share of the clean input (labor) of the U.S. manufacturing industry over the past decade. For the detailed procedure, see the online resource.

that the consumption elasticity of substitution is greater than the threshold level. However, as shown in Panel (b), if $EMU > 1$ then economic growth is sustainable even if σ is very low (and $\omega = 0$ as we assumed here). In this case, as predicted by Proposition 2, even highly inelastic consumer preferences and producer technology do not prevent pollution from beginning to decline along the optimal path.

8.3. Growth sacrifice caused by the pollution tax

Finally, Panel (a) in Figure 3 shows the rates of growth of real consumption $\left(\overset{\wedge}{c / e} \right)$ for $EMU = 0.8$. The rate of economic growth is always positive, although it falls below the potential growth rate over the short run. However, if $EMU > 1$, it recovers towards the potential growth rate over the long run. The growth sacrifices over the short and medium terms are rather small and growth recovers more quickly if the elasticity of substitution is larger. Even when σ is relatively low (i.e., 2), the growth sacrifice is not very large, reaching a maximum value of the order of 0.6 annual percentage points, although the growth rate begins recovering at a much later date than when $\sigma = 4$. The growth sacrifice is large if σ is less than one (i.e., $\sigma = 0.8$) and, more importantly, and as predicted by Proposition 1, the economy's growth rate converges to a lower but still positive rate of growth over the long run.

Panel (b) of Figure 3 illustrates the case when $EMU = 2$. If $\sigma < 1$, then the long run growth rate remains positive but falls below toward the technological growth rate ($\zeta = 0.005$). However, as predicted by Proposition 1, if $\sigma = 2$ then the rate of economic growth converges to the potential growth rate M / a and, moreover, the growth sacrifice imposed by environmental sustainability is smaller than the previous case and temporary. The maximum reduction of the rate of economic growth is in this case only about 0.5 percentage points. In the short run the growth sacrifice caused by the pollution tax is only 0.2 percentage points, from 2% annual growth when no environmental tax is implemented to about 1.8% when the tax imposed.

(a)
EMU = 0.8

(b)
EMU = 2
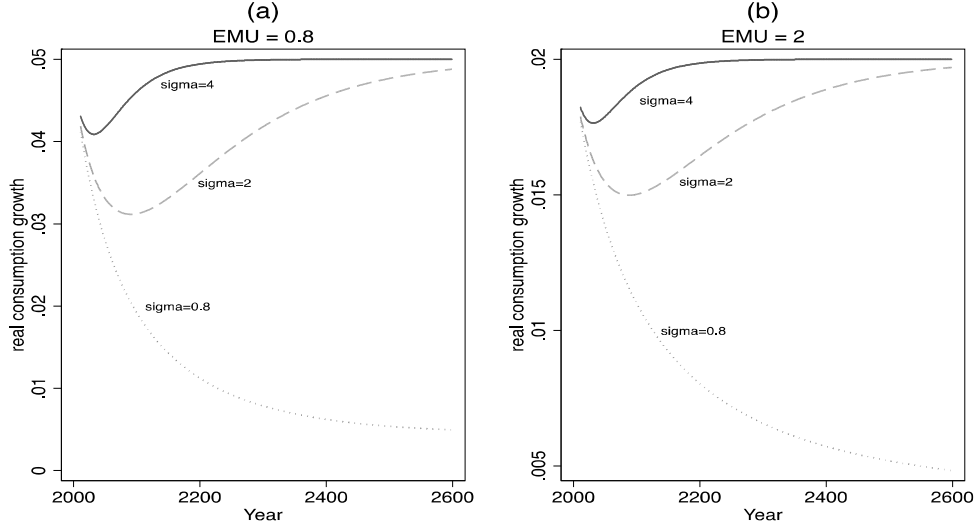
Source: Authors' calculation

Fig.3. Real consumption growth rates for different values of σ and *EMU*

8.4. Numerical simulation considering the stock effects

We now consider the possibility of irreversible disaster assuming Cobb-Douglas utility and production function, and that $EMU > 1$. Although there is no clear consensus on the structure of the carbon cycle, recent scientific studies find that the lifetime of carbon in the air spans a few centuries. According to IPCC (2007), about half of an increase of CO2 will be removed from the atmosphere within 30 years, implying a 1.6 percent regeneration rate of clean air per annum (IPCC, 2007). Then, Equation (19) implies that $x(t^*) = 0.016\underline{E}$.

Given the Cobb-Douglas specification, the cost share of clean input in production, S_k, and the consumer's budget share of the dirty final good, s, are constant. Assuming that service output and labor input are less pollution intensive than manufacturing output and energy intensive input, we use estimates for the share of clean input and clean final goods in world GDP for calibration purposes and set $S_k = 0.5$ and $s = 0.54$ (Guscina, 2006; World Bank, 2012). Using the same values for the other parameters (i.e., $a = 2$; $\zeta = 0.005$; $M = 0.04$; $\eta = 1$), we obtain from Equation (16) that $x(t) = x_0 \exp(-\vartheta t)$, where $\vartheta = 0.0085$, implying that the optimal pollution decreasing rate is equal to 0.85 percent per annum.

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Since there is no direct measure to gauge absolutely clean air stock, we construct the so-called relative clean air stock (RCAS) index to represent $E(t)$ in section 7. Let $Carbon_t$ and $Carbon^D$ represent the current global carbon stock in year t and the disaster-rendering magnitude of the global carbon stock, both measured in *ppm*. Define $RCAS$ index as follows;

$$E(t) = RCAS(t) = Carbon^D / Carbon_t.$$

For calibration purposes, we assume that the disaster-rendering level of the carbon stock is 650 *ppm*.[16] In addition, we set the initial value (year 2013) and pre-industrial value of global carbon stock level in the atmosphere at 395 *ppm* and 280 *ppm,* respectively (NOAA, 2013). Then the clean air stock index for the pre-industrial level that we consider environmentally pristine is $\bar{E} = 650/280 \approx 2.32$, while the current level and disaster-rendering level of clean air stock are $E_{2013} = 650/395 \approx 1.81$ and $\underline{E} = 650/650 = 1$, respectively.[17]

To solve for the corresponding critical level of emission, x_{2013}^c numerically, we first note that using Equation (19),

$$x_{2013}^c \exp(-\vartheta t^*) = \psi \underline{E} = \psi \qquad (21)$$

Also, from Equation (4') and (20), we have,

$$E(t^*) = \exp(\psi t^*)(E_{2013} - \int_0^{t^*} x_{2013} \exp(-\vartheta t)dt) = \underline{E} = 1.$$

Using the expression for the pollution emissions in the Cobb-Douglas case, $x(t) = x_0 \exp(-\vartheta t)$ and integrating, it follows that the previous expression can be written as:

$$\exp(\psi t^*)\left(E_{2013} + \frac{x_{2013}^c}{\vartheta + \psi}(\exp(-(\vartheta + \psi)t^*) - 1) \right) = 1. \qquad (22)$$

Solving Equations (21) and (22) using numerical methods gives the point for the year 2013 located in the envelope C, which corresponds to $x_{2013}^c = 0.043$ and $E_{2013} \approx 1.81$. We then generate the time profiles of pollution emissions and the stock of clean air under alternative scenarios.

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[16] Although the disaster-rendering magnitude of the stock of CO2 differs according to various experts, commonly accepted carbon concentration levels lie somewhere between 550 *ppm* and 750 *ppm,* implying a 3 Celsius degree and 4 Celsius degree increase, respectively (i.e., Glasby, 2006; Pearson et al., 2009).

[17] A pre-industrial level of carbon stock is often considered an environmentally clean air condition (i.e., Acemoglu et al., 2012).

We consider four alternative scenarios.

Scenario 1 (Optimistic case): The government is able to reduce emissions by 10 percent below the critical level, x_{2013}^c, and the rate of pollution emissions growth is to be regulated optimally according to Equation (16).

Scenario 2 (Sufficient case): The government takes measures to reduce emissions exactly to the critical level, x_{2013}^c, and the rate of pollution emissions growth is to be regulated optimally according to Equation (16).

Scenario 3 (Insufficient, late disaster case): The government is unable to reduce pollution emissions to the critical level, x_{2013}^c, and allows emission levels 10 percent higher than the critical level, x_{2013}^c, while still restricting the rate of pollution emissions growth optimally according to Equation (16).

Scenario 4 (Business as usual, early disaster case): Pollution emissions are 10 percent above the critical level, x_{2013}^c, and they grow by 3.1 percent per year, which corresponds to the historical growth rate of carbon emissions over the 2000-2010 time period (Peters et al., 2011).

Table 1

Time path of pollution emissions and clean air stock under different scenarios.

Year(t)	Scenario 1 Optimistic case ($\hat{x} = -0.0085$)		Scenario 2 Sufficient case ($\hat{x} = -0.0085$)		Scenario 3 Insufficient, late disaster case ($\hat{x} = -0.0085$)		Scenario 4 Business as usual, early disaster case ($\hat{x} = 0.031$)	
	$x(t)$	$E(t)$	$x(t)$	$E(t)$	$x(t)$	$E(t)$	$x(t)$	$E(t)$
2013	**0.0387**	1.809	**0.0430**	1.809	**0.0473**	1.809	**0.0473**	1.809
2027	0.0343	1.689	0.0381	1.625	0.0419	1.561	0.0706	1.063
2028	0.0340	1.682	0.0378	1.613	0.0416	1.544		
2062	0.0255	1.544	0.0283	1.275	0.0311	1.006		
2063	0.0253	1.543	0.0281	1.267				
2065	0.0248	1.5425	0.0276	1.252				
2066	**0.0246**	**1.5424**	0.0274	1.244				
2067	0.0244	1.5426	0.0271	1.237	**Environmental Disaster**		**Environmental Disaster**	
2129	0.0144	2.057	**0.0160**	**1.000**				
2130	0.0143	2.076	0.0159	1.00003				
2140	0.0134	2.311	0.0146	1.007				
2141			0.0144	1.009				
2240	**Pristine condition**		**0.0062**	2.293				
2241	**Pristine condition**							

Notes: 1) $x(t)$ and $E(t)$ denote the yearly index of pollution emissions and relative clean air stock, respectively. 2) For each scenario, Equation (4') is used to generate $E(t)$ over time starting from the initial year of 2013. *Source:* Author calculations.

Table 1 below shows the simulation results for the time profiles of $x(t)$ and $E(t)$ under the above scenarios. Under Scenario 1, sustainable development takes place. In this scenario the turnaround point of the clean air stock occurs in 2066, reaching an environmentally pristine condition by 2141. Under Scenario 2, sustainable development is also feasible, as the clean air stock never falls below the threshold level and starts growing in 2130. Under Scenario 3, an environmental disaster is unavoidable; by 2063, the stock of the clean air falls below the threshold level. An environmental disaster occurs despite the assumption that the government is able to regulate emissions growth according to the optimal rate of change. Lastly, under Scenario 4, an environmental disaster occurs by the year 2028.

9. Conclusion

Sustainable development can be achieved under a variety of plausible technological conditions using a pollution tax as the only policy instrument. If the often-used assumption regarding *EMU* being greater than one holds, then sustainable development is almost automatically satisfied as long as either the elasticity of substitution in production or in consumption is positive. An optimal pollution tax profile rules optimal pollution changes over time as characterized by equation (16) in the text and it is sufficiently high to set the initial pollution level below a critical level to avoid disastrous stock effects of pollution.

Moreover, even if the initial pollution tax is suboptimal level, sustainable development still takes place as long as the initial tax level is sufficient to set the initial pollution flow less than or equal to its critical level and that the rate of change of the tax over time be at the rate necessary to induce optimal pollution changes over time. Such a critical level is well defined once the initial level of renewable resource stock such as clean air is identified. Furthermore, the pollution tax affects the growth rate of the economy only modestly.

Sustainable development mainly becomes an issue when *EMU* is less than one. Sustainability may also occur in this case if consumer preferences between the clean and dirty goods are flexible enough, even if the production technology is highly inflexible. In contrast to the assumption of high producer flexibility made by the standard growth models, the assumption of consumer flexibility required in this case appears to be more adequately supported by empirical studies. This paper has demonstrated that neither strong production substitution nor technological optimism is necessary for environmentally sustainable growth.

Although the informational requirement for implementation of the of government intervention to ensure sustainable development is not formidable, it is not an easy task to mitigate the political and institutional obstacles to the implementation of optimal pollution taxes as part of the initial necessary measures to reduce emissions. This paper shows the scope of government intervention by characterizing a family of suboptimal sustainable growth paths.

References

Acemoglu, D., Aghion, P., Bursztyn L., Hemous, D., 2012. The environment and directed technological change. American Economic Review 102 (1), 131–166.

Aghion, P., Howitt P., 1997. Endogenous Growth Theory. Massachusetts: MIT press.

Arrow, K., Dasgupta P., Goulder L., Mumford, K., Oleson, K., 2010. Sustainability and the measurement of wealth. NBER Working Paper No.16599.

Attanasio, O.P., Browning, M. 1995. Consumption over the life cycle and over the business cycle. American Economic Review 85 (5), 1118–1137.

Banerjee, A., Solomon B.D., 2003. Eco-labeling for energy efficiency and sustainability: A meta-evaluation of US programs. Energy Policy 31 (2), 109–123.

Baylis, K., Fullerton, D., Karney, D. H., 2013. Leakage, welfare, and cost-effectiveness of carbon policy. American Economic Review, Papers and Proceedings.

Bovenberg, A.L., Smulders, S., 1995. Environmental quality and pollution-augmenting technological change in a two-sector endogenous growth model. Journal of Public Economics 57 (3), 369–391.

Bovenberg, A.L., de Mooij, R.A., 1997. Environmental tax reform and endogenous growth. Journal of Public Economics 63 (2), 207–237.

Bretschger, L., Smulders, S., 2012. Sustainability and substitution of exhaustible natural resources: How structural change affects long-term R&D-investments. Journal of Economic Dynamics and Control 36 (4), 536–549.

Brock, W., Taylor M., 2010. The green Solow model. Journal of Economic Growth 15 (2), 127–153.

Cole, M., Elliot, R., 2003. Determining the trade-environment composition effect: The role of capital, labor and environmental regulations. Journal of Environmental Economics and Management 46 (3), 363–383.

Copeland, B., Taylor, S., 2004. Trade, growth, and the environment. Journal of Economic Literature 42 (1), 7–71.

Cropper, M.L., 1976. Regulating Activities with Catastrophic Environmental Effects. Journal of Environmental Economics and Management 3 (1), 1–15.

Cropper, M.L, Oates, W.E., 1992. Environmental economics: A survey. Journal of Economic Literature 30 (2), 675–740.

Daly, H.E., 1992. From empty-world economics to full-world economics: Recognizing an historical turning point in economic development. In R. Goodland, H.E. Daly, S. El Serafy (eds.), Population, Technology, and Lifestyle (pp. 23–27). Washington D.C.: Island Press.

Field, B.C, Grebenstein, C., 1980. Capital-energy substitution in U.S. manufacturing. Review of Economics and Statistics 62 (2), 207–212.

Galarraga, I., González-Eguino, M., Markandya, A., 2011. Willingness to pay and price elasticities of demand for energy-efficient appliances: Combining the hedonic approach and demand systems. Energy Economics 33 (Supplement 1), S66–S74.

Glasby, G. P., 2006. Drastic reductions in utilizable fossil fuel reserves: an environmental imperative. Environment, development and sustainability 8 (1), 197–215.

Golosov, M., Hassler, J., Krusell, P., Tsyvinski, A., 2011. Optimal Taxes on Fossil Fuel In General Equilibrium. NBER Working Paper 17348.

Grossman G., Krueger, A., 1995. Economic growth and the environment. Quarterly Journal of Economics, 110 (2), 353–377.

Guscina, A., 2006. Effects of globalization on labor's share in national income [electronic resource]. International Monetary Fund.

Hassler, J., Krusell, P., Olovsson, C., 2012. "Energy-Saving Technical Change," NBER working paper 18456.

IPCC (Intergovernmental Panel on Climate Change). 2007. Climate Change 2007: The Physical Science Basis. Contribution of Working Group I to the Fourth Assessment Report of the Intergovernmental Panel on Climate Change [Solomon, S., D. Qin, M. Manning, Z. Chen, M. Marquis, K.B. Averyt, M.Tignor and H.L. Miller (eds.)]. Cambridge University Press, Cambridge, United Kingdom and New York, NY, USA.

Keller, K., Bolker, B. M., Bradford, D. F., 2004. Uncertain climate thresholds and optimal economic growth. Journal of Environmental Economics and Management 48 (1), 723–741.

Kemfert, C., Welsch, H., 2000. Energy-Capital-Labor substitution and the economic effects of CO2 abatement: evidence for Germany. Journal of Policy Modeling 22 (6), 641–660.

Kotchen, M.J., Moore, M.R., 2007. Private provision of environmental public goods: Household participation in green-electricity programs. Journal of Environmental Economics and Management 53 (1), 1–16.

Layard, R., Nickell, S., Mayraz, G., 2008. The marginal utility of income. Journal of Public Economics, 92 (8-9), 1846–1857.

Leizarowitz, A., Tsur, Y., 2012. Renewable resource management with stochastic recharge and environmental threats. Journal of Economic Dynamics and Control 36 (5), 736–753.

López, R. (1994). The environment as a factor of production: The effects of trade liberalization. Journal of Environmental Economics and Management 27 (2), 163–184.

Lin, B-H., Smith, T., Huang, C.L., 2008. Organic premiums of US fresh produce. Renewable Agriculture and Food Systems 23 (3), 208–216.

Nævdal, E., 2006. Dynamic optimisation in the presence of threshold effects when the location of the threshold is uncertain–with an application to a possible disintegration of the Western Antarctic Ice Sheet. Journal of Economic Dynamics and Control 30 (7), 1131-1158.

Nævdal, E., Oppenheimer, M., 2007. The economics of the thermohaline circulation—A problem with multiple thresholds of unknown locations. Resource and Energy Economics 29 (4), 262–283.

Nordhaus, W. D., 2007. A review of the Stern review on the economics of climate change. Journal of Economic Literature 45, 686–702.

National Oceanic and Atmospheric Administration (NOAA). (database). [Accessed on 5 April 2013].

Ogaki, M., Reinhart, C. M., 1998. Measuring intertemporal substitution: The role of durable goods. Journal of Political Economy 106 (5), 1078–1098.

Pearson, P. N., Foster, G. L., Wade, B. S., 2009. Atmospheric carbon dioxide through the Eocene-Oligocene climate transition. Nature 461 (7267), 1110–1113.

Peretto, P. F., 2009. Energy taxes and endogenous technological change. Journal of Environmental Economics and Management 57 (3), 269–283.

Peters, G. P., Marland, G., Le Que're', C., Boden, T., Canadell, J. G., & Raupach, M. R., 2011. Rapid growth in CO2 emissions after the 2008-2009 global financial crisis. Nature Climate Change 2 (1), 2–4.

Stokey, N. L., 1998. Are there limits to growth? International Economic Review 39 (1), 1–31.

Van der Werf, E., 2008. Production functions for climate policy modeling: An Empirical Analysis. Energy Economics 30 (6), 2964–2979.

Vissing-Jørgensen, A., 2002. Limited asset market participation and the elasticity of intertemporal substitution. Journal of Political Economy, 110 (4), 825–53.

Vissing-Jørgensen, A., Attanasio, O., 2003. Stock-Market Participation, Intertemporal substitution, and risk aversion. American Economic Review 93 (2), 383–391.

Vollebergh, H.R.J., Kemfert, C., 2005. The role of technological change for a sustainable development. Ecological Economics 54 (2-3), 133–147.

Weitzman M.L. 2007. A review of the Stern review on the economics of climate change, Journal of Economic Literature 45 (3), 703–724.

World Bank. 2012. Data retrieved August 2, 2012, from World Development Indicators Online database.

Appendix

Proofs of propositions and assertions in the text

Derivation of equation (9) (page 12):

Use Roy's identity to derive the demand for the dirty good from the indirect utility function as follows.

$$c_d = \frac{c}{e(1,p)} e_2(1,p). \tag{A.1}$$

Logarithmic time differentiation yields,

$$\hat{c}_d = \hat{c} + \hat{e}_2(1,p) - \hat{e}(1,p). \tag{A.2}$$

Totally differentiating both sides of first order condition Equation (5) with respect to time and using Equation (6), we have,

$$\hat{c} = \left(\frac{a-1}{a}\right)\hat{e} + \frac{M}{a}. \tag{A.3}$$

The second term of the right-hand side of Equation (A.2) can be written as,

$$\hat{e}_2 = \frac{d\log e_2}{dp}\frac{dp}{dt}. \tag{A.4}$$

Using the CES utility function we obtain,

$$\frac{d\log e_2}{dp} = \left(\frac{\sigma}{1-\sigma}\right)\frac{\gamma_d(1-\sigma)p^{-\sigma}}{\gamma_c + \gamma_d p^{1-\sigma}} - \frac{\sigma}{p} = \frac{\sigma}{p}\big(s(p)-1\big). \tag{A.5}$$

On the other hand, using Shephard's lemma on the expenditure function $e(1,p)$ we have,

$$\hat{e}(1,p) = \frac{pe_2}{e}\hat{p} = s(p)\hat{p}. \tag{A.6}$$

Using Equation (A.5) into Equation (A.4) and then using (A.3), (A.4) and (A.6) in (A.2) we find,

$$\hat{c}_d = \left(\frac{1-a}{a}\right)\big[M - s(p)\hat{p}\big] + \sigma(s(p)-1)\hat{p} - s(p)\hat{p} \tag{A.7}$$

$$= \frac{1}{a}M - \left[\frac{s(p)}{a} + (1-s(p))\sigma\right]\hat{p}.$$

Derivation of Equation (13) (page 13):

Logarithmic total differentiation of both sides of the first order condition Equation (8),

$$\eta\hat{x} - \hat{\lambda} = \hat{p} + g_D + \hat{b} + \left(\widehat{F_2(k_d, bx)}\right). \tag{A.8}$$

Also, since the function F is CES, we have,

$$\left(\widehat{F_2(k_d, bx)}\right) = \frac{\alpha}{\omega} \frac{\left(\widehat{\dfrac{k_d}{bx}}\right)}{\left[(1-\alpha)\left(\dfrac{k_d}{bx}\right)^{-\frac{\omega-1}{\omega}} + \alpha\right]} = \frac{S_k}{\omega}\left(\widehat{\dfrac{k_d}{bx}}\right). \tag{A.9}$$

Rearranging (A.8) and using (A.9) and $\hat{b} \equiv \zeta$, we arrive at

$$\hat{p} + \frac{S_k}{\omega}\left(\widehat{\dfrac{k_d}{bx}}\right) - \eta\hat{x} = M - \zeta - g_D. \tag{A.10}$$

Derivation of equations (14), (15) and (16) (page 13):

The system of Equations (11), (12) and (13) in matrix form can be written as,

$$\begin{bmatrix} z & S_k & 1 \\ 1 & -\dfrac{1}{\omega}(1-S_k) & 0 \\ 1 & \dfrac{1}{\omega}S_k & -\eta \end{bmatrix} \begin{bmatrix} \hat{p} \\ \left(\widehat{\dfrac{k_d}{bx}}\right) \\ \hat{x} \end{bmatrix} = \begin{bmatrix} \dfrac{M}{a} - g_d - \zeta \\ -g_d \\ M - g_d - \zeta \end{bmatrix}.$$

Using Cramer's rule and noting that the determinant ,

$$|W| = \begin{vmatrix} z & S_k & 1 \\ 1 & -\dfrac{1}{\omega}(1-S_k) & 0 \\ 1 & \dfrac{1}{\omega}S_k & -\eta \end{vmatrix} = \frac{1}{\omega}\left[(1-S_k)(1+z\eta) + S_k\right] + \eta S_k > 0,$$

we arrive at the solutions that are given in Equations (14), (15) and (16).

Proof of Proposition 1:

(i) The growth rate of real consumption is $\left(\dfrac{\hat{c}}{e}\right) = \hat{c} - \hat{e}$. Using Equations (A.3) and (A.6), it follows that

$$\left(\frac{\hat{c}}{e}\right) = \frac{1}{a}\left[M - s(p)\hat{p}\right]. \tag{A.11}$$

(ii) Equation (A.11) implies that real consumption grows over time as long as $\hat{p} < \dfrac{M}{s(p)}$. From Equation (14), we can decompose $\hat{p}$ as follows; $\hat{p} \equiv \hat{p}_0 + \hat{p}_b + \hat{p}_g$.

where $\hat{p}_0 = \dfrac{\dfrac{M}{\omega}(1-S_k)\left[\dfrac{\eta}{a}+1\right]}{|W|}$, $\hat{p}_b = \dfrac{-\zeta\left[(1-S_k)(\eta+1)\right]}{\omega|W|}$ and

$$\hat{p}_g = \frac{-g_d\left[(1-S_k)(\eta+1)+\omega S_k\left(\eta+\dfrac{1}{\omega}\right)\right]}{\omega|W|}.$$ Then since $\hat{p}_b < 0$ and $\hat{p}_g < 0$, we find that a

sufficient condition for $\hat{p} < \dfrac{M}{s(p)}$ to hold is,

$$\hat{p}_0 = \frac{(1/\omega)M(1-S_k)\left[(\eta/a)+1\right]}{(1/\omega)\left[(1-S_k)(1+z\eta)+S_k\right]+\eta S_k} < \frac{M}{s(p)}. \tag{A.12}$$

Rearranging (A.12) we have,

$$(1-S_k)\left(\frac{\eta}{a}+1\right)s(p) < \left[(1-S_k)(1+z\eta)\right]+S_k+\eta S_k\omega. \tag{A.13}$$

Since $(S_k + \eta S_k\omega) > 0$ and $z \equiv \dfrac{s(p)}{a} + (1-s(p))\sigma$, (A.13) is satisfied if the following inequality holds,

$$\frac{\eta s(p)}{a} + s(p) < 1 + \frac{\eta s(p)}{a} + (1-s(p))\sigma\eta, \tag{A.14}$$

or, equivalently if $0 < (1-s(p))(1+\sigma\eta)$, which is always true for $0 < s(p) < 1$. Thus, we have $\hat{p} < (M/s(p))$ at any finite point of time and for all finite σ and ω. That is, real consumption growth is positive along the equilibrium dynamic path.

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(iii) If $\omega > 1$, then $\lim_{t \to \infty} S_k = 1$ and $\lim_{t \to \infty} \hat{p} = -g_d$ for any $\sigma > 0$. If $\sigma < 1$, $\lim_{t \to \infty} s(p) = 0$. Suppose that $\omega < 1$ and $\sigma > 1$. Then we have $\lim_{t \to \infty} S_k = 0$ and the relative price of dirty goods monotonically increases over time under Assumption 2. It then follows that $\lim_{t \to \infty} s(p) = 0$. In either case we find that $s(p)\hat{p}$ approaches to zero. Thus, from (A.11) it follows that the growth rate of real consumption converges from below to M/a if either $\omega > 1$ or $\sigma > 1$, but not both. When $\omega > 1$, and $\sigma > 1$, then $\lim_{t \to \infty} \hat{p} = -g_d$ and $\lim_{t \to \infty} s(p) = 1$. It follows that $s(p)\hat{p}$ converges to $-g_d$ and the consumption growth rate converges to $(M + g_d)/a$.

(iv) If $\omega < 1$ and $\sigma < 1$, then $\lim_{t \to \infty} S_k = 0$ and $\lim_{t \to \infty} s(p) = 1$.

This implies that $\lim_{t \to \infty} \hat{p} = \dfrac{(1+\eta/a)M - (1+\eta)(\zeta + g_d)}{1 + z\eta} > 0$. But since $\lim_{t \to \infty} s(p) = 1$, we have that $\lim_{t \to \infty} z = 1/a$. It follows that $\lim_{t \to \infty} \hat{p} = M - \dfrac{(1+\eta)(\zeta + g_d)}{1 + (\eta/a)}$. Thus, using this expression in (A.11) and considering the fact that $\lim_{t \to \infty} s(p) = 1$ we have,

$$\lim_{t \to \infty}\left(\widehat{\frac{c}{e}}\right) = \left(\frac{1+\eta}{a+\eta}\right)(\zeta + g_d).$$

Finally, we show that $s(p)\hat{p}$ is increasing over time, meaning that $\left(\widehat{\dfrac{c}{e}}\right)$ converges towards the limit from above. Substituting the definitions of $|W|$ and z into Equation (14) we can write,

$$s\hat{p} = \frac{(1+\eta)\left[\dfrac{1+\eta/a}{1+\eta}M - (\zeta + g_d)\right] + \dfrac{S_k}{1 - S_k}(1+\eta\omega)g_d}{1 + \dfrac{s\eta}{a} + \dfrac{(1-s)}{s}\sigma\eta + \dfrac{S_k}{1 - S_k}(1+\eta\omega)}.$$

Clearly, this expression is increasing in s and decreasing in S_k. If $\sigma < 1$ it follows that s is increasing over time as p increases. Also, since k_d/bx increases over time, the assumption that $\omega < 1$ implies that S_k is falling. Thus, along the equilibrium growth path $s\hat{p}$ is increasing when

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g_d is sufficiently small. Hence, we have that $\left(\dfrac{\hat{c}}{e}\right) = \dfrac{1}{a}\left[M - s(p)\hat{p}\right]$ must be falling over time.

That is, the rate of growth of real consumption converges to a positive rate $\dfrac{1+\eta}{a+\eta}\zeta$ from above.

In other words, if $\sigma < 1$ and $\omega < 1$, then the rate of economic growth is declining over time. To

show that $M / a > \dfrac{1+\eta}{a+\eta}(\zeta + g_d)$ note that this inequality can be written as

$M + \eta M / a > (\zeta + g_d) + \eta(\zeta + g_d)$, which is true under Assumption 2. *QED*

Proof of Proposition 2:

Proposition 1 already shows that the growth rate of real consumption always remains positive for

any positive ω and σ. Here we show that positive growth is accompanied by a decreasing level

of pollution over the long run, that $\lim_{t \to \infty} \hat{x} < 0$ as long as $a > 1$. We first note from Equation (15)

that k_d / bx always increases over time which implies that $\lim_{t \to \infty} S_k = 1$ for $\omega > 1$, and $\lim_{t \to \infty} S_k = 0$

for $\omega < 1$. Then from Equation (14) and Assumption 2, we find that $\lim_{t \to \infty} \hat{p} > 0$ for $\omega < 1$, and

$\lim_{t \to \infty} \hat{p} < 0$ for $\omega > 1$.

Case 1: $\omega > 1$ and $\sigma > 1$

We have $\lim_{t \to \infty} s = 1$; $\lim_{t \to \infty} z = 1/a$; $\lim_{t \to \infty} S_k = 1$.

Plugging these values into Equation (16),

$\lim_{t \to \infty} \hat{x} = \dfrac{1}{(1+\omega\eta)}\left\{\left(\dfrac{M}{a} - \zeta\right) - \omega(M - \zeta) + g_d\left(\dfrac{1}{a} - 1\right)\right\}$. Assumption 2 implies that $\lim_{t \to \infty} \hat{x} < 0$ if

$a > 1$. This is also valid if technological change is absent, $\zeta = g_d = 0$.

Case 2: $\omega > 1$ and $\sigma < 1$

We have $\lim_{t \to \infty} s = 0$; $\lim_{t \to \infty} z = \sigma$; $\lim_{t \to \infty} S_k = 1$.

Plugging these values into Equation (16),

$\lim_{t \to \infty} \hat{x} = \dfrac{1}{(1+\omega\eta)}\left\{\left(\dfrac{M}{a} - \zeta\right) - \omega(M - \zeta) + g_d(\sigma - 1)\right\}$. Assumption 2 implies that $\lim_{t \to \infty} \hat{x} < 0$.

Case 3: $\omega < 1$ and $\sigma > 1$

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We have $\lim\limits_{t\to\infty} s = 0$; $\lim\limits_{t\to\infty} z = \sigma$; $\lim\limits_{t\to\infty} S_k = 0$.

Plugging these values into Equation (16),

$$\lim_{t\to\infty} \hat{x} = \frac{1}{(1+\sigma\eta)}\left\{\left(\frac{M}{a} - \zeta - g_d\right) - \sigma(M - \zeta - g_d)\right\}.$$

Since $a > 1$, we have $\left(\frac{M}{a} - \zeta - g_d\right)/(M - \zeta - g_d) < 1 < \sigma$, and $\lim\limits_{t\to\infty} \hat{x} < 0$.

Case 4: $\omega < 1$ and $\sigma < 1$

We have $\lim\limits_{t\to\infty} s = 1$; $\lim\limits_{t\to\infty} z = 1/a$; $\lim\limits_{t\to\infty} S_k = 0$.

Plugging these values into Equation (16), $\lim\limits_{t\to\infty} \hat{x} = \frac{1}{(1+\eta/a)}\left\{(g_d + \zeta)\left(\frac{1}{a} - 1\right)\right\} < 0$.

Case 5: $\omega \neq 1$ and $\sigma = 1$

We have $0 < s = \beta < 1$ and $z = \frac{\beta}{a} + (1-\beta) < 1$ for $a > 1$. We consider two cases.

If $\omega > 1$, then $\lim\limits_{t\to\infty} S_k = 1$ and $\lim\limits_{t\to\infty} \hat{x} = \frac{1}{1+\omega\eta}\left\{\left(\frac{M}{a} - \zeta\right) - \omega(M - \zeta) + g_d(z-1)\right\}$. Since $z < 1$,

Assumption 2 implies that $\lim\limits_{t\to\infty} \hat{x} < 0$. If $\omega < 1$, then $\lim\limits_{t\to\infty} S_k = 0$ and

$$\lim_{t\to\infty} \hat{x} = \frac{1}{1+\eta z}\left(M\left(\frac{1}{a} - z\right) - \xi(1-z) + g_d(z-1)\right).$$

Since $\frac{1}{a} < z < 1$, we have $\lim\limits_{t\to\infty} \hat{x} < 0$ for $a > 1$.

Case 6: $\omega = 1$ and $\sigma \neq 1$

Since $0 < S_k = \alpha < 1$, we have ;

$$\lim_{t\to\infty} \hat{p} = (\lim_{t\to\infty}\frac{1}{|W|\omega})\left[(1-\alpha)\left((\frac{M}{a} - \zeta)\eta + (M - \zeta)\right) - g_d(1+\eta)\right].$$ It follows that $\lim\limits_{t\to\infty} \hat{p} > (<)0$ if and

only if $g_d < (>)\dfrac{(1-\alpha)\left((\frac{M}{a} - \zeta)\eta + (M - \zeta)\right)}{1+\eta}$. We consider four alternative cases.

(i) $\sigma < 1$ and $g_d < \dfrac{(1-\alpha)\left[(\frac{M}{a}-\zeta)\eta+(M-\zeta)\right]}{1+\eta}$.

Since $\sigma < 1$, we have $\lim\limits_{t\to\infty} s(p)=1$ and $\lim\limits_{t\to\infty} z=1/a$.

It follows that $\lim\limits_{t\to\infty}\hat{x}=\dfrac{\left(\frac{1}{a}-1\right)\left(M\alpha+g_d+\zeta(1-\alpha)\right)}{\left((1-\alpha)\left(1+\frac{\eta}{a}\right)+\alpha(1+\eta)\right)}<0$ for $a>1$ regardless of magnitude of

$g_d>0$.

(ii) $\sigma > 1$ and $g_d < \dfrac{(1-\alpha)\left[(\frac{M}{a}-\zeta)\eta+(M-\zeta)\right]}{1+\eta}$. We have $\lim\limits_{t\to\infty} s(p)=0$ and $\lim\limits_{t\to\infty} z=\sigma$. It

follows that

$\lim\limits_{t\to\infty}\hat{x}=\dfrac{M(\frac{1}{a}-1)-(1-\alpha)(\sigma-1)(M-\zeta)+g_d(\sigma-1)}{(1-\alpha)(1+\sigma\eta)+(1+\eta)\alpha}$. The first term of the numerator is negative,

while the sum of second and third term becomes negative since

$-(1-\alpha)(\sigma-1)(M-\zeta)+g_d(\sigma-1)<-(1-\alpha)(\sigma-1)(M-\zeta)+\dfrac{(1-\alpha)\left[(\frac{M}{a}-\zeta)\eta+(M-\zeta)\right](\sigma-1)}{1+\eta}<0$

(iii) $\sigma < 1$ and $g_d > \dfrac{(1-\alpha)\left[(\frac{M}{a}-\zeta)\eta+(M-\zeta)\right]}{1+\eta}$ .We have $\lim\limits_{t\to\infty} s(p)=0$ and $\lim\limits_{t\to\infty} z=\sigma$.

It follows that $\lim\limits_{t\to\infty}\hat{x}=\dfrac{\left(\frac{1}{a}-\sigma\right)M+M\alpha(\sigma-1)+g_d(\sigma-1)+\zeta(1-\alpha)(\sigma-1)}{(1-\alpha)(1+\sigma\eta)+(1+\eta)\alpha}<0$ for $a>1$.

(iv) $\sigma > 1$ and $\quad g_d > \dfrac{(1-\alpha)\left((\dfrac{M}{a}-\zeta)\eta + (M-\zeta)\right)}{1+\eta}$. We have $\lim\limits_{t\to\infty} s(p) = 1$ and $\lim\limits_{t\to\infty} z = 1/a$. It

follows that $\lim\limits_{t\to\infty} \hat{x} = \dfrac{\left(\dfrac{1}{a}-1\right)\left(M\alpha + g_d + \zeta(1-\alpha)\right)}{\left((1-\alpha)\left(1+\dfrac{\eta}{a}\right) + \alpha(1+\eta)\right)} < 0$ for $a > 1$.

Case 7: $\omega = 1$ and $\sigma = 1$

We always have $0 < S_k = \alpha < 1$, $\quad 0 < s(p) = \beta < 1$, and $\quad z = \dfrac{\beta}{a} + (1-\beta) < 1$.

Then $\lim\limits_{t\to\infty} \hat{x} < 0$ if and only if $M\left(\dfrac{1}{a} - (1-\alpha)z - \alpha\right) - \zeta\left(1 - (1-\alpha)z - \alpha\right) < 0$.

Rearranging, we have,

$$M\left(\dfrac{1}{a} - (1-\alpha)z - \alpha\right) - \zeta\left(1 - (1-\alpha)z - \alpha\right) = \left[M\left(\dfrac{1}{a} - z\right) - \zeta(1-z)\right] + (M-\zeta)\alpha(z-1).$$

The first term is negative since $\dfrac{(M/a - \zeta)}{(M-\zeta)} < \dfrac{1}{a} < \dfrac{\beta}{a} + (1-\beta) = z$, and the second term is also

negative since $z < 1$. QED

Proof of Proposition 3:

(i) First we assume $\omega > 1$. For any $\sigma > 0$, Equation (18) applies with $g_d = 0$,

$$\lim\limits_{t\to\infty} \hat{x} = \dfrac{M\left(\dfrac{1}{a\omega} - 1\right) - \zeta\left(\dfrac{1}{\omega} - 1\right)}{\dfrac{1}{\omega} + \eta} < 0 \text{ if and only if } \omega > d(M,a;\zeta,0) = \dfrac{\dfrac{M}{a} - \zeta}{M - \zeta}. \text{ Since the minimum}$$

value of $d(M,a;\zeta,0)$ is $\dfrac{1}{a} > 1$ for $0 < a < 1$, we have $d(M,a;\zeta,0) > 1$.

(ii) Consider now the case where $\omega < 1$. If $\sigma > 1$, Equation (18) applies with $g_d = 0$,

$$\lim\limits_{t\to\infty} \hat{x} = \dfrac{M\left(\dfrac{1}{a} - \sigma\right) - \zeta(1-\sigma)}{(1+\sigma\eta)} < 0 \text{ if and only if } \sigma > d(M,a;\zeta,0) = \dfrac{\dfrac{M}{a} - \zeta}{M - \zeta} > 1 \text{ for } 0 < a < 1. \otimes$$

Proof of Corollary 2:

If we allow capital-augmenting technological change, $\dot{n}/n = \theta > 0$, in addition to pollution-augmenting and neutral technological change in the dirty sector, the equilibrium growth rates of

$\hat{p}$, $\left(\dfrac{\hat{nk_d}}{bx}\right)$ and $\hat{x}$ become as follow:

$$\hat{p} = \frac{1}{|W|\omega}\left[\begin{array}{l}\left[M(1-S_k)\left(\dfrac{\eta}{a}+1\right)\right]-g_d\left[(1-S_k)(\eta+1)+\omega S_k\left(\eta+\dfrac{1}{\omega}\right)\right]\\ -\zeta\left[(1-S_k)(\eta+1)\right]-\theta S_k(\eta+\dfrac{1}{\omega})\end{array}\right],\qquad (A.16)$$

$$\left(\frac{\hat{nk_d}}{bx}\right) = \frac{1}{|W|}\left[M\left(\frac{\eta}{a}+1\right)+g_d\eta(z-1)-\zeta(\eta+1)+\theta(z\eta+1)\right]>0 \qquad (A.17)$$

$$\hat{x} = \frac{1}{|W|\omega}\left\{M\left(\frac{1}{a}-z(1-S_k)-\omega S_k\right)+g_d(z-1)+\theta S_k(z-\omega)+\zeta\left(z(1-S_k)+\omega S_k-1\right)\right\}, \qquad (A.18)$$

where $|W|\equiv\dfrac{1}{\omega}\left[(1-S_k)(1+z\eta)+S_k\right]+\eta S_k>0$.

We prove Corollary 2 for all different cases of parameter combinations.

Case 1: $\omega>1$ and $\sigma>1$

By Equation (A.17) for $\omega>1$, we have $\lim_{t\to\infty} S_k=1$. Plugging this into Equation (A.16), we have;

$\lim_{t\to\infty}\hat{p}=-\dfrac{1}{1+\eta\omega}\left(\eta+\dfrac{1}{\omega}\right)(g_d+\theta)<0$. It follows that for $\sigma>1$, $\lim_{t\to\infty} s=1$, and $\lim_{t\to\infty} z=1/a$. Then

Equation (A.18) implies; $\lim_{t\to\infty}\hat{x}=\dfrac{1}{|W|\omega}\left\{\left(\dfrac{M}{a}-\zeta\right)-\omega(M-\zeta)+g_d\left(\dfrac{1}{a}-1\right)+\theta\left(\dfrac{1}{a}-\omega\right)\right\}$. Since

$\left(\dfrac{M}{a}-\zeta\right)-\omega(M-\zeta)<0$ for $a>1$ and $\omega>1$, it follows that $\lim_{t\to\infty}\hat{x}<0$.

Case 2: $\omega>1$ and $\sigma<1$

By Equation (A.17) for $\omega>1$, we have $\lim_{t\to\infty} S_k=1$. Plugging this into Equation (A.16), we have;

$\lim_{t\to\infty}\hat{p}=-\dfrac{1}{1+\eta\omega}\left(\eta+\dfrac{1}{\omega}\right)(g_d+\theta)<0$. It follows that for $\sigma<1$, $\lim_{t\to\infty} s=0$ and $\lim_{t\to\infty} z=\sigma$. Then

Equation (A.18) becomes; $\lim\limits_{t\to\infty}\hat{x}=\dfrac{1}{|W|\omega}\left\{\left(\dfrac{M}{a}-\zeta\right)-\omega(M-\zeta)+g_d(\sigma-1)+\theta(\sigma-\omega)\right\}$. We find

that $\lim\limits_{t\to\infty}\hat{x}<0$ if $\left(\dfrac{M}{a}-\zeta\right)<\omega(M-\zeta)$, which is always true for $a>1$.

Case 3: $\omega<1$ and $\sigma>1$

By Equation (A.17) for $\omega>1$, we have $\lim\limits_{t\to\infty}S_k=0$. Plugging this into Equation (A.16), we have;

$\lim\limits_{t\to\infty}\hat{p}=\dfrac{1}{\left(\lim\limits_{t\to\infty}|W|\omega\right)}\left(M\left(\dfrac{\eta}{a}+1\right)-(g_d+\zeta)(\eta+1)\right)>0.$ Therefore for $\sigma>1$, we have that $\lim\limits_{t\to\infty}s=0$

and $\lim\limits_{t\to\infty}z=\sigma$ so that $\lim\limits_{t\to\infty}\hat{p}=\dfrac{1}{1+\sigma\eta}\left(M\left(\dfrac{\eta}{a}+1\right)-(g_d+\zeta)(\eta+1)\right)>0.$ Then by Equation (A.18),

$\lim\limits_{t\to\infty}\hat{x}=\dfrac{1}{1+\sigma\eta}\left\{M\left(\dfrac{1}{a}-\sigma\right)+(g_d+\zeta)(\sigma-1)\right\}<0$ if and only if $\sigma>\dfrac{\dfrac{M}{a}-\zeta-g_d}{M-\zeta-g_d}=h_3(\zeta,g_d).$

For $a>1$, this requirement is automatically satisfied since $h_3(\zeta,g_d)<1$.

Case 4: $\omega<1$ and $\sigma<1$

From Equation (A.17) for $\omega>1$, we have $\lim\limits_{t\to\infty}S_k=0$. It follows that $\lim\limits_{t\to\infty}\hat{p}>0$. Since $\sigma<1$, we

have that $\lim\limits_{t\to\infty}s=1$ and $\lim\limits_{t\to\infty}z=1/a$, and therefore $\lim\limits_{t\to\infty}\hat{p}=M-\dfrac{1+\eta}{1+(\eta/a)}(\zeta+g_d)>0$ and

$\lim\limits_{t\to\infty}(\hat{c}/e)=\left(\dfrac{1+\eta}{a+\eta}\right)(\zeta+g_d).$ By Equation (A.18), $\lim\limits_{t\to\infty}\hat{x}=\dfrac{\left(\dfrac{1}{a}-1\right)(g_d+\zeta)}{\left(1+(\eta/a)\right)}<0$ for $a>1$.

Case 5: $\omega=1,\ \sigma\neq1$

Since $0<S_k=\alpha<1$ we have,

$\lim\limits_{t\to\infty}\hat{p}=(\lim\limits_{t\to\infty}\dfrac{1}{|W|\omega})\left[(1-\alpha)\left((\dfrac{M}{a}-\zeta)\eta+(M-\zeta)\right)-g_d(1+\eta)-\theta\alpha\left(\eta+\dfrac{1}{\omega}\right)\right].$

It follows that $\lim\limits_{t\to\infty}\hat{p}>(<)0$ if and only if

$g_d<(>)\dfrac{(1-\alpha)\left(\dfrac{M}{a}-\zeta)\eta+(M-\zeta)\right)-\theta\alpha(1+\eta)}{1+\eta}=\bar{g}$.

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We consider four different sub-cases.

5-1) $\sigma < 1$ and $g_d < \bar{g}$: We have $\lim\limits_{t\to\infty} s(p) = 1$ and $\lim\limits_{t\to\infty} z = 1/a$. It follows that

$$\lim_{t\to\infty} \hat{x} = \frac{\left(\frac{1}{a}-1\right)\left(M\alpha + g_d + \zeta(1-\alpha) + \theta\alpha\right)}{\left((1-\alpha)\left(1+\frac{\eta}{a}\right) + \alpha(1+\eta)\right)} < 0 \quad \text{for} \quad a > 1 \quad \text{regardless of magnitude of}$$

$g_d > 0$.

5-2) $\sigma > 1$ and $g_d < \bar{g}$: We have $\lim\limits_{t\to\infty} s(p) = 0$ and $\lim\limits_{t\to\infty} z = \sigma$. It follows that

$$\lim_{t\to\infty} \hat{x} = \frac{\left[M\left(\frac{1}{a}-1\right)\right] - \left[(1-\alpha)(\sigma-1)(M-\zeta)\right] + \left[(\sigma-1)(g_d + \theta\alpha)\right]}{(1-\alpha)(1+\sigma\eta) + (1+\eta)\alpha} \, .$$

The first term of the numerator is negative, while the sum of second and third term becomes negative since

$$-(1-\alpha)(\sigma-1)(M-\zeta) + (\sigma-1)(g_d + \theta\alpha) <$$

$$-(1-\alpha)(\sigma-1)(M-\zeta) + \left[\frac{(1-\alpha)\left(\left(\frac{M}{a}-\zeta\right)\eta + (M-\zeta)\right) - \theta\alpha(\eta+1) + \theta\alpha(1+\eta)}{1+\eta}\right](\sigma-1) < 0.$$

Therefore, $\lim\limits_{t\to\infty} \hat{x} < 0$.

5-3) $\sigma < 1$ and $g_d > \bar{g}$: We have $\lim\limits_{t\to\infty} s(p) = 0$ and $\lim\limits_{t\to\infty} z = \sigma$. It follows that

$$\lim_{t\to\infty} \hat{x} = \frac{\left(\frac{1}{a}-\sigma\right)M + M\alpha(\sigma-1) + g_d(\sigma-1) + \zeta(1-\alpha)(\sigma-1) + \theta\alpha(\sigma-1)}{(1-\alpha)(1+\sigma\eta) + (1+\eta)\alpha} < 0 \quad \text{for } a > 1.$$

5-4) $\sigma > 1$ and $g_d > \bar{g}$: We have $\lim\limits_{t\to\infty} s(p) = 1$ and $\lim\limits_{t\to\infty} z = 1/a$. It follows that

$$\lim_{t\to\infty} \hat{x} = \frac{\left(\frac{1}{a}-1\right)\left(M\alpha + g_d + \zeta(1-\alpha) + \theta\alpha\right)}{\left((1-\alpha)\left(1+\frac{\eta}{a}\right) + \alpha(1+\eta)\right)} < 0 \quad \text{for } a > 1.$$

Case 6: $\omega = 1$ and $\sigma = 1$

We always have $0 < S_k = \alpha < 1$, $0 < s(p) = \beta < 1$, and

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$z = \dfrac{\beta}{a} + (1-\beta) < 1$. Equation (A.18) implies that $\lim_{t\to\infty} \hat{x} < 0$ if and only if

$M\left(\dfrac{1}{a} - (1-\alpha)z - \alpha\right) - \zeta\left(1 - (1-\alpha)z - \alpha\right) + \theta\alpha(z-1) < 0$. Rearranging terms in the left-hand side,

we have,

$$M\left(\frac{1}{a} - (1-\alpha)z - \alpha\right) - \zeta\left(1 - (1-\alpha)z - \alpha\right) + \theta\alpha(z-1)$$

$$= \left[M\left(\frac{1}{a} - z\right) - \zeta(1-z)\right] + (M - \zeta + \theta)\alpha(z-1).$$

The first term is negative since $\dfrac{(M/a - \zeta)}{(M - \zeta)} < \dfrac{1}{a} < \dfrac{\beta}{a} + (1-\beta) = z$, and the second term is also

negative since $z < 1$. *QED*