AJAE Appendix for “Crop Yield Skewness Under Law of the Minimum Technology”

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Note: The material contained herein is supplementary to the article named in the title and published in the American Journal of Agricultural Economics (AJAE).

Uniform Distribution; Heterogeneity in Means:

The moments are

$$\mu_y = 0.5(0) + 0.25\tau + 0.25(1) = \frac{\tau+1}{4},$$

$$\sigma_y^2 = 0.5\left(0 - \frac{(\tau+1)}{4}\right)^2 + 0.25\left(\tau - \frac{(\tau+1)}{4}\right)^2 + 0.25\left(1 - \frac{(\tau+1)}{4}\right)^2$$
$$= \frac{(\tau+1)^2}{32} + \frac{(3\tau-1)^2}{64} + \frac{(3-\tau)^2}{64} = \frac{2\tau^2 + 4\tau + 2 + 9\tau^2 - 6\tau + 1 + 9 - 6\tau + \tau^2}{64}$$
$$= \frac{12\tau^2 - 8\tau + 12}{64} = \frac{3\tau^2 - 2\tau + 3}{16},$$

$$\zeta_y = 0.5\left(0 - \frac{(\tau+1)}{4}\right)^3 + 0.25\left(\tau - \frac{(\tau+1)}{4}\right)^3 + 0.25\left(1 - \frac{(\tau+1)}{4}\right)^3$$
$$= -\frac{(\tau+1)^3}{128} + \frac{(3\tau-1)^3}{256} + \frac{(3-\tau)^3}{256}$$
$$= -\frac{2\tau^3 + 6\tau^2 + 6\tau + 2}{256} + \frac{27\tau^3 - 27\tau^2 + 9\tau - 1}{256} + \frac{27 - 27\tau + 9\tau^2 - \tau^3}{256}$$
$$= \frac{24\tau^3 - 24\tau^2 - 24\tau + 24}{256} = \frac{3(\tau^3 - \tau^2 - \tau + 1)}{32} = \frac{3(1-\tau)^2(1+\tau)}{32},$$

$$\gamma_y = \frac{(3/32)(1-\tau)^2(1+\tau)}{\left(\frac{3\tau^2 - 2\tau + 3}{16}\right)^{3/2}} = \frac{6(1-\tau)^2(1+\tau)}{\left(3\tau^2 - 2\tau + 3\right)^{3/2}} > 0.$$

So, for (9),

$$\frac{d\gamma_y}{d\tau} = \frac{\left(3\tau^2 - 2\tau + 3\right)^{3/2} 6\left[(1-\tau)^2 - 2(1-\tau)(1+\tau)\right] - 9(1-\tau)^2(1+\tau)(6\tau-2)\left(3\tau^2 - 2\tau + 3\right)^{1/2}}{\left(3\tau^2 - 2\tau + 3\right)^3}$$

$$= \frac{\left(3\tau^2 - 2\tau + 3\right)6(1-\tau)\left[-1-3\tau\right] - 9(1-\tau)^2(1+\tau)(6\tau-2)}{\left(3\tau^2 - 2\tau + 3\right)^{2.5}}$$

$$= -6(1-\tau)\frac{\left(3\tau^2 - 2\tau + 3\right)\left[1+3\tau\right] + 3(1-\tau)(1+\tau)(3\tau-1)}{\left(3\tau^2 - 2\tau + 3\right)^{2.5}}$$

$$= -6(1-\tau)\frac{3\tau^2 - 2\tau + 3 + 9\tau^3 - 6\tau^2 + 9\tau + 3(1-\tau^2)(3\tau-1)}{\left(3\tau^2 - 2\tau + 3\right)^{2.5}} = -\frac{6(1-\tau)(16\tau)}{\left(3\tau^2 - 2\tau + 3\right)^{2.5}} < 0.$$

Uniform Distribution; Heterogeneity in Variances:

The moments are

$$\mu_y = 0.5(0) + 0.25\tau + 0.25(1-\tau) = 0.25,$$

$$\sigma_y^2 = 0.5\left(0 - 0.25\right)^2 + 0.25\left(\tau - 0.25\right)^2 + 0.25\left(1 - \tau - 0.25\right)^2$$

$$= \frac{\frac{2}{16} + \frac{(4\tau-1)^2}{16} + \frac{(3-4\tau)^2}{16}}{4} = \frac{2 + 16\tau^2 - 8\tau + 1 + 9 - 24\tau + 16\tau^2}{64}$$

$$= \frac{12 + 32\tau^2 - 32\tau}{64} = \frac{8\tau^2 - 8\tau + 3}{16}.$$

$$\zeta_y = 0.5\left(0 - 0.25\right)^3 + 0.25\left(\tau - 0.25\right)^3 + 0.25\left(1 - \tau - 0.25\right)^3$$

$$= \frac{-\frac{2}{64} + \frac{(4\tau-1)^3}{64} + \frac{(3-4\tau)^3}{64}}{4}$$

$$= \frac{-2 + \left(64\tau^3 - 48\tau^2 + 12\tau - 1\right) + \left(27 - 108\tau + 144\tau^2 - 64\tau^3\right)}{256}$$

$$= \frac{3 - 12\tau + 12\tau^2}{32} = \frac{3(1-2\tau)^2}{32}.$$

Turning to (10), use $\gamma_y = 6(1-2\tau)^2(8\tau^2 - 8\tau + 3)^{-3/2}$ to obtain

$$\frac{d\gamma_y}{d\tau} = 6\left[\frac{-4(8\tau^2-8\tau+3)^{3/2}(1-2\tau)-1.5(1-2\tau)^2(16\tau-8)(8\tau^2-8\tau+3)^{1/2}}{(8\tau^2-8\tau+3)^3}\right]$$

$$= 6(1-2\tau)\left[\frac{12(1-2\tau)^2-4(8\tau^2-8\tau+3)}{(8\tau^2-8\tau+3)^{2.5}}\right] = 24(1-2\tau)\left[\frac{3(1-2\tau)^2-(8\tau^2-8\tau+3)}{(8\tau^2-8\tau+3)^{2.5}}\right]$$

$$= 24(1-2\tau)\left[\frac{-4\tau+4\tau^2}{(8\tau^2-8\tau+3)^{2.5}}\right] = \frac{96(2\tau-1)\tau(1-\tau)}{\left(3-8\tau+8\tau^2\right)^{2.5}} \overset{sign}{=} \tau-0.5 \le 0.$$

Dependence in Uniform Distribution:

Use (11) to obtain the moments as

$$\mu_y = 0.25(1+\rho)(0)+0.25(1-\rho)(0)+0.25(1-\rho)(0)+0.25(1+\rho)(1) = 0.25(1+\rho),$$

$$\sigma_y^2 = 0.25(1+\rho)\left[0-0.25(1+\rho)\right]^2 + 0.25(1-\rho)\left[0-0.25(1+\rho)\right]^2$$
$$+0.25(1-\rho)\left[0-0.25(1+\rho)\right]^2 + 0.25(1+\rho)\left[1-0.25(1+\rho)\right]^2$$
$$= 0.25(3-\rho)\left[0.25(1+\rho)\right]^2 + 0.25(1+\rho)\left[0.75-0.25\rho\right]^2$$
$$= 0.25(1+\rho)(3-\rho)\left\{(0.25)^2(1+\rho)+(0.25)^2(3-\rho)\right\}$$
$$= \frac{(3-\rho)(1+\rho)}{16},$$

$$\zeta_y = 0.25(1+\rho)\left[0-0.25(1+\rho)\right]^3 + 0.25(1-\rho)\left[0-0.25(1+\rho)\right]^3$$
$$+0.25(1-\rho)\left[0-0.25(1+\rho)\right]^3 + 0.25(1+\rho)\left[1-0.25(1+\rho)\right]^3$$
$$= -0.25(3-\rho)\left[0.25(1+\rho)\right]^3 + 0.25(1+\rho)\left[0.75-0.25\rho\right]^3$$
$$= 0.25\left\{(1+\rho)(0.25)^3\left[3-\rho\right]^3 - (3-\rho)(0.25)^3(1+\rho)^3\right\}$$
$$= (0.25)^4(1+\rho)(3-\rho)\left\{(3-\rho)^2-(1+\rho)^2\right\}$$
$$= (0.25)^4(1+\rho)(3-\rho)\left\{8-8\rho\right\} = (0.25)^3(1+\rho)(3-\rho)2(1-\rho)$$
$$= (3-\rho)(1+\rho)(1-\rho)/32,$$

$$\gamma_y = \frac{(3-\rho)(1+\rho)(1-\rho)/32}{[(3-\rho)(1+\rho)]^{3/2}/(16)^{3/2}} = \frac{2(1-\rho)}{(3-\rho)^{1/2}(1+\rho)^{1/2}}.$$

Use this last expression to establish (12),

$$\frac{d\gamma_y}{d\rho}=\frac{(3-\rho)^{1/2}(1+\rho)^{1/2}(-2)-2(1-\rho)\left\{0.5\frac{(3-\rho)^{1/2}}{(1+\rho)^{1/2}}-0.5\frac{(1+\rho)^{1/2}}{(3-\rho)^{1/2}}\right\}}{(3-\rho)(1+\rho)}$$

$$=\frac{(3-\rho)^{1/2}(1+\rho)^{1/2}(-2)-\dfrac{2(1-\rho)^2}{(1+\rho)^{1/2}(3-\rho)^{1/2}}}{(3-\rho)(1+\rho)}$$

$$=-2\left\{\frac{(3-\rho)(1+\rho)+(1-\rho)^2}{(1+\rho)^{3/2}(3-\rho)^{3/2}}\right\}=-\frac{8}{(3-\rho)^{3/2}(1+\rho)^{3/2}}<0.$$

Moments of Distribution with Density:

(A1) $$g(y)=\frac{\text{Cos}(y)[1-\text{Sin}(y)]}{2}.$$

Integrate

$$\int_{-\pi/2}^{\pi/2} yg(y)dy=0.5\int_{-\pi/2}^{\pi/2} y\text{Cos}(y)[1-\text{Sin}(y)]dy$$

$$=\left[0.5\text{Cos}(x)+0.125x\text{Cos}(2x)+0.5x\text{Sin}(x)-0.0625\text{Sin}(2x)\right]_{-\pi/2}^{\pi/2}$$

(A2) $$=\left[0.125x\text{Cos}(2x)+0.5x\text{Sin}(x)\right]_{-\pi/2}^{\pi/2}$$

$$=0.125\left[(\pi/2)\text{Cos}(\pi)+2\pi\text{Sin}(\pi/2)-(-\pi/2)\text{Cos}(-\pi)-(-2\pi)\text{Sin}(-\pi/2)\right]$$

$$=0.125\left[(\pi/2)(-1)+2\pi(1)+(\pi/2)(-1)+2\pi(-1)\right]=0.125\left[-\pi\right]=-0.3927.$$

$$\int_{-\pi/2}^{\pi/2} y^2g(y)dy=0.5\int_{-\pi/2}^{\pi/2} y^2\text{Cos}(y)[1-\text{Sin}(y)]dy$$

$$=\left[x\text{Cos}(x)+0.0625(2x^2-1)\text{Cos}(2x)+0.5(x^2-2)\text{Sin}(x)-0.125x\text{Sin}(2x)\right]_{-\pi/2}^{\pi/2}$$

$$=\left[0.0625(2x^2-1)\text{Cos}(2x)+0.5(x^2-2)\text{Sin}(x)\right]_{-\pi/2}^{\pi/2}$$

(A3) $$=0.0625\left[\begin{array}{l}\left(\dfrac{\pi^2}{2}-1\right)\text{Cos}(\pi)+8\left(\dfrac{\pi^2}{4}-2\right)\text{Sin}(\pi/2)\\ -\left(\dfrac{\pi^2}{2}-1\right)\text{Cos}(-\pi)-8\left(\dfrac{\pi^2}{4}-2\right)\text{Sin}(-\pi/2)\end{array}\right]$$

$$=0.0625\left[\left(\frac{\pi^2}{2}-1\right)(-1)+8\left(\frac{\pi^2}{4}-2\right)(1)-\left(\frac{\pi^2}{2}-1\right)(-1)-8\left(\frac{\pi^2}{4}-2\right)(-1)\right]$$

$$=0.5\left[\left(\frac{\pi^2}{4}-2\right)+\left(\frac{\pi^2}{4}-2\right)\right]=\left(\frac{\pi^2}{4}-2\right)=0.467397.$$

$$\int_{-\pi/2}^{\pi/2} y^3 g(y)dy = 0.5\int_{-\pi/2}^{\pi/2} y^3\text{Cos}(y)[1-\text{Sin}(y)]dy$$

$$= \begin{bmatrix} 1.5(x^2-2)\text{Cos}(x)+0.0625x(2x^2-3)\text{Cos}(2x) \\ +0.5x(x^2-6)\text{Sin}(x)-0.09375(2x^2-1)\text{Sin}(2x) \end{bmatrix}_{-\pi/2}^{\pi/2}$$

$$= \left[0.0625x(2x^2-3)\text{Cos}(2x)+0.5x(x^2-6)\text{Sin}(x)\right]_{-\pi/2}^{\pi/2}$$

(A4)

$$= 0.0625\begin{bmatrix} \frac{\pi}{2}\left(\frac{\pi^2}{2}-3\right)\text{Cos}(\pi)+4\pi\left(\frac{\pi^2}{4}-6\right)\text{Sin}(\pi/2) \\ +\frac{\pi}{2}\left(\frac{\pi^2}{2}-3\right)\text{Cos}(-\pi)+4\pi\left(\frac{\pi^2}{4}-6\right)\text{Sin}(-\pi/2) \end{bmatrix}$$

$$= 0.0625\left[\frac{\pi}{2}\left(\frac{\pi^2}{2}-3\right)(-1)+\frac{\pi}{2}\left(\frac{\pi^2}{2}-3\right)(-1)\right]$$

$$= -0.0625\pi\left(\frac{\pi^2}{2}-3\right) = -0.379896.$$

(A5)

$$\gamma_y = \frac{E[(y-\mu_y)^3]}{\sigma_y^3} = \frac{E[y^3]-3\mu_y E[y^2]+2\mu_y^3}{\sigma_y^3}$$

$$= \frac{-0.3799-3(-0.3927)(0.4674)+2(-0.3927)^3}{\left[0.4674-(-0.3927)^2\right]^{3/2}} = \frac{0.17074-2(0.3927)^3}{0.175269} = 0.283.$$

Moments of Distribution with Density:

(A6) $$g(y) = \frac{4\text{Cos}^2(y)\left\{0.5-([y+\text{Cos}(y)\text{Sin}(y)])/\pi\right\}}{\pi}.$$

Integrate

(A7)

$$\int_{-\pi/2}^{\pi/2} yg(y)dy = \frac{4}{\pi}\int_{-\pi/2}^{\pi/2} y\text{Cos}^2(y)\left\{0.5-\frac{[y+\text{Cos}(y)\text{Sin}(y)]}{\pi}\right\}dy$$

$$= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} y\text{Cos}^2(y)dy - \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} y^2\text{Cos}^2(y)dy - \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} y\text{Cos}^3(y)\text{Sin}(y)dy.$$

Now

(A8) $$\frac{2}{\pi}\int_{-\pi/2}^{\pi/2} y\text{Cos}^2(y)dy = \frac{1}{4\pi}\left[2x^2+\text{Cos}(2x)+2x\text{Sin}(2x)\right]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi}\left[x\text{Sin}(2x)\right]_{-\pi/2}^{\pi/2} = 0;$$

$$\frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} y^2\mathrm{Cos}^2(y)dy = \frac{4}{\pi^2}\left[\frac{x^3}{6}+\frac{x\mathrm{Cos}(2x)}{4}+\frac{(2x^2-1)\mathrm{Sin}(2x)}{8}\right]_{-\pi/2}^{\pi/2}$$

(A9)
$$= \frac{4}{\pi^2}\left[\frac{x^3}{6}+\frac{x\mathrm{Cos}(2x)}{4}\right]_{-\pi/2}^{\pi/2} = \frac{4}{\pi^2}\left[\frac{\pi^3}{24}+\frac{\pi\mathrm{Cos}(\pi)}{8}+\frac{\pi\mathrm{Cos}(-\pi)}{8}\right]$$

$$= \frac{4}{\pi^2}\left[\frac{\pi^3}{24}-\frac{\pi}{4}\right] = \frac{\pi^2-6}{6\pi}.$$

$$\frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^3(y)\mathrm{Sin}(y)dy = \frac{4}{\pi^2}\left[\frac{\mathrm{Sin}(4x)}{128}-\frac{x\mathrm{Cos}(4x)}{32}+\frac{\mathrm{Sin}(2x)}{16}-\frac{x\mathrm{Cos}(2x)}{8}\right]_{-\pi/2}^{\pi/2}$$

$$= -\frac{1}{8\pi^2}\left[x\mathrm{Cos}(4x)+4x\mathrm{Cos}(2x)\right]_{-\pi/2}^{\pi/2}$$

(A10)
$$= -\frac{1}{8\pi^2}\left[\frac{\pi}{2}\mathrm{Cos}(2\pi)+\frac{\pi}{2}\mathrm{Cos}(-2\pi)+2\pi\mathrm{Cos}(\pi)+2\pi\mathrm{Cos}(-\pi)\right]$$

$$= -\frac{1}{8\pi^2}\left[\frac{\pi}{2}+\frac{\pi}{2}+2\pi(-1)+2\pi(-1)\right] = \frac{3}{8\pi}.$$

So

$$\int_{-\pi/2}^{\pi/2} yg(y)dy = -\frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} y^2\mathrm{Cos}^2(y)dy - \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^3(y)\mathrm{Sin}(y)dy$$

(A11)
$$= -\frac{\pi^2-6}{6\pi}-\frac{3}{8\pi} = \frac{1}{2\pi}\left(\frac{5}{4}-\frac{\pi^2}{3}\right) = -0.3247.$$

Also,

$$\int_{-\pi/2}^{\pi/2} y^2 g(y)dy = \frac{4}{\pi}\int_{-\pi/2}^{\pi/2} y^2\mathrm{Cos}^2(y)\left\{0.5-\frac{[y+\mathrm{Cos}(y)\mathrm{Sin}(y)]}{\pi}\right\}dy$$

(A12)
$$= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} y^2\mathrm{Cos}^2(y)dy - \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} y^3\mathrm{Cos}^2(y)dy - \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} y^2\mathrm{Cos}^3(y)\mathrm{Sin}(y)dy.$$

$$\frac{2}{\pi}\int_{-\pi/2}^{\pi/2} y^2\mathrm{Cos}^2(y)dy = \frac{2}{\pi}\left[\frac{x^3}{6}+\frac{x\mathrm{Cos}(2x)}{4}+\frac{(2x^2-1)\mathrm{Sin}(2x)}{8}\right]_{-\pi/2}^{\pi/2}$$

(A13)
$$= \frac{2}{\pi}\left[\frac{x^3}{6}+\frac{x\mathrm{Cos}(2x)}{4}\right]_{-\pi/2}^{\pi/2} = \frac{2}{\pi}\left[\frac{\pi^3}{24}+\frac{\pi\mathrm{Cos}(\pi)}{8}+\frac{\pi\mathrm{Cos}(-\pi)}{8}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{24}-\frac{\pi}{8}-\frac{\pi}{8}\right] = \frac{\pi^2-6}{12}.$$

$$\text{(A14)}\quad \begin{aligned}&\frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} y^3\text{Cos}^2(y)dy = \frac{4}{\pi^2}\left[\frac{x^4}{8}+\frac{3(2x^2-1)\text{Cos}(2x)}{16}+\frac{x(2x^2-3)\text{Sin}(2x)}{8}\right]_{-\pi/2}^{\pi/2}\\ &= \frac{4}{\pi^2}\left[\frac{3(2x^2-1)\text{Cos}(2x)}{16}\right]_{-\pi/2}^{\pi/2} = \frac{3}{4\pi^2}\left[(2x^2-1)\text{Cos}(2x)\right]_{-\pi/2}^{\pi/2} = 0.\end{aligned}$$

$$\text{(A15)}\quad \begin{aligned}&\frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} y^2\text{Cos}^3(y)\text{Sin}(y)dy\\ &= \frac{4}{\pi^2}\left[\frac{x\text{Sin}(2x)}{8}-\frac{(2x^2-1)\text{Cos}(2x)}{16}+\frac{x\text{Sin}(4x)}{64}-\frac{(8x^2-1)\text{Cos}(4x)}{256}\right]_{-\pi/2}^{\pi/2}\\ &= -\frac{4}{\pi^2}\left[\frac{(2x^2-1)\text{Cos}(2x)}{16}+\frac{(8x^2-1)\text{Cos}(4x)}{256}\right]_{-\pi/2}^{\pi/2} = 0.\end{aligned}$$

So

$$\text{(A16)}\quad \int_{-\pi/2}^{\pi/2} y^2 g(y)dy = \frac{\pi^2-6}{12} = 0.322466.$$

Also,

$$\text{(A17)}\quad \begin{aligned}&\int_{-\pi/2}^{\pi/2} y^3 g(y)dy = \frac{4}{\pi}\int_{-\pi/2}^{\pi/2} y^3\text{Cos}^2(y)\left\{0.5-\frac{[y+\text{Cos}(y)\text{Sin}(y)]}{\pi}\right\}dy\\ &= \overbrace{\frac{2}{\pi}\int_{-\pi/2}^{\pi/2} y^3\text{Cos}^2(y)dy}^{\text{0, odd function}} - \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} y^4\text{Cos}^2(y)dy - \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} y^3\text{Cos}^3(y)\text{Sin}(y)dy\\ &= -\frac{4}{\pi^2}\left\{\int_{-\pi/2}^{\pi/2} y^4\text{Cos}^2(y)dy + \int_{-\pi/2}^{\pi/2} y^3\text{Cos}^3(y)\text{Sin}(y)dy\right\}.\end{aligned}$$

$$\text{(A18)}\quad \begin{aligned}&\int_{-\pi/2}^{\pi/2} y^4\text{Cos}^2(y)dy = \left[\frac{x^5}{10}+\frac{x(2x^2-3)\text{Cos}(2x)}{4}+\frac{(2x^4-6x^2+3)\text{Sin}(2x)}{8}\right]_{-\pi/2}^{\pi/2}\\ &= \left[\frac{x^5}{10}+\frac{x(2x^2-3)\text{Cos}(2x)}{4}\right]_{-\pi/2}^{\pi/2}\\ &= \left[\frac{(\pi/2)^5}{5}+\frac{(\pi/2)(2(\pi/2)^2-3)\text{Cos}(\pi)}{4}-\frac{(-\pi/2)(2(\pi/2)^2-3)\text{Cos}(-\pi)}{4}\right]\\ &= \frac{(\pi/2)^5}{5}-\frac{(\pi/2)(2(\pi/2)^2-3)}{4}-\frac{(\pi/2)(2(\pi/2)^2-3)}{4}\\ &= \frac{\pi^5}{160}-\frac{\pi((\pi^2/2)-3)}{4} = 1.9126149-1.51958 = 0.393033.\end{aligned}$$

$$\int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^3(y)\text{Sin}(y)dy$$

$$= \left[\begin{array}{l} -\frac{x(2x^2-3)\text{Cos}(2x)}{16} + \frac{3(2x^2-1)\text{Sin}(2x)}{32} \\ -\frac{x(8x^2-3)\text{Cos}(4x)}{256} + \frac{3(8x^2-1)\text{Sin}(4x)}{128(8)} \end{array} \right]_{-\pi/2}^{\pi/2}$$

$$= -\left[\frac{x(2x^2-3)\text{Cos}(2x)}{16} + \frac{x(8x^2-3)\text{Cos}(4x)}{256} \right]_{-\pi/2}^{\pi/2}$$

$$= -\left[\begin{array}{l} \frac{(\pi/2)(2(\pi/2)^2-3)\text{Cos}(\pi)}{16} + \frac{(\pi/2)(8(\pi/2)^2-3)\text{Cos}(2\pi)}{256} \\ -\frac{(-\pi/2)(2(\pi/2)^2-3)\text{Cos}(-\pi)}{16} - \frac{(-\pi/2)(8(\pi/2)^2-3)\text{Cos}(-2\pi)}{256} \end{array} \right]$$

$$= -\left[\begin{array}{l} \frac{(\pi/2)(2(\pi/2)^2-3)(-1)}{16} + \frac{(\pi/2)(8(\pi/2)^2-3)}{256} \\ -\frac{(-\pi/2)(2(\pi/2)^2-3)(-1)}{16} - \frac{(-\pi/2)(8(\pi/2)^2-3)}{256} \end{array} \right]$$

(A19) $$= -\left[-\frac{\pi(2(\pi/2)^2-3)}{16} + \frac{\pi(8(\pi/2)^2-3)}{256} \right] = \frac{\pi(2(\pi/2)^2-3)}{16} - \frac{\pi(8(\pi/2)^2-3)}{256}$$

$$= \frac{\pi(\pi^2/2-3)}{16} - \frac{\pi(2\pi^2-3)}{256} = 0.379895 - 0.20542 = 0.17447.$$

So

(A20) $$\int_{-\pi/2}^{\pi/2} y^3 g(y)dy = -\frac{4}{\pi^2}\{0.393033 + 0.17447\} = -\frac{2.270012}{\pi^2} = -0.23000691.$$

$$\gamma_y = \frac{E[(y-\mu_y)^3]}{\sigma_y^3} = \frac{E[y^3] - 3\mu_y E[y^2] + 2\mu_y^3}{\sigma_y^3}$$

(A21) $$= \frac{-0.23000691 - 3(-0.3247)(0.322466) + 2(-0.3247)^3}{\left[(0.322466) - (-0.3247)^2\right]^{3/2}}$$

$$= \frac{0.08410722 - 2(0.3247)^3}{0.10111076} = 0.15624 > 0.$$

Moments of Distribution with Density $0.75\text{Cos}^3(\varepsilon_i)$ **:**

When the RA density function is $0.75\text{Cos}^3(\varepsilon_i)$, then the cumulative distribution function is

$0.5 + (9/16)\text{Sin}(\varepsilon_i) + (1/16)\text{Sin}(3\varepsilon_i)$, and yield density is

(A22) $$g(y)=\frac{3}{2}\mathrm{Cos}^3(y)\left[0.5-\frac{9}{16}\mathrm{Sin}(y)-\frac{1}{16}\mathrm{Sin}(3y)\right].$$

Integrate

$$\int_{-\pi/2}^{\pi/2} yg(y)dy=\frac{3}{2}\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^3(y)\left[0.5-\frac{9}{16}\mathrm{Sin}(y)-\frac{1}{16}\mathrm{Sin}(3y)\right]dy$$

(A23) $$=\frac{3}{4}\overbrace{\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^3(y)dy}^{\text{0, odd function}}-\frac{27}{32}\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^3(y)\mathrm{Sin}(y)dy-\frac{3}{32}\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^3(y)\mathrm{Sin}(3y)dy$$
$$=-\frac{27}{32}\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^3(y)\mathrm{Sin}(y)dy-\frac{3}{32}\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^3(y)\mathrm{Sin}(3y)dy.$$

$$\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^3(y)\mathrm{Sin}(y)dy=\left[\frac{\mathrm{Sin}(2x)}{16}-\frac{x\mathrm{Cos}(2x)}{8}+\frac{\mathrm{Sin}(4x)}{8(16)}-\frac{x\mathrm{Cos}(4x)}{32}\right]_{-\pi/2}^{\pi/2}$$

(A24) $$=-\left[\frac{x\mathrm{Cos}(2x)}{8}+\frac{x\mathrm{Cos}(4x)}{32}\right]_{-\pi/2}^{\pi/2}$$
$$=-\left[\frac{(\pi/2)\mathrm{Cos}(\pi)}{8}+\frac{(\pi/2)\mathrm{Cos}(2\pi)}{32}-\frac{(-\pi/2)\mathrm{Cos}(-\pi)}{8}-\frac{(-\pi/2)\mathrm{Cos}(-2\pi)}{32}\right]$$
$$=-\left[-\frac{(\pi/2)}{8}+\frac{(\pi/2)}{32}-\frac{(\pi/2)}{8}+\frac{(\pi/2)}{32}\right]=-\left[\frac{\pi}{32}-\frac{\pi}{8}\right]=\frac{3\pi}{32}.$$

$$\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^3(y)\mathrm{Sin}(3y)dy$$
$$=\left[\frac{3\mathrm{Sin}(2x)}{32}-\frac{3x\mathrm{Cos}(2x)}{16}+\frac{3\mathrm{Sin}(4x)}{8(16)}-\frac{3x\mathrm{Cos}(4x)}{32}+\frac{\mathrm{Sin}(6x)}{8(36)}-\frac{x\mathrm{Cos}(6x)}{48}\right]_{-\pi/2}^{\pi/2}$$

(A25) $$=-\frac{1}{48}\left[9x\mathrm{Cos}(2x)+\frac{9x\mathrm{Cos}(4x)}{2}+x\mathrm{Cos}(6x)\right]_{-\pi/2}^{\pi/2}$$
$$=-\frac{1}{48}\left[\begin{array}{l}\frac{9\pi}{2}\mathrm{Cos}(\pi)+\frac{9\pi}{4}\mathrm{Cos}(2\pi)+\frac{\pi}{2}\mathrm{Cos}(3\pi)\\ +\frac{9\pi}{2}\mathrm{Cos}(-\pi)+\frac{9\pi}{4}\mathrm{Cos}(-2\pi)+\frac{\pi}{2}\mathrm{Cos}(-3\pi)\end{array}\right]$$
$$=-\frac{1}{48}\left[-\frac{9\pi}{2}+\frac{9\pi}{4}-\frac{\pi}{2}-\frac{9\pi}{2}+\frac{9\pi}{4}-\frac{\pi}{2}\right]=-\frac{1}{48}\left[-9\pi+\frac{9\pi}{2}-\pi\right]=\frac{11\pi}{96}.$$

So

(A26) $$\int_{-\pi/2}^{\pi/2} yg(y)dy=-\frac{27}{32}\left(\frac{3\pi}{32}\right)-\frac{3}{32}\left(\frac{11\pi}{96}\right)=-\left[27+\frac{11}{3}\right]\frac{3\pi}{(32)^2}=-0.28225.$$

Also,

$$\int_{-\pi/2}^{\pi/2} y^2 g(y)dy = \frac{3}{2}\int_{-\pi/2}^{\pi/2} y^2 \mathrm{Cos}^3(y)\left[0.5 - \frac{9}{16}\mathrm{Sin}(y) - \frac{1}{16}\mathrm{Sin}(3y)\right]dy$$

$$= \frac{3}{4}\int_{-\pi/2}^{\pi/2} y^2 \mathrm{Cos}^3(y)dy - \frac{27}{32}\overbrace{\int_{-\pi/2}^{\pi/2} y^2 \mathrm{Cos}^3(y)\mathrm{Sin}(y)dy}^{\text{0, odd function}} - \frac{3}{32}\overbrace{\int_{-\pi/2}^{\pi/2} y^2 \mathrm{Cos}^3(y)\mathrm{Sin}(3y)dy}^{\text{0, odd function}}$$

$$= \frac{3}{4}\int_{-\pi/2}^{\pi/2} y^2 \mathrm{Cos}^3(y)dy$$

$$= \frac{3}{4}\left[\frac{3x\mathrm{Cos}(x)}{2} + \frac{x\mathrm{Cos}(3x)}{18} + \frac{3(x^2-2)\mathrm{Sin}(x)}{4} + \frac{(9x^2-2)\mathrm{Sin}(3x)}{108}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{3}{4}\left[\frac{3(x^2-2)\mathrm{Sin}(x)}{4} + \frac{(9x^2-2)\mathrm{Sin}(3x)}{108}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{3}{4}\left[\frac{3((\pi/2)^2-2)}{2} - \frac{(9(\pi/2)^2-2)}{54}\right] = \frac{3}{4}\left[\frac{3(\pi/2)^2}{2} - 3 - \frac{9(\pi/2)^2}{54} + \frac{2}{54}\right]$$

(A27) $$= \frac{3}{4}\left[\frac{4(\pi/2)^2}{3} - \frac{80}{27}\right] = (\pi/2)^2 - \frac{20}{9} = 0.245174709.$$

$$\int_{-\pi/2}^{\pi/2} y^3 g(y)dy = \frac{3}{2}\int_{-\pi/2}^{\pi/2} y^3 \mathrm{Cos}^3(y)\left[0.5 - \frac{9}{16}\mathrm{Sin}(y) - \frac{1}{16}\mathrm{Sin}(3y)\right]dy$$

(A28) $$= \frac{3}{4}\overbrace{\int_{-\pi/2}^{\pi/2} y^3 \mathrm{Cos}^3(y)dy}^{\text{0, odd function}} - \frac{27}{32}\int_{-\pi/2}^{\pi/2} y^3 \mathrm{Cos}^3(y)\mathrm{Sin}(y)dy - \frac{3}{32}\int_{-\pi/2}^{\pi/2} y^3 \mathrm{Cos}^3(y)\mathrm{Sin}(3y)dy$$

$$= -\frac{27}{32}\int_{-\pi/2}^{\pi/2} y^3 \mathrm{Cos}^3(y)\mathrm{Sin}(y)dy - \frac{3}{32}\int_{-\pi/2}^{\pi/2} y^3 \mathrm{Cos}^3(y)\mathrm{Sin}(3y)dy.$$

$$\int_{-\pi/2}^{\pi/2} y^3 \mathrm{Cos}^3(y)\mathrm{Sin}(y)dy$$
$$= \left[\frac{3(2x^2-1)\mathrm{Sin}(2x)}{32} - \frac{x(2x^2-3)\mathrm{Cos}(2x)}{16} + \frac{3(8x^2-1)\mathrm{Sin}(4x)}{8(128)} - \frac{x(8x^2-3)\mathrm{Cos}(4x)}{8(32)}\right]_{-\pi/2}^{\pi/2}$$
$$= -\left[\frac{x(2x^2-3)\mathrm{Cos}(2x)}{16} + \frac{x(8x^2-3)\mathrm{Cos}(4x)}{8(32)}\right]_{-\pi/2}^{\pi/2}$$
(A29)
$$= -\left[\begin{array}{l} -\frac{(\pi/2)(2(\pi/2)^2-3)}{16} + \frac{(\pi/2)(8(\pi/2)^2-3)}{8(32)} \\ +\frac{(-\pi/2)(2(\pi/2)^2-3)}{16} - \frac{(-\pi/2)(8(\pi/2)^2-3)}{8(32)} \end{array}\right]$$
$$= -\left[-\frac{\pi(2(\pi/2)^2-3)}{16} + \frac{\pi(8(\pi/2)^2-3)}{8(32)}\right] = \frac{\pi}{16}\left[2(\pi/2)^2 - 3 - \frac{(8(\pi/2)^2-3)}{16}\right]$$
$$= \frac{\pi}{16}\left[2(\pi/2)^2 - 3 - \frac{(\pi/2)^2}{2} + \frac{3}{16}\right] = \frac{\pi}{16}\left[\frac{3(\pi/2)^2}{2} - \frac{45}{16}\right] = 0.174475.$$

$$\int_{-\pi/2}^{\pi/2} y^3 \mathrm{Cos}^3(y)\mathrm{Sin}(3y)dy$$
$$= \left[\frac{9(2x^2-1)\mathrm{Sin}(2x)}{64} - \frac{3x(2x^2-3)\mathrm{Cos}(2x)}{32}\right]_{-\pi/2}^{\pi/2}$$
$$+ \left[\frac{9(8x^2-1)\mathrm{Sin}(4x)}{8(128)} - \frac{3x(8x^2-3)\mathrm{Cos}(4x)}{8(32)}\right]_{-\pi/2}^{\pi/2}$$
$$+ \left[\frac{(18x^2-1)\mathrm{Sin}(6x)}{8(216)} - \frac{x(6x^2-1)\mathrm{Cos}(6x)}{8(36)}\right]_{-\pi/2}^{\pi/2}$$
$$= -\left[\frac{3x(2x^2-3)\mathrm{Cos}(2x)}{32}\right]_{-\pi/2}^{\pi/2} - \left[\frac{3x(8x^2-3)\mathrm{Cos}(4x)}{8(32)}\right]_{-\pi/2}^{\pi/2} - \left[\frac{x(6x^2-1)\mathrm{Cos}(6x)}{8(36)}\right]_{-\pi/2}^{\pi/2}$$
$$= -\left[-\frac{3(\pi/2)(2(\pi/2)^2-3)}{32} + \frac{3(-\pi/2)(2(\pi/2)^2-3)}{32}\right]$$
$$-\left[\frac{3(\pi/2)(8(\pi/2)^2-3)}{8(32)} - \frac{3(-\pi/2)(8(\pi/2)^2-3)}{8(32)}\right]$$
$$-\left[-\frac{(\pi/2)(6(\pi/2)^2-1)}{8(36)} + \frac{(-\pi/2)(6(\pi/2)^2-1)}{8(36)}\right]$$
(A30)
$$= \frac{3\pi(2(\pi/2)^2-3)}{32} - \frac{3\pi(8(\pi/2)^2-3)}{8(32)} + \frac{\pi(6(\pi/2)^2-1)}{8(36)}$$
$$= 0.569843 - 0.61626 + 0.1505823 = 0.1041653.$$

So

(A31) $\int_{-\pi/2}^{\pi/2} y^3 g(y)dy = -\frac{27}{32}(0.174475) - \frac{3}{32}(0.1041653) = -0.147213281 - 0.0097655$
$= -0.156978781.$

Finally,

$$\gamma_y = \frac{E[(y-\mu_y)^3]}{\sigma_y^3} = \frac{E[y^3] - 3\mu_y E[y^2] + 2\mu_y^3}{\sigma_y^3}$$

(A32) $$= \frac{-0.156978781 - 3(-0.28225)(0.2451747) + 2(-0.28225)^3}{\left[(0.2451747) - (-0.28225)^2\right]^{3/2}}$$

$$= \frac{0.050622307 - 2(0.28225)^3}{0.0673312} = 0.083934 > 0.$$

Moments of Distribution with Density $[8/(3\pi)]\mathrm{Cos}^4(\varepsilon_i)$ **:**

When the RA density function is $[8/(3\pi)]\mathrm{Cos}^4(\varepsilon_i)$, then the cumulative distribution function is

$0.5 + (\varepsilon_i/\pi) + [2/(3\pi)]\mathrm{Sin}(2\varepsilon_i) + [1/(12\pi)]\mathrm{Sin}(4\varepsilon_i)$, and yield density is

(A33) $$g(y) = \frac{16}{3\pi}\mathrm{Cos}^4(y)\left[0.5 - \frac{y}{\pi} - \frac{2}{3\pi}\mathrm{Sin}(2y) - \frac{1}{12\pi}\mathrm{Sin}(4y)\right].$$

Integrate

$$\int_{-\pi/2}^{\pi/2} y g(y)dy = \frac{16}{3\pi}\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^4(y)\left[0.5 - \frac{y}{\pi} - \frac{2}{3\pi}\mathrm{Sin}(2y) - \frac{1}{12\pi}\mathrm{Sin}(4y)\right]dy$$

$$= \frac{8}{3\pi}\overbrace{\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^4(y)dy}^{\text{0, odd function}} - \frac{16}{3\pi^2}\int_{-\pi/2}^{\pi/2} y^2\mathrm{Cos}^4(y)dy - \frac{32}{9\pi^2}\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^4(y)\mathrm{Sin}(2y)dy$$

(A34) $$-\frac{4}{9\pi^2}\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^4(y)\mathrm{Sin}(4y)dy$$

$$= -\frac{16}{3\pi^2}\int_{-\pi/2}^{\pi/2} y^2\mathrm{Cos}^4(y)dy - \frac{32}{9\pi^2}\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^4(y)\mathrm{Sin}(2y)dy$$

$$-\frac{4}{9\pi^2}\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^4(y)\mathrm{Sin}(4y)dy.$$

Integrate

$$\int_{-\pi/2}^{\pi/2} y^2 \mathrm{Cos}^4(y)dy$$

$$= \left[\frac{x^3}{8} + \frac{x\mathrm{Cos}(2x)}{4} + \frac{x\mathrm{Cos}(4x)}{64} + \frac{(2x^2-1)\mathrm{Sin}(2x)}{8} + \frac{(8x^2-1)\mathrm{Sin}(4x)}{256}\right]_{-\pi/2}^{\pi/2}$$

(A35)

$$= \left[\frac{x^3}{8} + \frac{x\mathrm{Cos}(2x)}{4} + \frac{x\mathrm{Cos}(4x)}{64}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{(\pi/2)^3}{8} + \frac{(\pi/2)(-1)}{4} + \frac{(\pi/2)(1)}{64} - \frac{(-\pi/2)^3}{8} - \frac{(-\pi/2)(-1)}{4} - \frac{(-\pi/2)(1)}{64}$$

$$= \frac{2(\pi/2)^3}{8} + \frac{2(\pi/2)(-1)}{4} + \frac{2(\pi/2)(1)}{64} = \frac{\pi^3}{32} - \frac{\pi}{4} + \frac{\pi}{64}$$

$$= \frac{\pi}{32}\left(\pi^2 - 7.5\right) = 0.2326335.$$

$$\int_{-\pi/2}^{\pi/2} y\mathrm{Cos}^4(y)\mathrm{Sin}(2y)dy$$

$$= 2\left[\frac{5\mathrm{Sin}(2x)}{128} - \frac{5x\mathrm{Cos}(2x)}{64} + \frac{\mathrm{Sin}(4x)}{8(16)} - \frac{x\mathrm{Cos}(4x)}{32} + \frac{\mathrm{Sin}(6x)}{32(36)} - \frac{x\mathrm{Cos}(6x)}{32(6)}\right]_{-\pi/2}^{\pi/2}$$

(A36)

$$= -2\left[\frac{5x\mathrm{Cos}(2x)}{64} + \frac{x\mathrm{Cos}(4x)}{32} + \frac{x\mathrm{Cos}(6x)}{32(6)}\right]_{-\pi/2}^{\pi/2}$$

$$= -2\left[\frac{5(\pi/2)(-1)}{64} + \frac{(\pi/2)(1)}{32} + \frac{(\pi/2)(-1)}{32(6)} - \frac{5(-\pi/2)(-1)}{64} - \frac{(-\pi/2)(1)}{32} - \frac{(-\pi/2)(-1)}{32(6)}\right]$$

$$= -2\left[-\frac{5\pi}{64} + \frac{\pi}{32} - \frac{\pi}{32(6)}\right] = -\frac{2\pi}{32}\left[-\frac{5}{2} + 1 - \frac{1}{6}\right] = \frac{5\pi}{48}.$$

$$\int_{-\pi/2}^{\pi/2} y\text{Cos}^4(y)\text{Sin}(4y)dy$$

$$=\left[\begin{array}{l}\frac{\text{Sin}(2x)}{16}-\frac{x\text{Cos}(2x)}{8}+\frac{3\text{Sin}(4x)}{8(16)}-\frac{3x\text{Cos}(4x)}{32}\\+\frac{\text{Sin}(6x)}{4(36)}-\frac{x\text{Cos}(6x)}{24}+\frac{\text{Sin}(8x)}{16(64)}-\frac{x\text{Cos}(8x)}{16(8)}\end{array}\right]_{-\pi/2}^{\pi/2}$$

$$=-\left[\frac{x\text{Cos}(2x)}{8}+\frac{3x\text{Cos}(4x)}{32}+\frac{x\text{Cos}(6x)}{24}+\frac{x\text{Cos}(8x)}{16(8)}\right]_{-\pi/2}^{\pi/2}$$

$$=-\left[\begin{array}{l}\frac{(\pi/2)\text{Cos}(\pi)}{8}+\frac{3(\pi/2)\text{Cos}(2\pi)}{32}+\frac{(\pi/2)\text{Cos}(3\pi)}{24}+\frac{(\pi/2)\text{Cos}(4\pi)}{16(8)}\\+\frac{(\pi/2)\text{Cos}(-\pi)}{8}+\frac{3(\pi/2)\text{Cos}(-2\pi)}{32}+\frac{(\pi/2)\text{Cos}(-3\pi)}{24}+\frac{(\pi/2)\text{Cos}(-4\pi)}{16(8)}\end{array}\right]$$

$$=-\left[\begin{array}{l}\frac{(\pi/2)(-1)}{8}+\frac{3(\pi/2)(1)}{32}+\frac{(\pi/2)(-1)}{24}+\frac{(\pi/2)(1)}{16(8)}\\+\frac{(\pi/2)(-1)}{8}+\frac{3(\pi/2)(1)}{32}+\frac{(\pi/2)(-1)}{24}+\frac{(\pi/2)(1)}{16(8)}\end{array}\right]$$

(A37) $$=-\left[-\frac{\pi}{8}+\frac{3\pi}{32}-\frac{\pi}{24}+\frac{\pi}{16(8)}\right]=\pi\left[\frac{1}{32}+\frac{1}{24}-\frac{1}{16(8)}\right]=\frac{\pi}{8}\left[\frac{3}{16}+\frac{1}{3}\right]=\frac{5\pi}{192}.$$

So

$$\int_{-\pi/2}^{\pi/2} yg(y)dy=-\frac{16}{3\pi^2}\int_{-\pi/2}^{\pi/2} y^2\text{Cos}^4(y)dy-\frac{32}{9\pi^2}\int_{-\pi/2}^{\pi/2} y\text{Cos}^4(y)\text{Sin}(2y)dy$$

$$-\frac{4}{9\pi^2}\int_{-\pi/2}^{\pi/2} y\text{Cos}^4(y)\text{Sin}(4y)dy$$

(A38) $$=-\frac{16}{3\pi^2}(\ )-\frac{32}{9\pi^2}(\ )-\frac{4}{9\pi^2}(\ )=-\frac{4}{9\pi^2}\left[12(\ )+8(\ )+(\ )\right]$$

$$=-\frac{4}{9\pi^2}\left[12(0.2326335)+8\left(\frac{5\pi}{48}\right)+\left(\frac{5\pi}{192}\right)\right]$$

$$=-\frac{4}{9\pi^2}\left[2.791602+2.617991667+0.081812239\right]=-0.247287416.$$

Integrate

$$\int_{-\pi/2}^{\pi/2} y^2 g(y)dy = \frac{16}{3\pi}\int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^4(y)\left[0.5 - \frac{y}{\pi} - \frac{2}{3\pi}\text{Sin}(2y) - \frac{1}{12\pi}\text{Sin}(4y)\right]dy$$

(A39)

$$= \frac{8}{3\pi}\int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^4(y)dy - \frac{16}{3\pi^2}\overbrace{\int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y)dy}^{\text{0, odd function}} - \frac{32}{9\pi^2}\overbrace{\int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^4(y)\text{Sin}(2y)dy}^{\text{0, odd function}}$$

$$- \frac{4}{9\pi^2}\overbrace{\int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^4(y)\text{Sin}(4y)dy}^{\text{0, odd function}}$$

$$= \frac{8}{3\pi}\int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^4(y)dy = \frac{8(0.2326335)}{3\pi} = 0.197465614.$$

And

$$\int_{-\pi/2}^{\pi/2} y^3 g(y)dy = \frac{16}{3\pi}\int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y)\left[0.5 - \frac{y}{\pi} - \frac{2}{3\pi}\text{Sin}(2y) - \frac{1}{12\pi}\text{Sin}(4y)\right]dy$$

$$= \frac{8}{3\pi}\overbrace{\int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y)dy}^{\text{0, odd function}} - \frac{16}{3\pi^2}\int_{-\pi/2}^{\pi/2} y^4 \text{Cos}^4(y)dy - \frac{32}{9\pi^2}\int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y)\text{Sin}(2y)dy$$

(A40)

$$- \frac{4}{9\pi^2}\int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y)\text{Sin}(4y)dy$$

$$= -\frac{16}{3\pi^2}\int_{-\pi/2}^{\pi/2} y^4 \text{Cos}^4(y)dy - \frac{32}{9\pi^2}\int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y)\text{Sin}(2y)dy$$

$$- \frac{4}{9\pi^2}\int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y)\text{Sin}(4y)dy.$$

Now

$$\int_{-\pi/2}^{\pi/2} y^4 \mathrm{Cos}^4(y)dy = \left[\begin{array}{l} \frac{3x^5}{40} + \frac{(2x^2-3)x\mathrm{Cos}(2x)}{4} + \frac{(8x^2-3)x\mathrm{Cos}(4x)}{256} \\ + \frac{(2x^4-6x^2+3)\mathrm{Sin}(2x)}{8} + \frac{(32x^4-24x^2+3)\mathrm{Sin}(4x)}{1024} \end{array} \right]_{-\pi/2}^{\pi/2}$$

$$= \left[\frac{3x^5}{40} + \frac{(2x^2-3)x\mathrm{Cos}(2x)}{4} + \frac{(8x^2-3)x\mathrm{Cos}(4x)}{256} \right]_{-\pi/2}^{\pi/2}$$

$$= \frac{3(\pi/2)^5}{40} + \frac{(2(\pi/2)^2-3)(\pi/2)\mathrm{Cos}(2x)}{4} + \frac{(8(\pi/2)^2-3)(\pi/2)\mathrm{Cos}(4x)}{256}$$

$$+ \frac{3(\pi/2)^5}{40} - \frac{(2(\pi/2)^2-3)(-\pi/2)\mathrm{Cos}(2x)}{4} - \frac{(8(\pi/2)^2-3)(-\pi/2)\mathrm{Cos}(4x)}{256}$$

$$= \frac{3(\pi/2)^5}{20} + \frac{(2(\pi/2)^2-3)(\pi/2)(-1)}{4} + \frac{(8(\pi/2)^2-3)(\pi/2)}{256}$$

$$+ \frac{(2(\pi/2)^2-3)(\pi/2)(-1)}{4} + \frac{(8(\pi/2)^2-3)(\pi/2)}{256}$$

$$= \frac{3(\pi/2)^5}{20} - \frac{(2(\pi/2)^2-3)\pi}{4} + \frac{(8(\pi/2)^2-3)\pi}{256} \quad \text{(A41)}$$

$$= 1.434461214 - 1.519582264 + 0.205420414 = 0.120299364.$$

$$\int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y)\text{Sin}(2y)dy = 2\left[\frac{15(2x^2-1)\text{Sin}(2x)}{8(32)} - \frac{5x(2x^2-3)\text{Cos}(2x)}{128}\right]_{-\pi/2}^{\pi/2}$$

$$+2\left[\frac{3(8x^2-1)\text{Sin}(4x)}{8(128)} - \frac{x(8x^2-3)\text{Cos}(4x)}{8(32)}\right]_{-\pi/2}^{\pi/2}$$

$$+2\left[\frac{(18x^2-1)\text{Sin}(6x)}{216(32)} - \frac{x(6x^2-1)\text{Cos}(6x)}{36(32)}\right]_{-\pi/2}^{\pi/2}$$

$$= 2\left[-\frac{5x(2x^2-3)\text{Cos}(2x)}{128}\right]_{-\pi/2}^{\pi/2} + 2\left[-\frac{x(8x^2-3)\text{Cos}(4x)}{8(32)}\right]_{-\pi/2}^{\pi/2}$$

$$+2\left[-\frac{x(6x^2-1)\text{Cos}(6x)}{36(32)}\right]_{-\pi/2}^{\pi/2}$$

$$= 2\left[-\frac{5(\pi/2)(2(\pi/2)^2-3)(-1)}{128} - \frac{5(\pi/2)(2(\pi/2)^2-3)(-1)}{128}\right]$$

$$+2\left[-\frac{(\pi/2)(8(\pi/2)^2-3)(1)}{8(32)} - \frac{(\pi/2)(8(\pi/2)^2-3)(1)}{8(32)}\right]$$

$$+2\left[-\frac{(\pi/2)(6(\pi/2)^2-1)(-1)}{36(32)} - \frac{(\pi/2)(6(\pi/2)^2-1)(-1)}{36(32)}\right] \tag{A42}$$

$$= 2\left[\frac{5\pi(2(\pi/2)^2-3)}{128}\right] - \frac{\pi(8(\pi/2)^2-3)}{4(32)} + \frac{\pi(6(\pi/2)^2-1)}{18(32)}$$

$$= 0.474869457 - 0.410840829 + 0.075291158 = 0.139319786.$$

$$\int_{-\pi/2}^{\pi/2} y^3\mathrm{Cos}^4(y)\mathrm{Sin}(4y)dy = \left[\frac{3(2x^2-1)\mathrm{Sin}(2x)}{32} - \frac{x(2x^2-3)\mathrm{Cos}(2x)}{16}\right]_{-\pi/2}^{\pi/2}$$

$$+\left[\frac{9(8x^2-1)\mathrm{Sin}(4x)}{8(128)} - \frac{3x(8x^2-3)\mathrm{Cos}(4x)}{8(32)}\right]_{-\pi/2}^{\pi/2}$$

$$+\left[\frac{(18x^2-1)\mathrm{Sin}(6x)}{4(216)} - \frac{x(6x^2-1)\mathrm{Cos}(6x)}{4(36)}\right]_{-\pi/2}^{\pi/2}$$

$$+\left[\frac{3(32x^2-1)\mathrm{Sin}(8x)}{16(2048)} - \frac{x(32x^2-3)\mathrm{Cos}(8x)}{16(256)}\right]_{-\pi/2}^{\pi/2}$$

$$= -\left[\frac{x(2x^2-3)\mathrm{Cos}(2x)}{16}\right]_{-\pi/2}^{\pi/2} - \left[\frac{3x(8x^2-3)\mathrm{Cos}(4x)}{8(32)}\right]_{-\pi/2}^{\pi/2}$$

$$-\left[\frac{x(6x^2-1)\mathrm{Cos}(6x)}{4(36)}\right]_{-\pi/2}^{\pi/2} - \left[\frac{x(32x^2-3)\mathrm{Cos}(8x)}{16(256)}\right]_{-\pi/2}^{\pi/2}$$

$$= -\left[\frac{(\pi/2)(2(\pi/2)^2-3)(-1)}{16} + \frac{(\pi/2)(2(\pi/2)^2-3)(-1)}{16}\right]$$

$$-\left[\frac{3(\pi/2)(8(\pi/2)^2-3)(1)}{8(32)} + \frac{3(\pi/2)(8(\pi/2)^2-3)(1)}{8(32)}\right]$$

$$-\left[\frac{(\pi/2)(6(\pi/2)^2-1)(-1)}{4(36)} + \frac{(\pi/2)(6(\pi/2)^2-1)(-1)}{4(36)}\right]$$

$$-\left[\frac{(\pi/2)(32(\pi/2)^2-3)(1)}{16(256)} + \frac{(\pi/2)(32(\pi/2)^2-3)(1)}{16(256)}\right]$$

(A43)

$$= \frac{\pi(2(\pi/2)^2-3)}{16} - \frac{3\pi(8(\pi/2)^2-3)}{8(32)} + \frac{\pi(6(\pi/2)^2-1)}{4(36)} - \frac{\pi(32(\pi/2)^2-3)}{16(256)}$$

$$= \frac{\pi}{16}\left\{(2(\pi/2)^2-3) - \frac{3(8(\pi/2)^2-3)}{16} + \frac{(6(\pi/2)^2-1)}{9} - \frac{(32(\pi/2)^2-3)}{256}\right\}$$

$$= \frac{\pi}{16}\{1.934793864 - 3.138595398 + 1.533820177 - 0.296705866\} = 0.0065409428.$$

So

$$\int_{-\pi/2}^{\pi/2} y^3 g(y)dy = -\frac{16}{3\pi^2}\int_{-\pi/2}^{\pi/2} y^4 \mathrm{Cos}^4(y)dy - \frac{32}{9\pi^2}\int_{-\pi/2}^{\pi/2} y^3 \mathrm{Cos}^4(y)\mathrm{Sin}(2y)dy$$

$$-\frac{4}{9\pi^2}\int_{-\pi/2}^{\pi/2} y^3 \mathrm{Cos}^4(y)\mathrm{Sin}(4y)dy$$

(A44) $$= -\left[\frac{16}{3\pi^2}\left(0.120299364\right) + \frac{32}{9\pi^2}\left(0.139319786\right) + \frac{4}{9\pi^2}\left(0.0065409428\right)\right]$$

$$= -\frac{4}{9\pi^2}\left[12\left(0.120299364\right) + 8\left(0.139319786\right) + \left(0.0065409428\right)\right]$$

$$= -\frac{4}{9\pi^2}\left[1.443592368 + 1.114558288 + 0.0065409428\right] = -\frac{4(2.564691598)}{9\pi^2}$$

$$= -0.115492456.$$

Finally,

$$\gamma_y = \frac{E[(y-\mu_y)^3]}{\sigma_y^3} = \frac{E[y^3] - 3\mu_y E[y^2] + 2\mu_y^3}{\sigma_y^3}$$

(A45) $$= \frac{-0.115492456 - 3(-0.247287416)(0.197465614) + 2(-0.247287416)^3}{\left[(0.197465614) - (-0.247287416)^2\right]^{3/2}}$$

$$= \frac{0.030999828 - 2(0.247287416)^3}{0.050328421} = 0.015022322 > 0.$$

Demonstrating eqn. (21):

$$\gamma_y = \frac{E[y^3 \mid z] - 3E[y \mid z]E[y^2 \mid z] + 2\left(E[y \mid z]\right)^3}{\left(E[y^2 \mid z] - \left(E[y \mid z]\right)^2\right)^{1.5}};$$

(A46) $$= \frac{\frac{1+3z^4}{4} - 3\frac{1+z^2}{2}\frac{1+2z^3}{3} + 2\left(\frac{1+z^2}{2}\right)^3}{\left(\frac{1+2z^3}{3} - \left(\frac{1+z^2}{2}\right)^2\right)^{1.5}}$$

$$= \frac{\frac{1+3z^4}{4} - \frac{(1+z^2)(1+2z^3)}{2} + \frac{(1+z^2)^3}{4}}{\left(\frac{1+2z^3}{3} - \frac{(1+z^2)^2}{4}\right)^{1.5}}$$

$$= \frac{\left(1 - 4z + 6z^2 - 4z^3 + z^4\right)z^2 6\sqrt{3}}{\left(1 - 6z^2 + 8z^3 - 3z^4\right)^{1.5}}.$$

Demonstrating eqn. (22):

$$\gamma_y = \frac{\dfrac{1+3\lambda z^4}{4} - \dfrac{(1+\lambda z^2)(1+2\lambda z^3)}{2} + \dfrac{(1+\lambda z^2)^3}{4}}{\left(\dfrac{1+2\lambda z^3}{3} - \left(\dfrac{(1+\lambda z^2)}{2}\right)^2\right)^{1.5}}$$

$$\text{(A47)} \quad = \frac{\lambda z^2 - 4\lambda z^3 + 3\lambda^2 z^4 + 3\lambda z^4 - 4\lambda^2 z^5 + \lambda^3 z^6}{4\left(\dfrac{1+8\lambda z^3 - 6\lambda z^2 - 3\lambda^2 z^4}{12}\right)^{1.5}}$$

$$= 6\sqrt{3}\lambda z^2 \frac{(1-4z+3\lambda z^2 + 3z^2 - 4\lambda z^3 + \lambda^2 z^4)}{\left(1+8\lambda z^3 - 6\lambda z^2 - 3\lambda^2 z^4\right)^{1.5}}.$$