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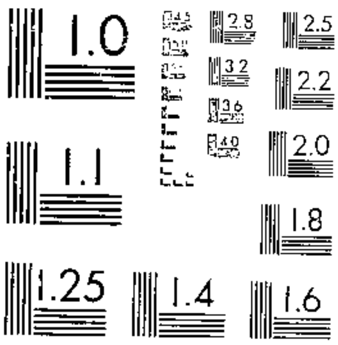

MICROCOPY RESOLUTION TEST CHART
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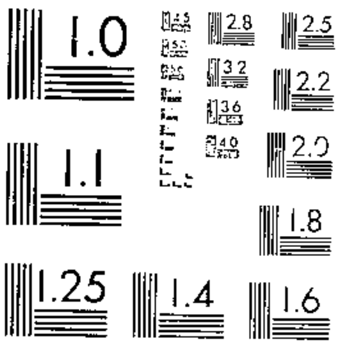

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USE OF THE EXPONENTIAL YIELD CURVE IN FERTILIZER EXPERIMENTS

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CONTENTS

INTRODUCTION

The objective of early experimental work with fertilizers was to find which of the necessary plant-food elements the soil could, and which it could not, supply in quantities sufficient for the needs of the growing crop.

It was soon learned that the plant-food elements most commonly deficient in the soil are nitrogen, phosphorus, and potassium. Accordingly, these three elements were widely used in fertilizer tests, the aim being to determine the relative degree of deficiency of each in specific cases. These tests gave results of sufficient value to permit the development of an extensive fertilizer industry. Other experiments have had for their object a comparison of different sources of the various plant-food elements. Such experiments are necessary in any system of experimentation with fertilizers.

Recognition of the fact that nitrogen, phosphoric acid, and potash are most frequently needed in fertilizers made it obvious that experiments of two kinds were needed. One kind had for its object the determining of the most profitable combination of fertilizer elements, that is, the most profitable fertilizer formula, for a given soil and crop;

[1] Doctor Spillman died July 11, 1931. In the uncompleted manuscript acknowledgment was made of the valuable aid rendered by Mrs. Florence O. Thomas in making the computations, and by S. W. Mendum, Senior Agricultural Economist, in offering suggestions as to arrangement and presentation of the subject matter of the report. After Doctor Spillman's death Mr. Mendum completed the manuscript.

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the other kind had for its object the determining of the most profitable quantity of this most profitable formula.

One group of experimenters gave particular attention to the problem of best fertilizer formulas, and devised an ingenious and effective system of experimentation which has given valuable information as to best formulas for given cases.

Others instituted series of experiments intended to determine the most profitable quantities of the various fertilizer elements and the various combinations to use in given cases.

The results of these two lines of experimentation are the basis of present fertilizer practice in the United States.

To cover the full range of fertilizer formulas and the full range of quantities of even a few formulas requires a large number of experimental plots. To obtain adequate answers to the problems of best formulas and best quantities of fertilizers to apply in a given case by the methods mentioned therefore entails large expenditures of both time and funds.

An equation expressing even approximately the relation between plant growth and quantity of plant food applied in fertilizers would enable the experimenter to carry on investigations of both these problems with a relatively small number of experimental plots. At the same time it would give more accurate answers to the two problems than could be otherwise obtained.

The economic bearing of such an equation is obvious. The ability to determine even approximately the formula for any quantity of fertilizer that will give the most profit, and the quantity of fertilizer made according to the best formula for that quantity that would result in the greatest profit per acre, should lead to less waste and greater profit in the use of fertilizer.

A large number of experiments have been performed in which the quantity of one or more growth factors was varied. When the results of these experiments are graphed, yields being used as ordinates and quantities of a growth factor as abscissas, a large proportion of the resulting curves are strikingly similar in form. (Fig. 1.) In fact, the proportion is so large as to suggest that in many cases those that do not give such a curve fail to do so because of large experimental errors in the work.

A curve that can be fitted satisfactorily to these experimental results would make it possible to calculate the yield from any quantity of the growth factor in question, the only experimental data needed being those required for finding accurate values for the constants of the equation.

In recent years it has been shown that either of the equations

$$Y = M - AR^x \qquad (1)$$

or

$$y = a + bx + cx^2 \qquad (A)$$

meets these requirements. Within the range of the data used in determining the constants in either of these equations, each gives a curve that fits experimental results satisfactorily. Over a considerable proportion of their range the two curves are closely similar in form.

It is shown later, however, that equation (1) has certain important advantages as compared with equation (A). It may be used with satisfactory results for calculating yields far beyond the limits of the experimental results used in determining the constants of the equation. This is not the case with equation (A).

An even more important advantage of equation (1) is that it may be written in a generalized form (p. 22) that permits it to be used, when its constants have been determined, for calculating the yield to be expected from any combination of fertilizer elements, in any quantity of fertilizer. Again, the generalized form of the equation by differentiation may be converted into a set of equations, one for each variable growth factor, by means of which the most profitable quantities of nitrogen and potash to use with any quantity of phos-

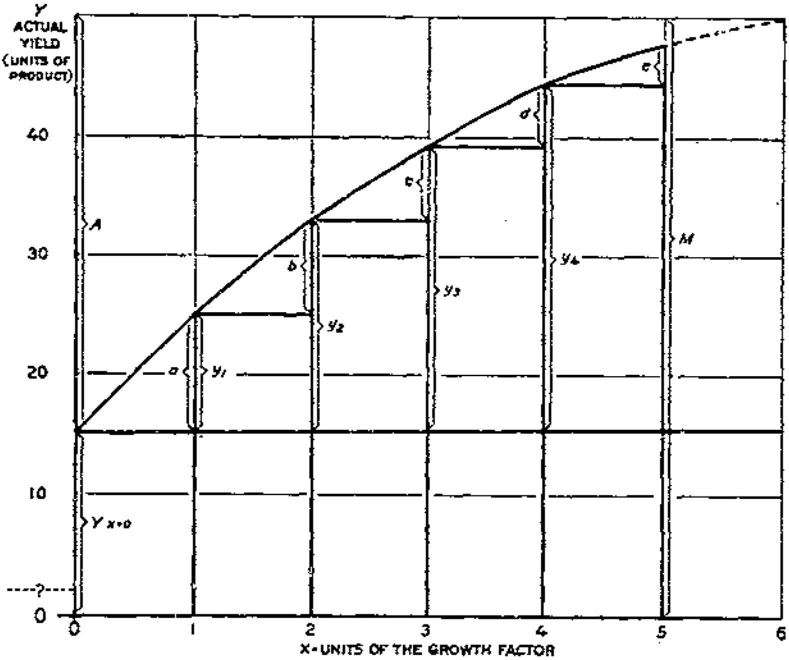

FIGURE 1.—EXPONENTIAL YIELD CURVE FOR A SINGLE VARIABLE GROWTH FACTOR

Y_0 is the yield when none of the factor is applied in fertilizer, y_1 the increase in yield due to 1 unit of the factor applied in fertilizer, y_2 the increase for 2 units, and so on. Y is the actual yield for x units of the factor, and M and A are the respective limits approached by Y and y as x increases indefinitely. Lines a, b, c, d, etc., are the increments in yield due to the first, second, third, fourth, etc., units applied. These increments tend to form a decreasing geometric series, of which R is the ratio.

phoric acid, as well as the most profitable quantity of this most profitable combination to use in a given case, can be determined.

These facts appear to justify at least the tentative adoption of equation (1) as the mathematical expression of the quantitative relation between plant growth and the quantity of a growth factor available. Further research may result in a more accurate expression for this relation. But the fact that equation (1) does permit satisfactory curve fitting, and thus greatly reduces the amount of experimental work required for at least an approximate solution to both the problem of best formula and the problem of best quantity of fertilizer having this formula, would seem to justify the tentative use of the equation as a basis for planning and interpreting experimental work with fertilizers.

The uses to which equation (1) and its generalized form adapted to two or more variable growth factors may be applied are set forth in the following pages.

On certain soils very small applications of a fertilizer constituent give no increase in yield. As the quantity applied increases, a point is finally reached beyond which the yield begins to increase, and the increase proceeds from that point to follow a well-defined curve of normal increase. The small quantity thus having no effect on yield has been referred to by soil chemists as "absorbed" nitrogen, phosphoric acid, or potash, as the case may be. Perhaps the term "occlusion" might be preferred for this phenomenon. The reason for the failure of the small quantity of the growth factor to produce any effect on yield is as yet not definitely known. The fact appears to be that it is not available to the growing plant. This bulletin gives a method of determining, by means of the yield curve, the quantity of a plant-food element thus rendered unavailable.

This determination has economic value. It sometimes happens that a farmer can not obtain as much fertilizer as he knows he needs, but must distribute what he can obtain over a considerable acreage. It would be a complete waste to apply less than the quantity taken up by the soil and held in a condition unavailable to the growing crop. With knowledge of the quantity that would be absorbed, or occluded, and thus rendered unavailable, and of the increase in yield to be expected from applications over and above this quantity, the fertilizer obtainable can be distributed at the rate that will give the highest net return.

Certain other soils are said to absorb a definite proportion of the potash applied to them, irrespective of the quantity applied. In these cases the method described herein is not applicable. There is a possibility, however, that when further work with the yield curve has been done, a method may be devised for measuring this type of absorption by means of the curve, especially if it should be demonstrated that the effect factor (*11*)[2] of potash is constant for all soils not exhibiting these absorption phenomena.

The fact that the yield curve offers a means of determining from the yield of a relatively small number of experimental plots, the quantities of available nitrogen, phosphoric acid, and potash in the soil, makes the curve available for determining (1) the quantity of available nitrogen added to the soil by a green-manure crop; (2) the effect of lime or other soil amendments, including tillage practices, on the availability of plant-food elements in the soil; (3) the rate at which each plant-food element is exhausted in any system of crop management; and (4) the rate at which plant-food elements accumulate in the soil when fertilizers are applied in excessive quantities, thus permitting judicious modification of fertilizer practice with a view to preventing extravagant use of fertilizers which may result in accumulations that might become injurious to the crop.

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[2] Italic numbers in parentheses refer to Literature Cited, p. 60.

YIELD CURVE FOR A SINGLE VARIABLE GROWTH FACTOR

The form of yield curve employed in this bulletin is known as the exponential yield curve. Its derivation and the reasons for preferring it are discussed later (p. 51). The curve assumes different shapes for different numbers of variable growth factors. The form assumed when a single growth factor is varied is shown in Figure 1. The equation of this form is

$$Y=M-AR^{x}, \tag{1}$$

in which Y is the yield obtained when x units of the growth factor are applied in fertilizers, the unit being any convenient quantity of the factor. M is the limit approached by Y as x increases indefinitely, or the theoretical maximum yield possible with any number of units of the growth factor. A is the theoretical maximum increase in yield obtainable by increasing x indefinitely. R is the ratio of a decreasing geometric series the terms of which are the respective increments in yield due to successive unit increments in x. In Figure 1, lines a, b, c, d, e, etc., represent these increments; R is therefore the ratio of the series a, b, c, d, e, etc. This means that if b is a given percentage of a, then c tends to be the same percentage of b, d this same percentage of c, and so on.

The value of R in any given case depends on the size of unit in which x is measured, on the nature of the variable growth factor, and on the conditions of the experiment.

In Figure 1, $Y_{x=0}$ is the yield when none of the growth factor is supplied in fertilizers; that is, it is the yield due to the quantity of the growth factor available in the unfertilized soil. M is the limit approached by the curve as x increases indefinitely. The line at height M is an asymptote to the curve. The question mark at the lower left corner of the figure merely calls attention to whatever quantity of the growth factor may be available in the soil.

DETERMINATION OF THE CONSTANTS OF THE EXPONENTIAL CURVE

Several methods are available for finding the value of the constants of the yield equation. These vary in the reliability of the values found. Some of them are given below.

To illustrate these methods, and to compare the results they give, some results obtained by the Michigan Agricultural Experiment Station in applying varying quantities of potash (K_2O) to potatoes grown on muck soil are used. In addition to potash, each plot received phosphoric acid (P_2O_5) at the rate of 300 pounds of 16 per cent superphosphate per acre. One hundred pounds of 50 per cent muriate of potash (50 pounds of K_2O) is taken as the unit of x.

The yields per acre of four plots thus fertilized were:

Plot 1 (no potash)	91 bushels.
Plot 2 (1 unit of potash)	251 bushels.
Plot 3 (2 units of potash)	331 bushels.
Plot 4 (3 units of potash)	381 bushels.

GRAPHIC METHOD

A crude but simple and often useful method of finding at least a rough approximation to the value of the constants of the yield equation is as follows:

First, graph the experimental results, as is done in Figure 2 for the data given above. After fixing each point representing a yield, draw through these points as smooth a curve as may be and as nearly as possible of the form shown in Figure 1. If the yields correspond closely to the theory of the curve, as they appear to do in Figure 2, the curve will pass through, or very near to, each point; but if the yields are irregular, the experimental errors being large, the curve should be drawn of the general form of Figure 1, more or less steeply according to the location of the dots, and as nearly as possible in such manner as to make the squares of the deviations a minimum. The deviations here referred to are the vertical distances of the yield points from the curve.

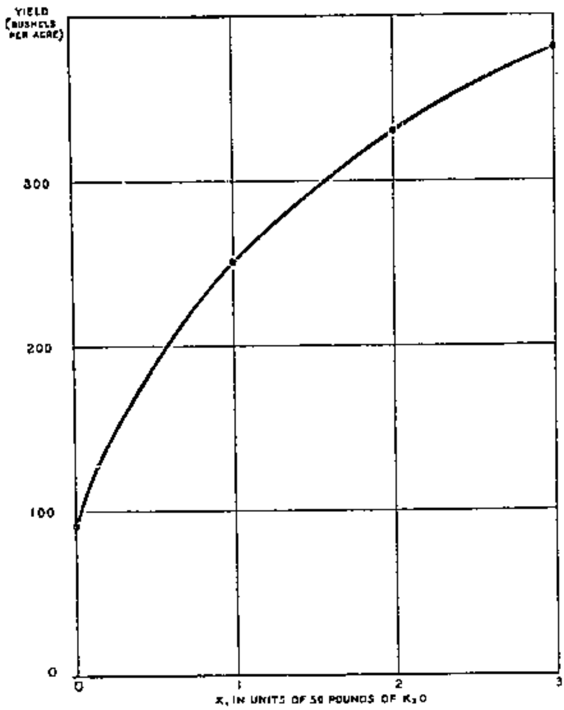

FIGURE 2.—PRELIMINARY DETERMINATION OF R

The actual yields are plotted, and a smooth curve of the form of Figure 1 is drawn as nearly as may be through the points.

Equation (1) may now be written for three points on the curve of Figure 1, these points being chosen in the following way:

One of the points should have the smallest and another the largest abscissa for which the yield is known, or can be read from the curve; the third point should have an abscissa half way between those of the other two.

For the curve of Figure 2, the points that meet these requirements are those whose abscissas are 0, 1.5, and 3. The corresponding values of Y are 91, 295 (read from the curve), and 381.

Using equation (1) for these three points, the observation equations are

$$91 = M - A \tag{A}$$
$$295 = M - AR^{1.5} \tag{B}$$
$$381 = M - AR^{3} \tag{C}$$

To evaluate R, subtract (A) from (B) and (B) from (C), giving

$$204 = A - AR^{1.5} = A(1 - R^{1.5}) \tag{D}$$
$$86 = AR^{1.5} - AR^{3} = AR^{1.5}(1 - R^{1.5}) \tag{E}$$

Dividing (E) by (D),

$$R^{1.5} = \frac{86}{204} = 0.42157.$$

Then

$$1.5 \text{ Log } R = \log 0.42157 = \bar{1}.624,8697 = -0.375,1303$$
$$\text{Log } R = -0.250,0869 = \bar{1}.749,9131$$

whence

$$R = 0.56223.$$

Since

$R^{1.5} = 0.42157$, it follows that $1 - R^{1.5} = 0.57843$.

From (D), it is seen that

$$A = \frac{204}{0.57843} = 352.68.$$

The value of M may now be obtained from any one of the equations (A) to (C). From (A)

$$M = 91 + A = 91 + 352.68 = 443.68$$

The values of R^x are then found to be:

For plot 1 (no potash)	1.00000
For plot 2 (1 unit of potash)	.56223
For plot 3 (2 units of potash)	.31610
For plot 4 (3 units of potash)	.17772

Using the indicated values of M, A, and R^x in equation (1) the calculated yields are: Plot 1, 91 bushels; plot 2, 245.39 bushels; plot 3, 332.2 bushels; and plot 4, 381 bushels. The calculated yields differ from the observed yields by 5.61 bushels on plot 2 and by 1.2 bushels on plot 3. Squaring each and adding, the sum of the squares of the differences between calculated and observed yields on the four plots (two of which are zero) is 32.9121. Values of M, A, and R, which give the smallest sum of squared differences or residuals are the most probable values. The roughness of the method and size of the sum (32.9121) suggest that better values for the constants may be found.

LOGARITHMIC METHOD

The logarithmic method may be used for determining the values of M, A, and R of the exponential yield curve in cases where the successive observations result in positive increments as x increases. If any observation is a smaller number than the previous observation this method can not be used, as a negative number as such has no logarithm.

Letting z represent the increment of Y due to a unit increment of x, equation (1), $Y = M - AR^x$, becomes

$$Y + z = M - AR^{x+1} \tag{A}$$

whence by subtraction

$$z = AR^x - AR^{x+1} = AR^x(1-R) = A(1-R)R^x$$

and

$$\text{Log } z = \log [A(1-R)] + x \log R \tag{B}$$

Equation (B) is solved by the method of least squares, for which the form used in Table 1 is convenient. The observations are entered in the columns at the left. The values of z are then entered opposite the corresponding values of x and the observation equations

(C), (D), and (E) are made up by substituting the proper values for log *z* and for *x*.

TABLE 1.—*Example of work for determining R and A by the logarithmic method*

[$\text{Log } z = \log [A(1-R)] + x \log R$] (B)

Plot No.	x	Y	z	Log z	Coefficient of log $[A(1-R)]$	Coefficient (x), of log R	Reference
1	0	91	160	2.2041200	1	0	(C)
2	1	251	80	1.9030900	1	1	(D)
3	2	331	50	1.6989700	1	2	(E)
4	3	381					

Sum of (C), (D), and (E) $5.8061800 = 3 \quad +3$ (F)
Sum of (D) and 2 times (E) $5.3010300 = 3 \quad +5$ (G)

Subtract (F) from (G) $-0.5051500 = \quad +2$

$$\text{Log } R = -.2525750 = \bar{1}.7474250;$$
$$R = 0.559017$$
$$1 - R = 0.440983$$

$$3 \log [A(1-R)] = 5.8061800 - 3 \log R$$ (F)
$$= 5.8061800 + 0.7474250$$
$$= 6.5639050$$
$$\log [A(1-R)] = 2.1879683$$

But

$$\log (1-R) = \bar{1}.6444218$$

Hence

$$\log A = 2.5435465$$

and

$$A = 349.58$$

The normal equation (F) for log $[A(1-R)]$ is obtained by adding the three observations as they stand, since the coefficient of this unknown is 1 in each observation equation.

The normal equation (G) for log R is obtained by multiplying each observation equation through by the coefficient of log R in that equation and adding the resulting equations.

Equations (F) and (G) are then solved by the usual methods of algebra, and the values of A and of R are obtained therefrom, as indicated in Table 1.

The value of M is found by writing an observation equation based on equation (1) for each value of x, thus

$$91 = M - A$$
$$251 = M - AR$$
$$331 = M - AR^2$$
$$381 = M - AR^3$$

Since the coefficient of M is 1 in each of these equations, the normal equation for M is the sum of the four as they stand, or

$$1{,}054 = 4M - A(1 + R + R^2 + R^3)$$

whence

$$M = \frac{1}{4}\,[1{,}054 + A(1 + R + R^2 + R^3)]$$

But the values of A (349.58) and of R (0.559017) have been found. (Table 1.)

The value of the parenthesis is

$$1.000000+0.559017+0.312500+0.174693=2.046210$$

Therefore

$$M=\frac{1}{4}\,[1{,}054+349.58(2.046210)]=442.33$$

Using these values of M, A, and R the computed yields of the four plots are 92.75 bushels on plot 1, 246.91 bushels on plot 2, 333.09 bushels on plot 3, and 381.26 bushels on plot 4. The corresponding residuals from the observed yields were respectively 1.75, −4.09, 2.09, and 0.26. The sum of the squares of these residuals is 24.2263 as compared with the 32.9121 obtained by the graphic method. The logarithmic method thus gives better results than does the graphic method.

The value of an unknown determined by the method of least squares, called its most probable value, is the arithmetical mean of its value in each of the observation equations. Now the arithmetical mean of n quantities is $\frac{1}{n}$ of their sum.

The value of an unknown determined from the most probable value of its logarithm is therefore the geometric, not the arithmetical, mean of the measurements, direct or indirect, made on the unknown; for when the logarithms of n quantities are added, the sum is the logarithm of their product; when the sum is divided by n the quotient is the logarithm of the nth root of their product, which nth root is the geometric mean of the n quantities. The only condition under which the arithmetical and geometric means of a series of numbers are equal is that all numbers in the series be equal. This condition arises in statistical work only when there are no errors of observation, a condition that practically never occurs. Hence the value of a quantity obtained from the most probable value of its logarithm is not the most probable value of the quantity.

There is thus an error in the logarithmic method here outlined. The greater the errors of observation in the data employed, the greater is the magnitude of this error. With fairly good observed values the error is not large. To illustrate:

Arithmetical mean:

$$\frac{1}{4}\,(49+52+45+54)=50$$

Geometric mean:

$$\sqrt[4]{49.\,52.\,45.\,54}=49.88$$

When the errors of observation are such as to render some of the values of z negative, the logarithmic method is not applicable at all, for a negative quantity, as such, has no logarithm.

NEW METHOD

The author's new method applies the principle of least squares directly to observation equations based on the equation (1), $Y=M-AR^x$. Development of the necessary normal equations in general terms is explained. Solution of the normal equations—finding the most probable values of M, A, and R—has been reduced to plain arithmetic. Finding the value of R, however, can only be done by successive approximations, so that the work involved is formidable and should not be undertaken without adequate equipment for making the computations.

Using a, b, c, d ---- to represent observed values of x, and Y_a, Y_b, Y_c, Y_d ---- to represent corresponding values of Y, the several observation equations may be written:

$$\left.\begin{aligned} Y_a&=M-AR^a\\ Y_b&=M-AR^b\\ Y_c&=M-AR^c\\ Y_d&=M-AR^d \end{aligned}\right\} \tag{A}$$

and so on.

Normal equations for M, A, and R are worked out applying the general rule, of which the example used under the logarithmic method is a special case. This general rule for finding the normal equation for any variable in a set of observation equations is: Multiply each equation through by the derivative of that equation with respect to the variable in question and add the resulting equations. The derivatives of equation (1) with respect to M, A, and R are:

For M the derivative is 1.
For A the derivative is R^x.
For R the derivative is AxR^{x-1}.

The normal equation for M is therefore the sum of the observation equations (A) as they stand; this gives

$$\Sigma Y=nM-A\Sigma R^x,$$

from which, transposing and dividing through by n

$$M=\frac{1}{n}[\Sigma Y+A\Sigma R^x] \tag{2}$$

In these expressions ΣY is the sum of the quantities represented by Y_a, Y_b, Y_c, Y_d ----, n is the number of observation equations, and $A\Sigma R^x$ is the sum of the several values R^a, R^b, R^c, R^d ---- multiplied by A.

The normal equation for A is obtained by multiplying the first observation equation through by R^a (the value of R^x, the derivative of A in (A) above,) the second by R^b, and so on, and adding the resulting equations. The resulting normal equation may be reduced to the form

$$A=\frac{n\Sigma YR^x-\Sigma Y\Sigma R^x}{(\Sigma R^x)^2-n\Sigma R^{2x}} \tag{3}$$

The normal equation for R, is obtained by multiplying the first observation equation through by AaR^{a-1}, (the derivative for R in (A) above), the second by AbR^{b-1}, and so on, and adding the resulting equations. The normal equation so written may be reduced for convenience of subsequent quantitative computation to the form

$$A'=\frac{n\Sigma YxR^x-\Sigma Y\Sigma xR^x}{\Sigma R^x\Sigma xR^x-n\Sigma xR^{2x}} \tag{4}$$

The A' of equation (4) and the A of equation (3) are identical in value; the prime mark is used in equation (4) as a convenient means of distinguishing between the two formulas.

SOLUTION OF THE NORMAL EQUATIONS

The problem of finding the most probable values of M, A, and R in equation (1) now resolves itself into that of finding that value of R that will make A' equal to A. This can be done only by the method of trial and error. The solution is demonstrated with the data considered under the graphic and logarithmic methods.

The procedure found to be most convenient for solving the equations is here given, using Table 2 in the demonstration. The quantities for which values are wanted as steps in the computation are entered on the blank form, and the values are entered as they are arrived at. Six sections were used in this problem and will usually be enough for finding the value of R. The actual working sheets should provide for as many lines as there are observation equations; for economy of printing only the four needed in this 4-plot problem are shown in Table 2. The symbols not previously used will be explained later; they are all used in other phases of the general problem and are provided for in the table.

The table of values of R^x, prepared by Y. Kutsunai of the Hawaiian Sugar Experiment Station (Table 18), reduces the labor of computation materially.

LOCATING THE VALUE OF R

The most probable values of M, A, and R are those that render the sum of the squares of the residuals a minimum. The residuals are the remainders obtained by subtracting the observed values from the calculated values of Y. For the problem now in hand the sum obtained by the graphic method was 32.9121; that obtained by the logarithmic method was 24.2263. Hence the values of M, A, and R obtained by the logarithmic method approach more nearly the values sought than those obtained by the graphic method.

The values of R found by these two methods were—

By the graphic method, $R=0.56223$.

By the logarithmic method, $R=0.559017$.

Since the second value of R is smaller than the first and the sum of the squared residuals is smaller than for the other, it is probable that the most probable value of R is smaller than 0.559017. Computation by the author's method may then start with $R=0.55$. (The alternative to use of either of the above methods or both for approximating the value of R as a preliminary to computation by the author's new method is more trials by the more onerous method.)

In the upper left section of Table 2 trial is made with $R=0.55$. Opposite the given values of x are placed the observed values of Y, then the corresponding values of R^x, the values of R^3 and higher powers of R being obtained from Table 18. The figures in the column headed xR^x are the indicated products of the figures in the first and third columns. The figures in the column headed R^{2x} are the squares of those in the column headed R^x. By addition the values of ΣY, ΣR^x, ΣxR^x, and ΣR^{2x} are obtained.

The next step is to find the value of A and of A', equations (3) and (4). The numerator of equation (3), N, consisting of a positive term, $n\Sigma YR^x$, and a negative term, $-\Sigma Y\Sigma R^x$, is worked. Then the denominator, D, is worked out, the division performed, and the value of A $(=346.161475)$ set down. The value of A' is found in the same way.

TABLE 2.—*Arrangement[1] of work for computing R, M, and A by the new method. Same data as in graphic and logarithmic methods*

$Y=M-AR^x$ $A=\frac{N}{D}$ $N=n\Sigma YR^x-\Sigma Y\Sigma R^x$ $D=\Sigma R^x\Sigma R^x-n\Sigma R^{2x}$

$M=\frac{1}{n}(\Sigma Y-A\Sigma R^x)$ $A'=\frac{N'}{D'}$ $N'=n\Sigma YxR^x-\Sigma Y\Sigma xR^x$ $D'=\Sigma R^x\Sigma xR^x-n\Sigma xR^{2x}$

n=number of observations=4 $E=A-A'$ $q=(\log M-\log A)/\log R$

Item	x	Y	R^x	xR^x	R^{2x}	R^x	xR^x	R^{2x}
1	0	91	1.0	0.0	1.0	1.0	0.0	1.0
2	1	251	.55	.55	.3025	.54	.54	.2916
3	2	331	.3025	.6050	.091506	.2916	.5832	.085031
4	3	381	.166375	.499125	.027681	.157464	.472392	.024795
Σ		1,054	2.018875	1.654125	1.421687	1.989064	1.595592	1.401426

Item		
N	1,570.2655 −2,127.894250=−557.628750	1,532.213536−2,096.473456=−564.259920
D	4.075856− 5.686748=− 1.610892	3.956376− 5.605704=− 1.649328
N′	2,113.8865 −1,743.447750= 370.438750	2,034.242208−1,681.753968= 352.488240
D′	3.339472− 2.274220= 1.065252	3.173735− 2.144188= 1.029547
A	346.161475	342.115043
A′	347.747528	342.372170
E	−1.586053	−.257127

Item	x	Y	R^x	xR^x	R^{2x}	R^x	xR^x	R^{2x}
1	0	91	1.0	0.0	1.0	1.0	0.0	1.0
2	1	251	.538	.538	.289444	.539	.539	.290521
3	2	331	.289444	.578888	.083778	.290521	.581042	.084402
4	3	381	.155721	.467163	.024249	.156591	.469773	.024521
Σ		1,054	1.983165	1.584051	1.397471	1.986112	1.589815	1.399444

Item		
N	1,524.694660−2,090.255910=−565.561250	1,528.450488−2,096.473456=−564.259920
D	3.932943− 5.589884=− 1.656941	3.944641− 5.597776=− 1.653135
N′	2,018.556124−1,669.589754= 348.966370	2,026.389660−1,675.665010= 350.724650
D′	3.141345− 2.118898= 1.022447	3.157551− 2.131552= 1.025999
A	341.328539	341.721372
A′	341.305095	341.837224
E	.023444	−.115852

Item	x	Y	R^x	xR^x	R^{2x}	R^x	xR^x	R^{2x}
1	0	91	1.0	0.0	1.0	1.0	0.0	1.0
2	1	251	.5381	.5381	.289552	.5382	.5382	.289659
3	2	331	.289552	.579104	.083840	.289659	.579318	.083902
4	3	381	.155808	.467424	.024276	.155895	.467685	.024303
Σ		1,054	1.983460	1.584628	1.397668	1.983754	1.585203	1.397864

Item		
N	1,525.070640−2,090.566840=−565.496200	1,525.445296−2,090.876716=−565.431420
D	3.934114− 5.590672=− 1.656558	3.935280− 5.591456=− 1.656176
N′	2,019.340272−1,670.197912= 349.142360	2,020.121772−1,670.803962= 349.317810
D′	3.143046− 2.120240= 1.022806	3.144653− 2.121488= 1.023165
A	341.368186	341.407800
A′	341.357364	341.409069
E	.010822	−.001269

x	R^x	AR^x	M	Y (calculated)	Y (observed)	e	e^2
0	1.0	341.40	432.81	91.41	91	0.41	0.1681
1	.538188	183.74	432.81	249.07	251	−1.93	3.7249
2	.289646	98.89	432.81	333.92	331	2.92	8.5264
3	.155884	53.22	432.81	379.59	381	−1.41	1.9881
	1.983718						14.4075

		Logarithms
R	0.538188	−0.2690660
A	341.401	2.5332635
M	432.81	2.6362973

Unit of x 50 pounds of K_2O. q=0.3829 units=19.145 pounds per acre.

[1] The author's work sheets were mimeographed blank forms, with spaces provided for 8 values of x. The blank lines have been omitted in printing. See text for development of the work recorded on this form. Table 18 and a computing machine are practical necessities.

To obtain the value of the positive term, $n\Sigma YR^x$, of N, for example, set up 91 (=91 times 1.0) in the computing machine; add to it 251 times 0.55; add to that 331 times 0.3025; and then 381 times 0.166375; the sum so accumulated (the ΣYR^x) is then multiplied by n, in this case 4. The result, 1,570.265500 is recorded as the first term of N in the line below the columns of Table 2.

The second (negative) term of N is $\Sigma Y \Sigma R^x = 1{,}054$ times $2.018875 = 2{,}127.894250$. The algebraic sum of the two terms of N is -557.628750.

The reader should now be able to follow the procedure in obtaining the values of D, N', and D'.

Note that the negative terms of both N and D are larger than the positive terms. If the positive terms are computed first they may be subtracted from the negative terms without resetting the negative terms in the machine. The negative terms of N' and D' are smaller than the positive and may be advantageously computed first.

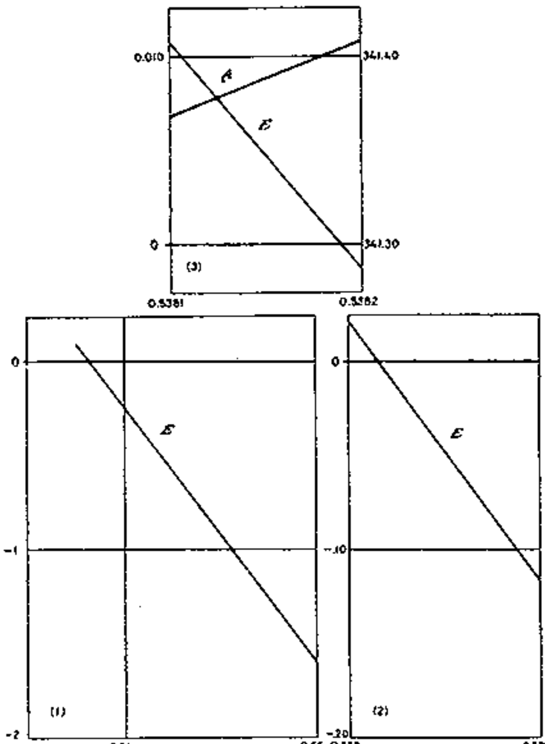

FIGURE 3.—STEPS IN THE DETERMINATION OF A AND R

In section 1, the values of E found in Table 2 are plotted. The line crosses the zero line at about 0.538. The value for E at 0.538 is positive (sec. 2), hence too small; so the value at 0.539 was computed, the line drawn through these points indicating a value between 0.5381 and 0.5382. Plotting the values of E and of A at these values and drawing the lines (sec. 3) gives the true values sought, $R=0.538188$ (E is 0 at this point), and $A=341.401$ for this value of R.

The difference E, between A and A' in this case, with $R=0.55$ is -1.586053. When the correct value of R is obtained, E will be zero. Under the conditions of this problem a negative value of E indicates that the value of R under test is too large. (See discussion of critical values of R below for conditions indicating the opposite situation.) Accordingly the value of $R=0.54$ is tried. (The computations are shown in the upper right section of Table 2.) This time E becomes -0.257127, much nearer zero, but still negative, indicating that R is somewhat less than 0.54.

With two values of E available, some work may be saved by graphing the values as in Figure 3 (1) before proceeding with further trials. A line drawn through the two computed values of E crosses the zero line at about 0.538. Hence the next value of R tested is 0.538. It should be noted here that the graph of E is somewhat curved, being convex downward. This curvature of the graph of E is the more marked the fewer the decimal places in the value of R under test.

Because of this curvature, the straight line used does not give the exact value of R, as the true graph of E probably crosses the zero line somewhat to the right of 0.538.

The value of E when R is placed at 0.538 comes out 0.023444, positive, indicating that R is greater than 0.538. The value of E when R is placed at 0.539 is -0.115852, negative, indicating a value of R less than 0.539. These last two values of E are graphed on a larger scale in Figure 3 (2), the E line crossing the zero line at about 0.53818.

Ordinarily it would not be necessary to carry the computation of R beyond this point, but since a comparison is desired between methods 1, 2, and 3 the values $R=0.5381$ and $R=0.5382$ are tested with the results given in Table 2. The line drawn through the values of E for these values of R crosses the zero line at a point indicating a value of 0.538188 for R. (Fig. 3 (2).)

The last two values of A are also graphed in Figure 3 (3). At $R=0.538188$ A is 341.401. These are taken as the most probable values of A and of R. These two accepted values are then used to find the most probable value of M. The steps in finding M are: Compute the values of R^x using the accepted value of R, entering them on the form (at the bottom); multiply each by A and enter them; add this column (giving $A\Sigma R^x$). Add to this sum the sum of the observed yields (ΣY); divide by n. The value of M comes out 432.81.

The calculated values of Y are obtained subtracting each value in the column headed AR^x from 432.81. The differences between the calculated and the observed yields (column headed "e") are entered, squared, and the squares totaled ($\Sigma e^2=14.4075$). The smaller this sum is, the more accurate is the fit. The values of M, A, and R found by the new method are thus considerably more satisfactory than those found by the simpler methods, for the sum of the squared residuals is much smaller.

There are numerous other methods of finding the approximate values of M, A, and R, but those given are sufficient for practical purposes.

CRITICAL VALUES OF R

The work of finding the value of R is complicated in some cases by the fact that for certain values of R the value of D' (Table 2) becomes zero, thus making A' infinite.

When the values of x used include zero, and consist of the consecutive numbers 0, 1, 2, 3, etc., D' is always positive, so that the complication mentioned does not occur. In these cases E is positive for all values of R less than the true value, and negative for all values greater than the true value.

But if the given values of x do not include zero, and consist of the consecutive numbers 1, 2, 3, 4, etc., then there is always a value of R, here called its critical value, for which $D'=0$. The critical values of R for series of x values commonly occurring in experimental work are as follows:

Series of x values	Critical value of R
1, 2, 3, 4	0.621173
1, 2, 3, 4, 5	.661689
1, 2, 3, 4, 5, 6	.694225
1, 2, 3, 4, 5, 6, 7	.720886
1, 2, 3, 4, 5, 6, 7, 8	.743354
1, 2, 3, 4, 5, 6, 7, 8, 9	.762477

Let R_m represent the most probable value of R,
R_t the value under test, and
R_c the critical value.

Figure 4 and the relations stated in tabular form below will aid in determining whether a given positive or negative value of E indicates that R_t is too large or too small.

In each of the three drawings of Figure 4 the abscissas are values of R, while the ordinates are values of A and A'. The most probable value of R, symbolized by R_m, is the abscissa of the point, X, at which the graph of A' crosses that of A. At this point $A'=A$ and $E=0$. The critical value of R, symbolized by R_c, is the abscissa of the point at which D' (not shown in the drawing) becomes zero and A' becomes infinite.

It is readily seen in Figure 4 (1), which represents cases having no critical value for R, that is, in which D' does not become zero for any value of R, that a positive E shows R_t to be too small, while a negative E shows R_t to be too large.

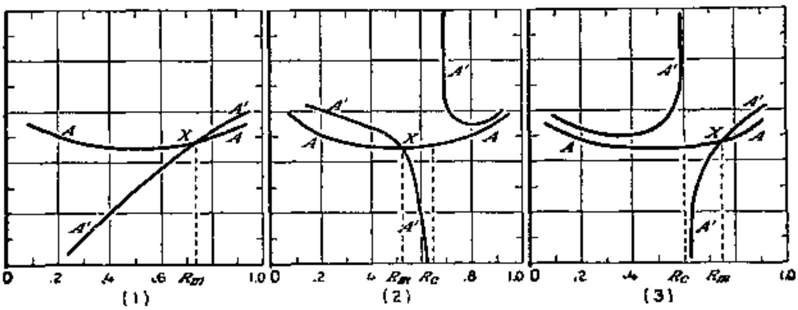

FIGURE 4.—GRAPHS OF THE VALUES OF A AND A'

A and A' are equal at some value of R, called its most probable value (R_m). In experimental series which do not include a value for $x=0$ there is always a critical value of R (R_c) which renders A' nfinite. Section 2 illustrates cases in which R_m is smaller than R_c; section 3, cases when R_m is larger than R_c. Section 1 covers those cases in which the values of x are consecutive numbers beginning with 0.

In Figure 4 (2) and Figure 4 (3) the relations between R_t and R_m may be stated as follows:

When preliminary tests (by the graphic or the logarithmic method indicate that R_m is less than R_c (fig. 4 (2)), then a negative E indicates that R_t is too small and a positive E indicates that R_t is too large.

When R_m is greater than R_c (fig. 4 (3)), then a negative E indicates that R_t is too large and a positive E indicates that R_t is too small.

Occasionally R_m lies so near to R_c that the value of R_m determined by preliminary tests may lie on the wrong side of R_c. Suppose, for instance, that the preliminary value of R_m is slightly greater than R_c, (fig. 4 (3)), while the true value of R_m is slightly less than R_c (fig. 4 (2)). In such a case a few trials of R values greater than R_c will show that as R_t decreases, A' rapidly increases, which shows that Figure 4 (2) and not Figure 4 (3) applies.

If for values of R_t slightly less than R_c the value of A' increases rapidly for slight increases in R_t, then R_m is greater than R_c.

For irregular series of x values it is necessary to work out the values of D' for a series of R values to determine the critical value of R, if any such value exists.

AVAILABLE PLANT FOOD IN THE SOIL

The quantity of a plant-food element available in the soil without any fertilizer application, designated as q, may be computed at this stage. The value of q is the value of x in equation (1) when Y is zero; that is, q equals x in the equation $0 = M - AR^x$, which reduces to

$$x = \frac{\log M - \log A}{\log R} = q \qquad (5)$$

This value of x is the point at which the yield curve crosses the x axis. This point lies to the left of the origin; hence q comes out negative. The negative sign is to be disregarded, for the origin was arbitrarily placed.

The data of this example show that q is 19.145 pounds of potash (K_2O) per acre. (Table 2.)

DEMONSTRATION USING EXPERIMENTAL RESULTS WITH THREE VARIABLE FACTORS

The preceding discussion relates to a case in which a single plant-food element (potassium as K_2O) was varied. In what follows the same principles are applied to a case in which varying quantities of nitrogen, phosphoric acid, and potash were applied to the same crop, the elements being varied one at a time; that is, plots receiving different quantities of nitrogen all received the same quantity of phosphoric acid and the same quantity of potash, those receiving different quantities of phosphoric acid received the same quantity of nitrogen and the same quantity of potash, and so on.

The data used below are from a series of experiments with fertilizers for tobacco, conducted at Tifton, Ga., by W. W. Garner and his associates, of the Bureau of Plant Industry, in cooperation with the State College of Agriculture and the Coastal Plain Experiment Station of that State. These data were kindly supplied by Doctor Garner.

The essential facts are given in Table 3.

TABLE 3.—*Fertilizers applied and yields obtained in the tobacco experiments at Tifton, Ga.*

[Rate of fertilizer application, 1,000 pounds per acre]

Plot No.	Analysis of fertilizer applied			Yields of tobacco per acre in—					
	NH_3	P_2O_5	K_2O	1924	1925	1926	1927	1928	1929
	Per cent	*Per cent*	*Per cent*	*Pounds*	*Pounds*	*Pounds*	*Pounds*	*Pounds*	*Pounds*
1	5	8	5	1,235	1,571	1,219	1,329	1,182	1,452
2	4	8	5	1,281	1,572	1,224	1,201	1,051	1,262
3	3	8	5	1,206	1,499	1,157	1,252	975	1,073
4	2	8	5	1,004	1,398	1,092	1,351	930	972
5	0	8	5	867	1,129	914	1,427	693	960
6 [1]	0	0	0	519	769	484	593	401	501
7	3	12	5	1,115	1,653	1,230	1,306	973	1,133
8	3	10	5	1,214	1,670	1,150	1,165	1,005	1,134
9	3	9	5	1,237	1,550	1,047	1,236	979	1,163
10	3	7	5	1,214	1,707	1,173	1,256	1,067	1,086
11	3	6	5	1,223	1,620	1,091	1,226	928	1,276
12	3	4	5	1,285	1,660	1,098	1,206	895	1,100
13	3	0	5	1,210	1,407	935	550	509	445
14 [1]	0	0	0	785	770	720	530	495	518
15	3	8	8	1,293	1,733	1,230	1,441	1,132	1,039
16	3	8	6	1,150	1,735	1,187	1,318	1,040	1,138
17	3	8	4	1,109	1,752	1,004	1,211	1,026	930
18	3	8	3	1,115	1,518	1,102	1,240	943	1,190
19	3	8	2	1,184	1,347	1,074	1,160	856	909
20	3	8	0	890	1,018	697	981	578	834

[1] Check plot.

The yields shown in Table 3 are graphed in Figure 5. The graphs in the first column are based on the yields from plots 1 to 5, in which ammonia is the variable; those in the second column relate to plots 7 to 13, in which phosphoric acid is the variable; and those in the last column relate to plots 15 to 20, in which the variable is potash. The numbers attached to the lines indicate the year, the first year being 1924. The yields on all plots were extraordinarily high in the second year (1925). Where these yields were used they were reduced by a

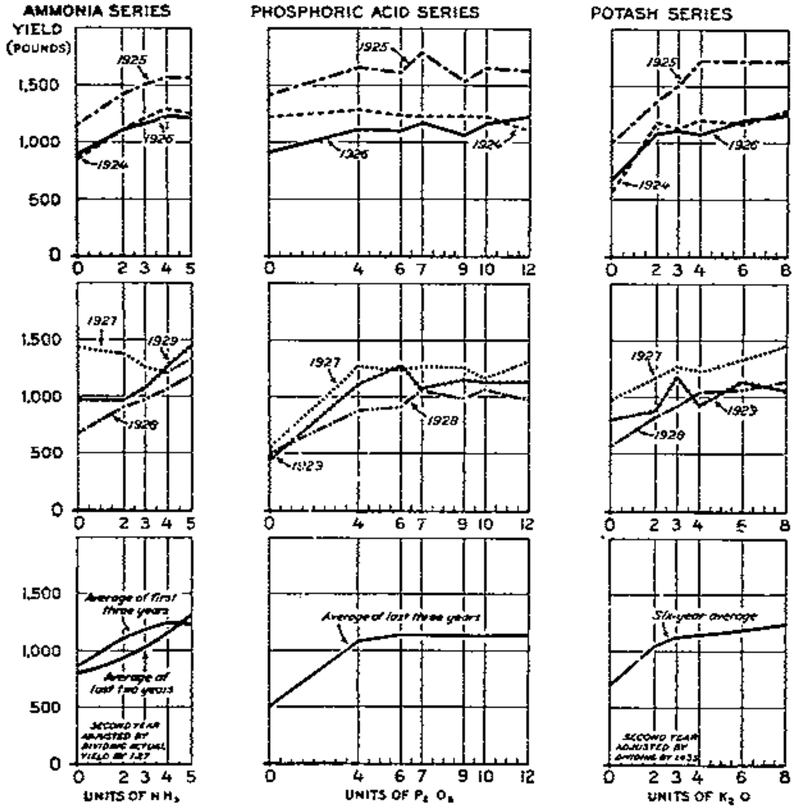

FIGURE 5.—YIELDS ON TOBACCO EXPERIMENTAL PLOTS AT TIFTON, GA., 1924–1929

These are the yields shown in Table 3 and the averages used in the computations. The separation into groups of three years is to avoid confusion of lines. The reasons for selecting the several averages are given in the text. The rate of application of fertilizer was 1,000 pounds per acre, and the unit of x (the growth factor) was 1 per cent, or 10 pounds, of each of the three plant foods.

factor which reduced the average of all nitrogen plots in 1925 to the average of the first and third years. This was done to avoid exaggeration of yields due to a very unusual season in 1925. The graphs in each column are separated into groups of three each to avoid confusion from so many overlapping lines and to bring out certain relations that are discussed below.

The graphs show a number of things of great interest. In the nitrogen series, curves 1 and 2 are almost identical in form, whereas curve 3 departs from this form very slightly. The graphs for later years show, strikingly, a cumulative effect of the nitrogen. This effect begins to show plainly the fourth year, and is very marked in each year following, more so the sixth than the fifth year.

The hypothesis is suggested that after the experiment had continued for about three years the resulting cumulative effect of the increased organic matter, presumably from the decaying roots and stubble of the larger crops on the more heavily fertilized plots, began to affect the yield on the latter plots. The yields in these later years on the plots receiving much nitrogen were influenced not only by the nitrogen applied from year to year but also by the increasing reserve of organic matter in the soil.

In the lower part of the first column of Figure 5 a curve is shown of the average yields the fifth and sixth years. This curve has the form of the lower part of a curve representing the effect of varying two or more growth factors. (Fig. 11.) In this case the factors concerned are presumably nitrogen in fertilizer and plant food made available by the decay of remains from previous crops, the latter increasing from year to year. Had the series included plots receiving more nitrogen, this curve (including the part extending beyond the limits shown in the drawing) would presumably have approximately the form of that of Figure 11.

If the above interpretation is correct, it is obvious that a formula intended to express solely the relation between yield and quantity of fertilizer applied would not apply to the results of an experiment continued for many years, at least so far as nitrogen is concerned.

In the computations that follow, the ammonia results used are the average results for the first three years of the experiment (the second-year results being adjusted as described above), because the yields in later years presumably do not represent merely the relation between yield and current applications of fertilizer.

The phosphoric acid plots (graphs in second column of fig. 5) show the reverse side of the same picture. The soil on which these plots were located had been heavily fertilized with phosphoric acid for many years. Judging by the first-year results, the soil was supplied with all the available phosphates the plants could use, so far as effect on yield is concerned, for the plot receiving no phosphates that year yielded about as well as the others.

But as time went on, the situation changed markedly. The yield on the plot receiving no phosphoric acid fell off rapidly as the store of available phosphoric acid in the soil was reduced. The yields on this plot were very low and almost identical in each of the last three years. Apparently the rate of exhaustion of phosphoric acid on the no-phosphate plot (No. 13) was very rapid during the first four years, and by that time the phosphoric acid supply had nearly reached the minimum that the soil could develop regularly.

Doctor Garner stated that the effect of phosphates on the quality of tobacco is so marked that it pays to use more of this fertilizer constituent than is necessary from the standpoint of yield alone. This effect appears to be produced by influencing the date of maturity of the crop. To secure the best quality of product the crop must be harvested while the weather is still quite warm. Heavy applications of phosphates hasten maturity, and hence result in a product of higher quality.

In general, the longer the growing season of a crop the greater the yield, other things being equal, though exceptions to this rule occur. There is some evidence in the graph at the bottom of the middle section of Figure 5 that yields are slightly lower on those plots receiving the heaviest applications of phosphoric acid. This may be due to the increasing earliness of the crop with increasing doses of phosphates.

So far as yields alone are concerned, the results from the phosphate plots for the first year of the experiment are the only ones that give a true picture of the fertilizer relations of the soil at the time the experiment began. The yields that year indicate that the crop had all the phosphoric acid it could use in making increased yields. The curve for the average of the last three years (bottom of middle section of fig. 5) indicates, however, that it took about 60 pounds of phosphoric acid (P_2O_5) a year to maintain maximum yields. Quality of product may demand more than 60 pounds, possibly at the expense of slightly reduced yields due to earlier maturity of the crop.

In the computations that follow, use is made of average yields on plots 11, 12, and 13 of the phosphoric acid series for the last three years, for the reason that these yields permit an estimate of the reduction in yield due to earlier maturity caused by large phosphate applications, as will be seen later.

The potash curves in Figure 5 show neither the cumulative effect of heavy applications seen in the ammonia curves nor the exhaustive effect of light applications seen in the phosphoric acid curves. For this reason the average results for the six years were used in the computations, with the second-year results adjusted as previously stated.

The computed values found in what follows are not to be regarded as final, since these experiments were planned for a different purpose. Particularly the absence of a complete series of check plots affects the results from the standpoint of the present purpose. Nevertheless the results illustrate very satisfactorily the methods of computation required for a series of plots fertilized as suggested later in this paper.

It will be observed that the nitrogen data in Table 3 are given in units of ammonia. These data could easily be converted into terms of nitrogen, since a pound of ammonia contains fourteen-seventeenths of a pound of nitrogen. But this would necessitate carrying out extensive calculations with fractional values of x. The computations are therefore carried out on the ammonia basis. In the final results the conversion of pounds of ammonia into pounds of nitrogen may easily be accomplished.

AMMONIA SERIES

(Plots 1 to 5)

The yields used for the ammonia series were the average yields of plots 1 to 5 for the first three years of the experiment, with second-year yields reduced as previously explained. A preliminary reading on the value of R was obtained by the graphic method, and the values of M, A, and R in the equation $Y=M-AR^x$ were then calculated by the author's new method.

The results were

$$R=0.69787$$
$$A=436.02$$
$$M=1323.44$$

The unit of x used was 10 pounds of ammonia (NH_3) per acre. To check the values found, calculated and observed yields were compared as follows:

Plot	x	R^x	AR^x	Y (calculated)	Y (observed)	ϵ
5	0	1.0	436.02	887.42	890	−2.58
4	2	.48702	212.35	1,111.09	1,095	16.09
3	3	.33988	148.19	1,175.25	1,181	−5.75
2	4	.23719	103.42	1,220.02	1,248	−27.98
1	5	.16553	72.17	1,251.27	1,231	20.27

The greatest residual is slightly more than 2 per cent of the corresponding observed yield, which must be considered fair agreement between theory and practice.

PHOSPHORIC ACID SERIES

(Plots 7 to 13)

The effect of phosphates in increasing yields is shown by those plots receiving 0, 40, and 60 pounds of phosphoric acid (P_2O_5). Higher applications gave no further increase; indeed, there appears to be a slight decrease in yield from them. This is presumably due to the effect of heavy applications in hastening the maturity of the crop. By using the yields from 0, 40, and 60 pounds of P_2O_5 some measure of this hastening effect can be obtained. The constants of the yield equation are, therefore, calculated from the yields of those plots receiving 0, 40, and 60 pounds of P_2O_5.

The observation equations based on these plots are:

$$\text{Plot 13, } x=0, \quad 504 = M - A \tag{A}$$

$$\text{Plot 12, } x=4, \quad 1{,}067 = M - AR^4 \tag{B}$$

$$\text{Plot 11, } x=6, \quad 1{,}143 = M - AR^6 \tag{C}$$

There being three equations between the three unknowns, they may be solved directly by the methods of algebra.

Subtracting (A) from (B) and (B) from (C),

$$563 = A(1 - R^4) \tag{D}$$

$$76 = AR^4(1 - R^2) \tag{E}$$

Dividing (E) by (D),

$$\frac{R^4}{1+R^2} = 0.13499112$$

The solution of this equation gives $R = 0.66412$, whence $R^4 = 0.19453$.

Substituting this value of R^4 in (D), and dividing through by $(1 - R^4)$,

$$A = \frac{563}{0.80547} = 698.97,$$

whence, using equation (A) since $Y_0 = 504.00$, $M = 1202.97$[3].

The yields of each of the plots in the phosphoric acid series may now be calculated by the equation

$$Y = M - AR^x \tag{1}$$

as follows

Plot No.	x	R^x	AR^x	Calculated Y	Observed Y	ϵ
13	0	1.00000	698.97	504.00	504	0
12	4	.19453	135.97	1,067.00	1,067	0
11	6	.08580	59.97	1,143.00	1,143	0
10	7	.05698	39.83	1,163.14	1,133	30.14
9	9	.02513	17.57	1,185.40	1,126	59.40
8	10	.01669	11.67	1,191.30	1,121	70.30
7	12	.00736	5.14	1,197.83	1,137	60.83

[3] In equation (1), M is the theoretical maximum yield (value of Y), while A is the theoretical maximum increase in yield from $x=0$; hence, if Y_0 represent the yield at $x=0$, $M = A + Y$.

The residuals (e) in the last column presumably show the decrease in yield due to earlier maturity from the heavier applications of phosphates. This is doubtless more than compensated by increase in quality, and consequently in price.

POTASH SERIES

(Plots 15 to 20)

The potash plots (last section of fig. 5) show neither the cumulative effect of large doses as the ammonia plots do, nor the gradual depletion on the lightly fertilized plots as the phosphoric acid plots do. The yields, however, are relatively irregular, and the values of M, A, and R calculated from them are less reliable than are those from the other series. In the absence of the trends above referred to, the 6-year average yield of each plot was used in the computations, the yields in 1925 being reduced to make the average that year comparable with the average of the first and third years.

The constants were evaluated by the author's new method, with the following result:

$$R=0.64364$$
$$A=492.61$$
$$M=1{,}224.85$$

In these computations 10 pounds of potash (K_2O) was used as the unit of x.

From the above values the yield of each plot may be calculated from the usual yield equation, as follows:

Plot No.	x	R^x	AR^x	Calculated Y	Observed Y	e
20	0	1.00000	492.61	732.24	731	1.24
19	2	.41427	204.07	1,020.78	1,021	−.22
18	3	.26664	131.35	1,093.50	1,110	−16.50
17	4	.17162	84.54	1,140.31	1,125	15.31
16	6	.07110	35.02	1,189.83	1,175	14.83
15	8	.02945	14.51	1,210.34	1,225	−14.66

Having worked out the constants of equation (1) for each of the fertilizer elements separately, the next step is to apply the results to equation (7), in which all three elements are variable. This is done in the following pages.

Data are now at hand for computing the quantities of available plant-food elements in the soil of this experimental field. These data, from the three preceding series, are:

Ammonia series: $M=1{,}323.44$, $R=0.69787$, and $A=436.02$.
Phosphoric acid series: $M=1{,}202.97$, $R=0.66412$, and $A=698.97$.
Potash series: $M=1{,}224.85$, $R=0.64364$, and $A=492.61$.

Substituting these values in equation (5), and remembering that the unit of each element is 10 pounds, we find that the values of q (the quantity available in the soil) for each of the series are

Ammonia series, 30.89 pounds NH_3.
Phosphoric acid series, 13.25 pounds P_2O_5.
Potash series 20.66 pounds K_2O.

ALL THREE VARIABLES TAKEN TOGETHER

The three values of A found above are the limiting values of Y, (1), when amomnia alone varies, and phosphoric acid and potash are held constant at 80 and 50 pounds per acre, respectively; (2), when phosphoric acid varies and ammonia and potash remain constant at 30 and 50 pounds per acre, respectively; and so on.

It is necessary now to find the value of A which represents the limiting value of Y when all three of the fertilizer constituents vary. The computations involve finding the value of $(1-R^x)$ for many values of x. To save labor, a table of values of $(1-R^x)$ for all values of x likely to occur in such computations is appended. (Table 19). It is used in a manner similar to that in which a logarithmic table is used.[4] Values of x are given in the first column and the corresponding values of $1-R^x$ in the columns to the right. The figure in the second decimal place in the value of x is placed at the head of a column of values of $1-R^x$, as in a table of logarithms. The table is calculated for $R=0.8$. Hence it is necessary to convert the units of ammonia, phosphoric acid, and potash thus far used into new units that will give each R the value 0.8.

If R represent the ratio of any series of yield increments due to successive unit increases in a given growth factor, then the number, u, of such units that must be used as a new unit in order that R shall equal 0.8 is the value of u in the equation $R^u=0.8$.

Passing to logarithms, $u \log R=\log 0.8$, whence

$$u=\frac{\log 0.8}{\log R}=\frac{-0.0969100}{\log R} \qquad (6)$$

The values found for R in the preceding work, with their logarithms, are

Ammonia series; $R=0.69787$; and $\log R=-0.1562255$.
Phosphoric acid series: $R=0.66412$; and $\log R=-0.1777534$.
Potash series: $R=0.64364$; and $\log R=-0.1913570$.

Substituting these values of $\log R$ in equation (6) above, we obtain the following values of u: Ammonia series, 0.6203; phosphoric acid series, 0.5452; potash series, 0.5064. Since the old units of each growth factor are 10 pounds each, the equivalent new units are respectively

Ammonia series, 6.203 pounds.
Phosphoric acid series, 5.452 pounds.
Potash series, 5.064 pounds.

The most general form of the yield equation, when all three fertilizer constituents vary, is

$$y=A(1-R^{n+a})\ (1-R^{p+b})\ (1-R^{k+c}) \qquad (7)$$

The derivation of this equation is given later. The significance of the quantities involved is as follows:

y = yield per acre.

A = limit approached by y as a, b, and c increase.

R = the ratio of the series of increments in yield for successive unit increments in a, b, or c, the size of the unit in each case being such as to make $R=0.8$.

n, p, and k = the respective quantities, in the above units, of nitrogen (ammonia in the case under consideration), phosphoric acid, and potash available in the soil.

a, b, and c = the respective quantities of these three fertilizer elements in the fertilizer applied.

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[4] The use of this table is explained on p. 60.

To use equation (7) it is necessary to express n, p, k, a, b, and c in terms of the new units mentioned above. To aid in doing this, some of the data developed in preceding pages are here brought together.

For n and a the new unit is 6.203 pounds of NH_3.
For p and b the new unit is 5.452 pounds of P_2O_5.
For k and c the new unit is 5.064 pounds of K_2O.
The value of n, 30.89 pounds, is 4.98 new units.
The value of p, 13.25 pounds, is 2.43 new units.
The value of k, 20.66 pounds, is 4.08 new units.

Table 4 shows in the first group of three columns the plant-food elements applied in fertilizer to each of Garner's plots, in units of 10 pounds (1 per cent of 1,000 pounds). In the next group the same quantities are shown in pounds (per acre, of course). In the third group these quantities are expressed in the new units, namely, units of the magnitude required to make the ratio (R) of each series equal 0.8.

TABLE 4.—*Plant food available on each plot in the tobacco experiments*

Plot No.	Plant food applied, in terms of—									Total available[1] (in new units)		
	Units of 10 pounds			Pounds			New units					
	NH_3, a	P_2O_5, b	K_2O, c	NH_3, a	P_2O_5, b	K_2O, c	NH_3, a	P_2O_5, b	K_2O, c	NH_3, $n+a$	P_2O_5, $p+b$	K_2O, $k+c$
1	5	8	5	50	80	50	8.06	14.67	9.87	13.04	17.10	13.95
2	4	8	5	40	80	50	6.45	14.67	9.87	11.43	17.10	13.95
3	3	8	5	30	80	50	4.84	14.67	9.87	9.82	17.10	13.95
4	2	8	5	20	80	50	3.22	14.67	9.87	8.20	17.10	13.95
5	0	8	5	0	80	50	0	14.67	9.87	4.98	17.10	13.95
6[2]	0	0	0	0	0	0	0	0	0	4.98	2.43	4.08
7	3	12	5	30	120	50	4.84	22.01	9.87	9.82	24.44	13.95
8	3	10	5	30	100	50	4.84	18.34	9.87	9.82	20.77	13.95
9	3	9	5	30	90	50	4.84	16.51	9.87	9.82	18.94	13.95
10	3	7	5	30	70	50	4.84	12.84	9.87	9.82	15.27	13.95
11	3	6	5	30	60	50	4.84	11.01	9.87	9.82	13.44	13.95
12	3	4	5	30	40	50	4.84	7.34	9.87	9.82	9.77	13.95
13	3	0	5	30	0	50	4.84	0	9.87	9.82	2.43	13.95
14[2]	0	0	0	0	0	0	0	0	0	4.98	2.43	4.08
15	3	8	8	30	80	80	4.84	14.67	15.80	9.82	17.10	19.88
16	3	8	6	30	80	60	4.84	14.67	11.85	9.82	17.10	15.93
17	3	8	4	30	80	40	4.84	14.67	7.90	9.82	17.10	11.98
18	3	8	3	30	80	30	4.84	14.67	5.92	9.82	17.10	10.00
19	3	8	2	30	80	20	4.84	14.67	3.95	9.82	17.10	8.03
20	3	8	0	30	80	0	4.84	14.67	0	9.82	17.10	4.08

[1] Quantity available in the soil plus quantity applied in fertilizer.
[2] Check plots; no fertilizer applied.

In the last group of three columns the numbers are obtained by adding the value of n, which is 4.98 in the new units, to each number in the a column of the preceding group, the value of p, or 2.43, to each number in the b column, and the value of k, or 4.08, to each number in the c column.

The last group of three columns thus shows the total quantity of each food element available to the crop on each of the 20 plots. It includes both the plant-food elements in fertilizers applied (a, b, c), and those available in the soil before the fertilizers were applied (n, p, k).

It is now possible by means of equation (7), Table 4, and Table 19, to find a value of the A of equation (7) for each of the 20 plots. In the case of plot 1, for instance, equation (7) becomes

$$y = A(1 - R^{13.04})\ (1 - R^{17.10})\ (1 - R^{13.95}) \qquad \text{(A)}$$

the value of R being 0.8. From Table 19 it will be found that

$$1-R^{13.04}=0.94551$$
$$1-R^{17.10}=0.97798$$
$$1-R^{13.95}=0.95553$$

The calculated yield for this plot (p. 20) is 1,251.27 pounds. Substituting this value and the above values of the three parentheses in equation (A) above, transposing both members, and then dividing through by the parentheses

$$A=1{,}251.27/(0.94551)\ (0.97798)\ (0.95553)=1{,}416.15 \text{ pounds}$$

TABLE 5.—*Computation of the value of A for the plots of the tobacco experiments*

$$\left[A=\frac{\text{Calculated yield}}{\text{Product}}=\frac{y}{(1-R^{n+a})\ (1-R^{p+b})\ (1-R^{k+c})}.\ \text{Data of Table 4}\right]$$

Plot No.	Calculated yield,[1] y (pounds)	$1-R^{n+a}$ [2]	$1-R^{p+b}$ [2]	$1-R^{k+c}$ [2]	Product	A (pounds)
1	1,251.27	0.94551	0.97798	0.95553	0.88357	1,416.15
2	1,220.02	.92196	.97798	.95553	.85156	1,416.06
3	1,175.25	.88823	.97798	.95553	.83004	1,415.90
4	1,111.09	.83991	.97798	.95553	.78489	1,415.60
5	887.42	.67085	.97798	.95553	.62690	1,415.57
6		.67085	.41855	.59765	.16781	
Average						1,415.86
7	1,107.83	.88823	.99572	.95553	.84510	1,417.38
8	1,191.30	.88823	.99020	.95553	.84049	1,417.39
9	1,185.40	.88823	.98539	.95553	.83633	1,417.38
10	1,163.14	.88823	.96687	.95553	.82061	1,417.41
11	1,143.00	.88823	.95017	.95553	.80644	1,417.34
12	1,067.00	.88823	.88697	.95553	.75280	1,417.38
13	504.00	.88823	.41855	.95553	.35524	1,418.76
14		.67085	.41855	.59765	.16781	
Average						1,417.58
15	1,210.34	.88823	.97798	.98810	.85830	1,410.01
16	1,189.83	.88823	.97798	.97141	.84384	1,410.02
17	1,140.31	.88823	.97798	.93097	.80871	1,410.04
18	1,093.50	.88823	.97798	.89263	.77540	1,410.24
19	1,020.73	.88823	.97798	.83335	.72391	1,410.09
20	732.24	.88823	.97798	.59765	.51916	1,410.43
Average						1,410.14
Grand average						1,414.53

[1] These yields are those previously calculated. See text for each series of plots.
[2] This is $(1-R^x)$ for the series. See last group of columns of Table 4 for the several values of x to be used in entering Table 19.

In a similar manner the value of A for each of the remaining plots was obtained, the results for all except the two check plots being shown in Table 5.

The slight variations in the value of A in the ammonia series, from 1,415.57 to 1,416.15, are due to omission of decimals in the preceding computations. A similar remark applies to the A of the phosphate series, plots 7 to 13, and to that in the potash series, plots 15 to 20.

There are thus three values of A, namely—

From the ammonia series	1,415.86
From the phosphate series	1,417.58
From the potash series	1,410.14

the extreme difference between which is about 0.5 per cent of their magnitude.

In arriving at each of these values, three factors were multiplied together. In each case, one factor is derived from data wholly independent of the others. The fact that the three values agree so closely indicates a high degree of fitness in the theory of the yield equation

$$y = A\ (1 - R^{n+a})\ (1 - R^{p+b})\ (1 - R^{k+c}) \qquad (7)$$

from which the values are derived.

FINAL CALCULATED YIELDS

Using the average of the three values of A in the above equation, and multiplying this by each of the "products" in the next to last column, Table 5, the final calculated yield of each of the experimental plots is obtained. In Table 6 these are compared with the observed yields.

TABLE 6.—*Final calculated yield of each plot compared with the observed yield*

Plot No.	Calculated Y	Observed Y	e	100 e/Y	Plot No.	Calculated Y	Observed Y	e	100 e/Y
1	1,250	1,231	19	1.5	11	1,141	1,143	−2	−.2
2	1,219	1,248	−29	−2.3	12	1,065	1,067	−2	−.2
3	1,174	1,181	−7	−.6	13	502	504	−2	−.4
4	1,110	1,095	15	1.4	14 [1]				
5	887	890	−3	−.3	15	1,214	1,225	−11	−.9
6 [1]					16	1,194	1,175	19	1.6
7	1,195	1,137	58	5.1	17	1,144	1,125	19	1.7
8	1,189	1,121	68	6.1	18	1,097	1,110	−13	−1.2
9	1,183	1,126	57	5.1	19	1,024	1,021	3	.3
10	1,161	1,133	28	2.5	20	734	731	3	.4

[1] Check plot.

The column headed e in Table 6 shows the difference between the final calculated yields and the observed yields. In the next column these differences are expressed as percentages of the observed yields.

Plots 7 to 10 of the phosphoric acid series show what is presumably the reduction in yield due to the effect of large doses of phosphate, a reduction probably much more than compensated for by increase in quality of product (tobacco). Thus, on plot 7 the calculated yield is 58 pounds more than the observed yield. This is presumably the reduction in yield due to the earlier maturity caused by the large application of phosphoric acid.

Of the remaining plots, 13 show residuals of less than 2 per cent, 8 of them less than 1 per cent. The remaining residual is 2.3 per cent.

This must be regarded as nothing less than remarkably close agreement between calculated and observed yields.

YIELD OF CHECK PLOTS

Two plots, Nos. 6 and 14, received no fertilizer applications. The yield calculated for these plots by the yield formula is only 238 pounds, whereas the actual average yield for the last three years on plot 6 was 503 pounds, and on plot 14, 498 pounds. The reason for this discrepancy is that the lower part of the phosphate curve represents yield increases due in part to current applications and in part to varying rates of phosphate exhaustion. Extrapolations downward by this curve do not give correct results. But these disturbing conditions do not exist for the upper part of this curve, so that extrapolation upwards should give correct results.

If the soil had not been so fully stocked with available phosphates at the beginning of the experiment, and if the first-year results alone had been used in the computations, then the yield of the check plots could have been calculated by the yield equation.

A hint as to the rate of phosphate exhaustion on plot 13, which received no phosphates, but did receive ammonia at the rate of 30 pounds and potash at the rate of 50 pounds per acre, is seen in Figure 6, where a comparison is given between the yields on plot 13, plot 11, which received phosphoric acid at the rate of 60 pounds per acre, along with 30 pounds of ammonia and 50 pounds of potash, and plot 14, which received no fertilizers.

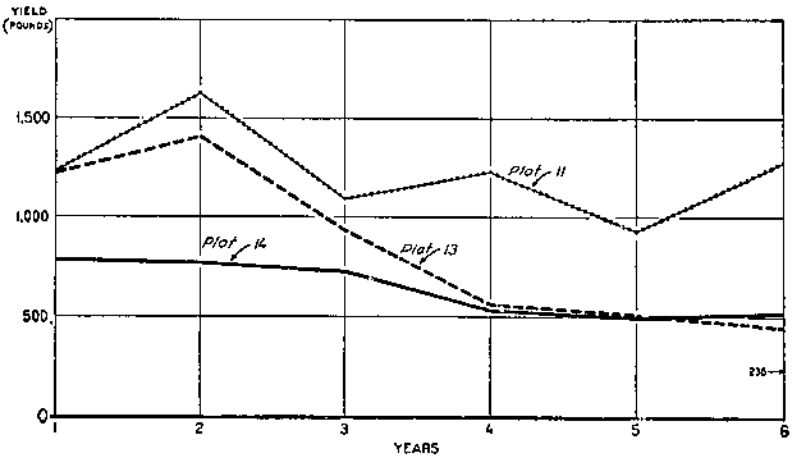

FIGURE 6.—YIELDS OF TOBACCO ON THREE PLOTS IN THE PHOSPHORIC ACID SERIES

Plot 13 gives a hint as to the rate of exhaustion of phosphoric acid. Had it received no ammonia or potash, its yield in the sixth year presumably would have been 238 pounds, indicated by the arrow. Plot 14 received no fertilizer. Plot 11 received complete fertilizer, 60 pounds phosphoric acid along with 30 pounds of ammonia and 50 pounds of potash.

By the fourth year the yield on plot 13 had fallen about to the level of that on plot 14, and remained there during the remainder of the period. Had it received no ammonia or potash the last three years, its yield the last year would presumably have been 238 pounds, indicated by the arrow.

The exhaustion of phosphates in the case of plot 13 was presumably much greater than in the case of plot 14, for during the early years of the experiment the yields on plot 13 were much greater than on plot 14, neither plot having received any phosphates.

While, as stated above, the lower part of the phosphate curve (middle section, fig. 5, at bottom) does not give a true picture because of the nature of the data on which it is based, the same can not be said of the other curves or of the upper part of this one. The upper part of each of the three curves presents what is presumably a true picture. This being the case, extrapolation upwards by means of the yield formula should give reliable results. The close agreement of the maximum possible yields for the three series tends to confirm this statement.

For a series of experiments planned as suggested later in this bulletin, and carried out on a uniform soil, the equation, if the theory on which it is based is true, should enable one to extrapolate either upward or downward indefinitely. The formula, of course, ceases to apply when the fertilizer application becomes so large as to be injurious to the crop. The effect of phosphates in hastening maturity may also cause departures of calculated from observed yields. The curve therefore offers a means of determining the reduction in yield due to the earlier maturity.

OPTIMUM FERTILIZER FORMULAS

A method of determining the most profitable quantity of ammonia (NH_3) and of potash (K_2O) to use with any given quantity of phosphoric acid (P_2O_5) follows.

Certain of the items of cost per unit of product are proportional to area. These include plowing, preparation of seed bed, planting, tillage, and sometimes part of the work of harvesting. Thus, in the case of sugar beets or potatoes, the cost of running the digger is proportional to acreage.

It is true that the amount of work done in preparing seed bed and in tillage, spraying, etc., may vary widely, and the amount and character of such work does affect the yield; but on most farms there are fixed standards for such tasks, and the work done is strictly proportional to acreage. The formulas developed below apply only to cases in which work of this kind is standardized and varies with acreage.

Certain unit costs vary strictly with yield per acre. Thus, grading, sacking or crating, hauling to storage or to market, and the like depend on yield, not on acreage.

Certain other costs are intermediate in character between the above two classes. Thus, in husking corn, for instance, the amount of labor required depends partly on area and partly on yield per acre. It costs slightly more per bushel to husk a field of corn yielding 40 bushels per acre than one yielding 60 or 80 bushels.

In considering these intermediate items from the standpoint of optimum applications of the various fertilizer ingredients it should be remembered that, in most cases, only slight variations in yield are involved. Thus, it may be a question whether a fertilizer application necessary to obtain a yield of 79 bushels or 80 bushels of corn would be most profitable. The variation in cost per bushel as between these yields for the one item of husking is too small to measure. There will therefore be no serious error if such costs are regarded as varying with yield. This point is discussed later in some detail.

In the case of some crops there are other factors of cost that vary partly with area and partly with yield. A heavy crop of sugarcane, for instance, requires less tillage and weeding than a light crop (*8*). Even in this case, however, when the question is between a yield of 19 tons and one of 20 tons, the difference in cost of this item is slight, and may be neglected without serious error.

In what follows the argument proceeds as if all items of cost may be regarded as varying with either area or yield per acre. In some cases this will involve a small degree of error, but not sufficient to vitiate the results, as will be seen later. In the case of sugar beets, where the harvesting is done by contract, at so much per acre, which is a common practice, the error does not enter.

In the subsequent discussion use is made of the following symbols:

a=units of available ammonia (NH_3) in fertilizer applied.
b=units of available phosphoric acid (P_2O_5) in fertilizer.
c=units of available potash (K_2O) in fertilizer.
$a'=a+q'$.
$b'=b+q''$.
$c'=c+q'''$.
C=sum of acre costs proportional to area.
H=costs per unit proportional to yield (the harvesting and marketing costs).
k=units of available potash in an acre of soil.
m=0.434,2945,–the modulus of the common system of logarithms.
M=maximum limit of y.
n=units of available ammonia in an acre of soil.
p=units of available phosphoric acid in an acre of soil.
P=profit per acre.
q'=units of NH_3 absorbed per acre.
q''=units of P_2O_5 absorbed per acre.
q'''=units of K_2O absorbed per acre.
$Q=m/v\ M(-\log R)$.
r'=cost of a unit of ammonia (including cost of application).
r''=cost of a unit of phosphoric acid (including cost of application).
r'''=cost of a unit of potash.
R=ratio of series of increments in y due to successive unit increments of nitrogen, phosphoric acid, or potash, =0.8.
$R'=R^{x_1}=R^{n+a}$.
$R''=R^{x_2}=R^{p+b}$.
$R'''=R^{x_3}=R^{k+c}$.
$s=r'a'+r''b'+r'''c'$=cost of fertilizer per acre.
$s'=r'R''+r''(1-R'')$.
$s'''=r'''R''+r''(1-R'')$.
$S'=1-R'$.
$S''=1-R''$.
$S'''=1-R'''$.
$v=V-H$=value of a unit of product less unit costs proportional to yield (harvesting and marketing costs).
V=value at market of a unit of product.
x'=units of NH_3 per acre=$n+a$.
x''=units of P_2O_5 per acre=$p+b$.
x'''=units of K_2O per acre=$k+c$.
y=yield per acre.

Let P=profit per acre. Then

$$\begin{aligned} P &= Vy-Hy-r'a'-r''b'-r'''c'-C \\ &= vy-r'a'-r''b'-r'''c'-C \end{aligned} \tag{A}$$

Considering first the most profitable application of ammonia, the problem is to find the value of a' that will render P a maximum. This value is obtained by placing equal to zero the partial derivative of P with respect to a'.

Differentiating (A) with respect to a', and noting that, since $a'=a+q'$, $\mathrm{d}a'=\mathrm{d}a$,

$$\frac{\mathrm{d}P}{\mathrm{d}a}=v\frac{\mathrm{d}y}{\mathrm{d}a}-r'=0 \tag{B}$$

Now $y=MS'S''S'''$, in which M has the same significance as the A of formula (7) on page 22. Substituting this value of y in (B) and performing the indicated differentiation, the result may be reduced to the form shown in equation (C) below.

$$R'S''S''' = \frac{mr'}{vM(-\log R)} \tag{C}$$

in which $m = 0.4342945$, the modulus of the common system of logarithms.

Letting $Q = m/vM(-\log R)$, this may be written

$$R'S''S''' = Qr' \tag{D}$$

The corresponding equations for P_2O_5 and K_2O, obtained in a similar manner, are

$$S'R''S''' = Qr'' \tag{E}$$

and

$$S'S''R''' = Qr''' \qquad \text{(F) (8)}$$

To solve these equations for R' and R''' in terms of R'', multiply (D) by (S'), (E) by (S''), and (F) by (S'''); and divide (D) by R', (E) by R'', and (F) by R'''. This gives

$$S'S''S''' = Qr'(S')/R' \tag{G}$$
$$S'S''S''' = Qr''(S'')/R'' \tag{H}$$
$$S'S''S''' = Qr'''(S''')/R''' \tag{I}$$

Equating the second members of (G) and (H),

$$Qr'(S')/R' = Qr''(S'')/R''$$

Multiplying through by $R'R''$, and dividing through by Q,

$$r'R''(S') = r''R'(S'')$$

Restoring the values of S' and S'',

$$r'R'' - r'R'R'' = r''R' - r''R'R''$$

Transposing, and changing signs

$$r'R'R'' + r''R' - r''R'R'' = r'R''$$

Factoring, $R'(r'R'' + r'' - r''R'') = r'R''$.

Dividing,

$$R' = \frac{r'R''}{r'R'' + r''(1-R'')} = \frac{r'R''}{s'} \tag{9}$$

Equating the second members of (H) and (I),

$$Qr''\ (S'')/R'' = Qr'''(S''')/R'''$$

Proceeding as in the case of R', the value of R''' is found to be

$$R''' = \frac{r'''R''}{r'''R'' + r''(1-R'')} = \frac{r'''R''}{s'''} \tag{10}$$

Equations (9) and (10) now permit the determination of the most profitable quantities of NH_3 and K_2O to use with any quantity of P_2O_5 within the toxic limit of these substances. The data required are as follows:

$r' = \$1.088 = 6.203$ pounds of NH_3 at $\$0.1754$ = cost of a unit of NH_3.
$r'' = \$0.393 = 5.452$ pounds of P_2O_5 at $\$0.0720$ = cost of a unit of P_2O_5.

r'''=\$0.385=5.064 pounds of K_2O at \$0.0760=cost of a unit of K_2O.
u_1=a unit of NH_3=6.203 pounds.
u_2=a unit of P_2O_5=5.452 pounds.
u_3=a unit of K_2O=5.064 pounds.
n=4.98;
p=2.43;
k=4.08.

The values are worked out for several cases in Table 7, the procedure of which is illustrated by the column headed "For b=20 pounds"; that is, if 20 pounds of phosphoric acid, P_2O_5 are to be used, what are the corresponding most profitable quantities of ammonia, NH_3, and of potash, K_2O, to use?

Convert pounds of P_2O_5 into units: 20÷5.452, b	=	3.67
Write p, in units	=	2.43
Add these, giving the x of the yield equations, or $x''=p+b$	=	6.10
In Table 19 find $(1-R^x)$ at 6.10, or $1-R''$	=	.74364
whence, subtracting from 1, R''	=	.25636
Multiply R'' by r', which is 1.088, giving $r'R''$	=	.27892
Multiply $(1-R'')$ by r'', which is 0.393, giving $r''(1-R'')$	=	.29225
Add $r'R''$ to $r''(1-R'')$ giving s'	=	.57117
Dividing $r'R''$ by s' gives R'	=	.48833
whence, subtracting from 1, $1-R'$, or S'	=	.51167
From Table 19 read the value of $1-R'$, which is x' or $n+a$ units	=	3.21
whence, by subtraction (n=4.98) a, in units	=	−1.77
Converting units to pounds, a in pounds	=	−11.0

Since a is negative, it does not pay to use any ammonia, NH_3, in this case when only 20 pounds of phosphoric acid are applied.

The correct quantity of potash is calculated in the lower section of the table. The reader should now be able to follow the procedure indicated. When 20 pounds of phosphoric acid are used, 10.6 pounds of potash are indicated as the most profitable application.

TABLE 7.—*Computation of most profitable quantities of ammonia and of potash to use with given quantities of phosphoric acid*

Step in computation [1]	For b=5 pounds	For b=10 pounds	For b=20 pounds	For b=40 pounds	For b=60 pounds	For b=80 pounds	For b=100 pounds	For b=120 pounds
b, in units	0.92000	1.83	3.67	7.34	11.01	14.67	18.34	22.01
$b+p$ (p=2.43)	3.35000	4.26	6.10	9.77	13.44	17.10	20.77	24.44
$1-R''$	.52647	.61349	.74364	.88697	.95011	.97798	.99029	.99572
R''	.47353	.38651	.25636	.11303	.04989	.02202	.00971	.00428
$r'R''$ (r'=1.088)	.51520	.42052	.27892	.12298	.05428	.02396	.01056	.00466
$r''(1-R'')$	.20690	.24110	.29225	.34858	.37339	.38435	.38918	.39132
$s'[=r'R''+r''(1-R'')]$	.72210	.66162	.57117	.47156	.42767	.40831	.39974	.39598
R' ($r'R''/s'$)	.71346	.63559	.48833	.26079	.12692	.05868	.02642	.01177
$1-R'$	.28654	.36441	.51167	.73921	.87308	.94132	.97358	.98823
$n+a, =x'$, in units	1.51	2.03	3.21	6.02	9.25	12.71	16.28	19.91
a, in units	−3.47	−2.95	−1.77	1.04	4.27	7.73	11.30	14.93
a, in pounds	−21.5	−18.3	−11.0	6.5	26.5	47.9	70.1	92.6
$r'''R''$	.18231	.14881	.09870	.04352	.01928	.00846	.00374	.00165
$s'''[=r'''R''+r''(1-R'')]$	.38921	.38991	.39095	.39210	.39260	.39283	.39292	.39297
R''' ($=r'''R''/s'''$)	.46841	.38165	.25246	.11099	.04893	.02159	.00952	.00420
$1-R'''$	.53159	.61835	.74754	.88901	.95107	.97841	.99048	.99580
$k+c, =x'''$, in units	3.40	4.32	6.17	9.86	13.52	17.19	20.86	24.52
c, in units	−.68	.24	2.09	5.78	9.44	13.11	16.78	20.44
c, in pounds	−3.4	1.2	10.6	29.3	47.8	66.4	85.0	103.5

[1] Find in the previous work or in Table 19 or compute as indicated in text.

Figure 7 was constructed from data in Table 7 and additional data similarily calculated. The abscissas of the figure are pounds of P_2O_5 per acre. For any one abscissa, the ordinates of the two curves show the quantities, in pounds per acre, of NH_3 and K_2O to use for greatest profit. Thus, at 70 pounds of P_2O_5 the ordinate of the NH_3 curve is 37.3; of the K_2O curve, 57. The exact quantities of NH_3 and K_2O to use with any quantity of P_2O_5 are easily calculated by the method of Table 7.

The method of converting quantities of fertilizer constituents per acre into fertilizer formulas may be illustrated from the data in any column of Table 7. In the column headed "For $b=60$ pounds" for instance, it is shown that the optimum quantities of ammonia and

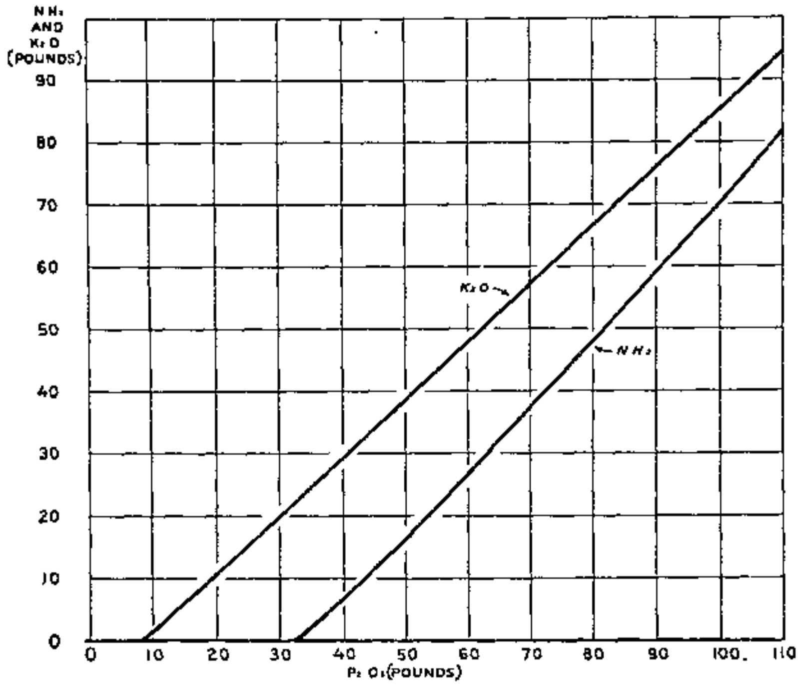

FIGURE 7.—QUANTITY OF AMMONIA AND OF POTASH ASSOCIATED WITH PHOSPHORIC ACID IN THE OPTIMUM FERTILIZER ANALYSIS FOR TOBACCO AT TIFTON, GA.

The computation showed the optimum fertilizer formula for tobacco in this series of experiments. With this chart the optimum quantities of ammonia and potash to use with any desired application of phosphoric acid may be read directly. Thus, ammonia will not pay if less than 33 pounds of P_2O_5 are used.

potash to use with 60 pounds of phosphoric acid in this case (tobacco at Tifton, Ga.) are

Phosphoric acid	60 pounds
Ammonia	26.5 pounds
Potash	47.8 pounds

To find the quantity and analysis (formula) of fertilizer containing these quantities of the three constituents, with, say, 12 per cent of phosphoric acid, proceed as follows:

60 pounds of $P_2O_5=12.0$ per cent of 500 pounds.
26.5 pounds of $NH_3=5.3$ per cent of 500 pounds.
47.8 pounds of $K_2O=9.6$ per cent of 500 pounds.

Hence in this case 500 pounds of a 5.3–12–9.6 fertilizer will contain plant-food elements in the right proportion to make the profit greatest from a fertilizer containing 60 pounds of phosphoric acid.

The foregoing problem has been worked out on the assumption that none of the plant food in fertilizers is absorbed, or occluded, by the soil. When the phenomenon of absorption is present, it is allowed for at this point by adding to the 60 pounds of P_2O_5, 26.5 pounds of NH_3, and 47.8 pounds of K_2O above, the quantity, in pounds per acre, of each of the plant-food elements absorbed. Thus, if 7.2 pounds of P_2O_5 is absorbed, use $60+7.2=67.2$ pounds of P_2O_5 instead of 60 pounds. Similarly for the other elements. This, of course, changes the formula of the fertilizer, which would then be as follows:

67.2 pounds of P_2O_5 = 12 per cent of 560 pounds.
26.5 pounds of NH_3 = 4.73 per cent of 560 pounds.
47.8 pounds of K_2O = 8.54 per cent of 560 pounds.

The formula then becomes 4.7–12–8.5 instead of 5.3–12–9.6.

OPTIMUM QUANTITY OF FERTILIZER

In the preceding section a method was given for finding the most profitable quantities of ammonia (NH_3) and potash (K_2O) to use with any quantity of phosphoric acid (P_2O_5). It remains now to find the most profitable quantity of fertilizer having the optimum analysis (formula) for that quantity to use in any given case.

Equation (8), page 29, obtained by differentiating P with respect to c is

$$(1-R')\ (1-R'')\ R'''=Qr''' \qquad (8)$$

Substituting in equation (8) the values of R' and R''' from (9) and (10),

$$\left[1-\frac{r'R''}{r''\ (1-R'')+r'R''}\right][1-R'']\left[\frac{r'''R''}{r''\ (1-R'')+r'''R''}\right]=Qr''',$$

which may be reduced to

$$\frac{r''S''\cdot R''S''}{s'\cdot s'''}-Q=O \qquad (11)$$

This is an equation of the third degree in R'', best solved by the method of trial and error. The work involved in solving it for a specific case is outlined later.

Being of the third degree, the equation has three roots. One of them is imaginary, another is negative, while the third root represents a maximum value of P (profit per acre). It is this last value that is sought.

Let T represent the fraction in equation (11); then

$$T-Q=O \qquad (12)$$

The problem now is to solve equation (12).

The solution of this equation is given in Table 8. Note that $R''=R^{p+b}$, in which $R=0.8$, $p=2.43$ (units of 5.452 pounds of P_2O_5), and b is the quantity of P_2O_5 applied in fertilizer, in units of 5.452

pounds. Assigning b a value in pounds, and then reducing this value to units, the value of R'' is fixed.

TABLE 8.—*Computation of T of equation (12) for the tobacco experiments* [1]

$$T=\frac{r''(1-R'')\cdot r'''(1-R'')}{[r'R''+r''(1-R'')]\,[r'''R''+r''(1-R'')]}$$ value of Q is 0.02263

[The values of the quantities are those previously used in this demonstration]

	Step in computation [2]	First trial	Second trial	Third trial
1	b, in pounds	101.1	101.2	101.3
2	b, in units	18.54	18.56	18.58
3	$p+b$, in units	20.97	20.99	21.01
4	$1-R''$ (Table 19)	.99071	.99075	.99080
5	R''	.00929	.00925	.00920
6	$r''(1-R'')$	.38935	.38936	.38938
7	$R''(1-R'')$	.00920	.00910	.00912
8	Step 6×step 7	.00358	.00357	.00355
9	$r'R''$	.01011	.01006	.01001
10	$r'''R''$	.00358	.00356	.00354
11	$r''(1-R'')+r'R''$	.39946	.39942	.39939
12	$r''(1-R'')+r'''R''$	.39293	.39292	.39292
13	Step 11×step 12	.15690	.15694	.15693
14	T=step 8÷step 13	.02281	.02275	.02263
15	R'=step 9÷step 11			.02506
16	$1-R'$			.97494
17	$n+a$, in units			16.52
18	a, in units			11.54
19	a, in pounds			71.6
20	R'''=step 10÷step 12			.00901
21	$1-R'''$			.99099
22	$k+c$, in units			21.105
23	c, in units			17.025
24	c, in pounds			86.2

[1] The procedure, however, is general in application.
[2] Find in the previous work or in Table 10 or compute as indicated.

Before explaining the procedure of the table, it may be stated that the value of T was calculated for a series of values of b (in pounds) by the procedure of the table and Figure 8 was constructed from data thus obtained. In that figure the value of T may be read off for any value of b (in pounds) from 0 to 120. The solution of equation (12) involves finding a value of b for which $T=Q$, indicated at X in Figure 8. The value of Q is $m/vA(-\log R)$(p. 28). Since $m=0.4342945$, and for the special case under consideration (tobacco at Tifton, Ga.) $v=0.14$, $A=1414.53$ (p. 24), and $-\log R=0.0969100$; the value of Q is 0.02263. Figure 8 shows that $T=Q$ at about $b=101$. This gives a starting point for Table 8, the procedure of which is like that of Table 7. The first value of b tried in the table was 101; the second was 101.1, the column calculated on the basis of $b=101$ being omitted to economize space. For $b=101.1$, the resulting value of T is 0.02281, a little too large. The next trial was for $b=101.2$, giving $T=0.02275$. The last trial, with $b=101.3$, gave $T=0.02263$, which is exactly the value sought.

When the value of b that renders $T=Q$ is found, the proper quantities of ammonia and of potash to use with that quantity of phosphoric acid may be computed as indicated in the lower part of Table 8, steps 15 to 24. Thus it appears that the application per acre giving the maximum profit in this case is

NH_3	71.6 pounds.
P_2O_5	101.3 pounds.
K_2O	86.2 pounds.

Conversion of these quantities of plant foods into terms of a fertilizer is accomplished by deciding on some percentage of one of them, and computing the associated percentages of the others. Thus, if the fertilizer to be used is based on 12 per cent of phosphoric acid, 844.2 pounds of fertilizer will be needed in this case, for $101.3 \div 0.12 = 844.2$. This quantity of fertilizer must have 8.5 per cent of ammonia to furnish 71.6 pounds ($71.6 \div 8.442$), and 10.2 per cent of potash to furnish 86.2 pounds ($86.2 \div 8.442$). Hence 844.2 pounds of 8.5–12–10.2 fertilizer will give the largest profit per acre in this case.

If the phenomenon of plant-food absorption is present, it is taken care of at this point by adding to the above 71.6 pounds of NH_3, 101.3 pounds of P_2O_5, and 86.2 pounds of K_2O, the quantity in pounds per acre, of each element absorbed. This modifies the formula, as well as the quantity per acre of fertilizer to apply. The profit per acre is reduced by the cost of the plant-food elements absorbed.

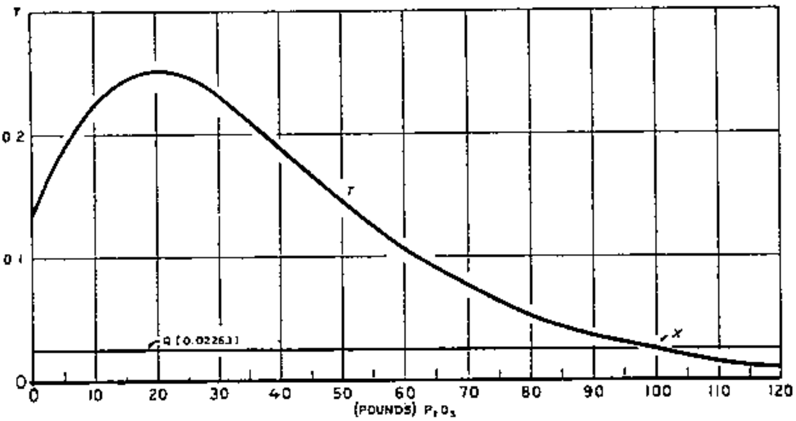

FIGURE 8.—VALUE OF T FOR DIFFERENT APPLICATIONS OF PHOSPHORIC ACID IN THE TOBACCO EXPERIMENTS AT TIFTON, GA.

As developed in the text, the point of intersection, X, at which T equals Q, shows the quantity of phosphoric acid per acre to use for greatest profit per acre for tobacco at Tifton, Ga. This quantity varies for different crops and for different soils. The quantities of ammonia and potash to be applied with this quantity of phosphoric acid are found directly from Figure 7, or computed from Table 8.

If the fertilizer is to contain 10 per cent of P_2O_5, then

$101.3 = 10$ per cent of 1,013 pounds.
$71.6 = 7.1$ per cent of 1,013 pounds.
$86.2 = 8.5$ per cent of 1,013 pounds.

That is, 1,013 pounds of a 7.1–10–8.5 fertilizer is most profitable. These results do not take into consideration the effect of heavy applications of phosphoric acid in hastening maturity of the crop, with resulting increase in quality, and presumably decrease in yield. They are therefore only approximate in the case considered. This difficulty does not arise in cases in which phosphoric acid does not have the effects mentioned.

The item of harvesting.—It has already been pointed out that it costs slightly less per pound to harvest a crop when the acre yield is large than when it is small. Thus, it should cost slightly less per pound to harvest a tobacco crop yielding 1,200 pounds per acre than one yielding 1,000 pounds.

In the case of tobacco, the one item that may vary in this manner is that of cutting and hauling the crop to the barn. In the preceding computations this variation was ignored.

To find what effect such variation may have on the results of the computations, note that, in the case of tobacco, the total cost of this item is ordinarily about 1.2 cents per pound. Suppose the item decreases to 1 cent because of the larger yield from adequate applications of fertilizer. This would change the value of v used in the computations from \$0.14 to \$0.142. Since $Q=m/vA\ (-\log R)$, this change in v reduces the value of Q from 0.02263 to 0.02195. The corresponding value of T, arrived at by the procedure of Table 8, is obtained when a, b, and c have the values shown below. The values of a, b, and c when $v=\$0.14$ are shown for comparison, as are also the yields corresponding to these fertilizer combinations.

$v=$	\$0.14	\$0.142.
$a=$	71.6	74.4 pounds of NH_3.
$b=$	101.3	102.0 pounds of P_2O_5.
$c=$	86.2	86.8 pounds of K_2O.
Optimum $y=$	1,353.86	1,354.08 pounds of tobacco per acre.

It is obvious that an increase in yield amounting to $1{,}354.08-1{,}353.86=0.22$ pound could not measurably affect the cost per pound of harvesting. Hence the inaccuracy due to changing cost per pound of harvesting is negligible, and the results arrived at are dependable when the experimental data are adequate and there are no complications due to effect of the fertilizer on quality of product or date of maturity of the crop.

Effect of market price.—In the formulas, v is the value per pound of crop product less cost of harvesting and marketing. It has been assumed in the computations that $v=14$ cents in the case under consideration. The magnitude of v affects the results because of the presence of v as a factor in the denominator of Q, to which T should be equal. The values of $Q=\frac{0.003168}{v}$, for several values of v are:

v	$Q(=T)$
\$0.10	0.03168
\$0.12	.02640
\$0.14	.02263
\$0.16	.01980

The corresponding values of b, read from the curve of Figure 8, and of a and c read from Figure 7, are:

When v is \$0.10, b is 94.0 pounds, a is 63.5 pounds, and c is 79.0 pounds.
When v is \$0.12, b is 99.5 pounds, a is 69.5 pounds, and c is 84.5 pounds.
When v is \$0.14, b is 101.0 pounds, a is 71.5 pounds, and c is 86.0 pounds.
When v is \$0.16, b is 102.5 pounds, a is 73.0 pounds, and c is 87.5 pounds.

Converting these into fertilizer containing 10 per cent of P_2O_5:

$v=\$0.10$	94.0 pounds $P_2O_5=10$ per cent of 940 pounds.
	63.5 pounds $NH_3=6.8$ per cent of 940 pounds.
	79.0 pounds $K_2O=8.4$ per cent of 940 pounds.
$v=\$0.12$	99.5 pounds $P_2O_5=10$ per cent of 995 pounds.
	69.5 pounds $NH_3=7$ per cent of 995 pounds.
	84.5 pounds $K_2O=8.5$ per cent of 995 pounds.
$v=\$0.14$	101.0 pounds $P_2O_5=10$ per cent of 1,010 pounds.
	71.5 pounds $NH_3=7.1$ per cent of 1,010 pounds.
	86.0 pounds $K_2O=8.5$ per cent of 1,010 pounds.
$v=\$0.16$	102.5 pounds $P_2O_5=10$ per cent of 1,025 pounds.
	73.0 pounds $NH_3=7.1$ per cent of 1,025 pounds.
	87.5 pounds $K_2O=8.5$ per cent of 1,025 pounds.

These results may be summarized thus:

v	Most profitable fertilizer application
$0.10	940 pounds of 6.8–10–8.4 goods.
$0.12	995 pounds of 7.0–10–8.5 goods.
$0.14	1,010 pounds of 7.1–10–8.5 goods.

From this the conclusion appears to be justified that, with a range in price of tobacco from 10 to 16 cents a pound (market price less cost of harvesting and marketing), the standard application of fertilizer to tobacco at Tifton, Ga., should be from 900 to 1,000 pounds of about 7–10–8.5 fertilizer.

As before stated, this does not take into account the effect of heavy applications of P_2O_5 in hastening the maturity of the crop, and consequently increasing the quality of the product.

In spite of the limitations just mentioned, these results should be of value in arriving at correct fertilizer practice for the locality concerned.

OPTIMUM ACREAGE TO WHICH TO APPLY FERTILIZER COSTING A FIXED AMOUNT

Occasions not infrequently arise in which farmers for one reason or another are unable to procure as much fertilizer as they would like to use. It is therefore desirable to have formulas that make it possible to determine the optimum acreage to which a given amount (value) of fertilizer should be applied. In working out these formulas the following symbols are employed:

a = units of available NH_3 in fertilizer.
b = units of available P_2O_5 in fertilizer.
c = units of available K_2O in fertilizer.
$a' = a + q'$.
$b' = b + q''$.
$c' = c + q'''$.
g = optimum acreage.
C = sum of costs proportional to area other than fertilizer.
K = total cost of fertilizer.
k = units of available K_2O in an acre of soil.
$\text{Log}_e R$ = natural logarithm of $R = -0.223144$.
M = upper limiting value of y.
n = units of available NH_3 in an acre of soil.
p = units of available P_2O_5 in an acre of soil.
p' = profit from an acre of the crop.
$P = gp'$ = profit from the entire crop.
q' = units of NH_3 absorbed by an acre of soil.
q'' = units of P_2O_5 absorbed by an acre of soil.
q''' = units of K_2O absorbed by an acre of soil.
r' = cost of a unit of NH_3.
r'' = cost of a unit of P_2O_5.
r''' = cost of a unit of K_2O.
$R = 0.8$.
$R' = R^{n+a}$.
$R'' = R^{p+b}$.
$R''' = R^{k+c}$.
$s = r'a' + r''b' + r'''c'$, cost of fertilizer per acre.
$s' = r'R' + r''(1 - R'')$.
$s''' = r'''R'' + r''(1 - R'')$.
$S' = 1 - R'$.
$S'' = 1 - R''$.
$S''' = 1 - R'''$.
v = value of a unit of crop product less cost of harvesting and marketing.
y = yield per acre.
y_0 = yield per acre when no fertilizer is applied.

Three cases are to be considered.

CASE 1. Only the acreage to be planted that will give a maximum profit from the crop.

The profit per acre is expressed by the formula

$$p' = vy - \frac{K}{g} - C$$

The profit on the entire crop is

$$P = gp' = gvy - K - gC \qquad \text{(A)}$$

The acreage to be planted, g, equals the total cost of the fertilizer divided by the amount applied per acre; that is, $g = K/(r'a' + r''b' + r'''c') = K/s$. The yield per acre is $y = MS'S''S'''$.

Substituting these values of y and g in equation (A),

$$P = \frac{KvMS'S''S''' - KC}{s} - K \qquad \text{(B)}$$

To obtain the values of a, b, and c that render P a maximum differentiate equation (B) with respect to a, then with respect to b, and then with respect to c, and place each of the resulting derivatives equal to zero. When this is done the three resulting equations are:

$$\text{Log}_e RsR'S''S''' + r'S'S''S''' = \frac{r'C}{vM} \qquad \text{(C)}$$

$$\text{Log}_e RsS'R''S''' + r''S'S''S''' = \frac{r''C}{vM} \qquad \text{(D)}$$

$$\text{Log}_e RsS'S''R''' + r'''S'S''S''' = \frac{r'''C}{vM} \qquad \text{(E)}$$

Eliminating S''' from equations (C) and (D), the value of R' is found to be

$$R' = r'R''/s' \qquad \text{(F)}$$

In a similar manner from equations (D) and (E) it is found that

$$R''' = r'''R''/s''' \qquad \text{(G)}$$

Note that the results in equations (F) and (G) are identical with those obtained previously.

To find the value of R'' substitute in equation (E) the values of R' and R''' from equations (F) and (G). When the resulting equation is reduced to its simplest form it is

$$vM \log_e Rsr''R''S''^2 + vMr''^2S''^3 - Cs's''' = 0 \qquad \text{(H)}$$

The solution of equation (H) is demonstrated in Table 9, using the data of Doctor Garner's experimental results with tobacco. The reader should now be able to follow the computations according to the indicated directions in the explanatory column of the table.

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Table 9.—*Computation of the optimum acreage to which to apply fertilizer costing a fixed amount*

[$100 to be spent for fertilizer. v=$0.14. Data are those of the tobacco experimental results]

	Step in computation	First trial	Second trial	Third trial	Fourth trial	Fifth trial	Sixth trial[1]
1	b, in units	10.0	11.0	11.1	11.2	11.3	11.24
2	$p+b$, in units	12.43	13.43	13.53	13.63	13.73	13.67
3	$1-R^{p+b}$	.93757	.95005	.95116	.95223	.95329	.95266
4	R^{p+b}	.06243	.04995	.04884	.04777	.04671	.04734
5	$r'R^{p+b}$	.067024	.054340	.053138	.051974	.050820	.051506
6	$r''(1-R^{p+b})$	.368465	.373370	.373806	.374226	.374643	.374395
7	Add steps 5 and 6	.436389	.427710	.426944	.426200	.425463	.425901
8	Divide step 5 by step 7, giving R^{n+a}	.155650	.127001	.124461	.121947	.119446	.120034
9	$1-R^{n+a}$	.844350	.872999	.875539	.878053	.880554	.879966
10	$n+a$, in units	8.34	9.25	9.34	9.43	9.53	9.47
11	a, in units	3.36	4.27	4.36	4.45	4.55	4.49
12	a' $(=a+q')$, in units	3.36	4.27	4.36	4.45	4.55	4.49
13	$r'''R^{p+b}$	.024036	.019235	.018803	.018391	.017983	.018226
14	Add steps 6 and 13	.392501	.392605	.392609	.392617	.392626	.392621
15	Divide step 13 by step 14, giving R^{k+c}	.061238	.048993	.047892	.046842	.045802	.046421
16	$1-R^{k+c}$	.938762	.951007	.952108	.953158	.954198	.953579
17	$k+c$, in units	12.52	13.52	13.62	13.72	13.82	13.70
18	c, in units	8.44	9.44	9.54	9.64	9.74	9.68
19	c' $(=c+q''')$, in units	8.44	9.44	9.54	9.64	9.74	9.68
20	b' $(=b+q'')$, in units	10.00	11.00	11.10	11.20	11.30	11.24
21	$r'a'$	3.66	4.645700	4.744	4.841600	4.950400	4.885120
22	$r''b'$	3.93	4.323	4.362	4.401000	4.440900	4.417320
23	$r'''c'$	3.25	3.634	3.673	3.711400	3.749900	3.726800
24	Add steps 21, 22, and 23	10.84	12.60	12.78	12.954600	13.141200	13.029240
25	Multiply step 24 by (vM $\log_e R$)	−470.021	−556.795	−564.749	−572.46505	−580.710929	−595.763405
26	r'' (step 4) (step 3)²	.021567	.017718	.017365	.017023	.016682	.016885
27	Multiply step 25 by step 26	−10.331046	−9.865204	−9.806866	−9.745073	−9.687420	−9.721765
28	(Step 3)² vMr'''^2	25.207748	26.227821	26.320	26.409022	26.497543	26.444606
29	(Step 7) (step 14) ($-C$)	−17.1283	−16.792344	−16.762206	−16.733337	−16.704784	−16.721768
30	Add step 27 and step 29	−27.459346	−26.657638	−26.533	−26.478410	−26.392204	−26.443533
31	Add step 28 and step 30	−2.251598	−.430	−.213	−.069388	.105339	.001160

[1] The number of units of b (=11.24) is approximated by graphing the values of step 31 at 11.2 and 11.3. When the correct value of b is reached, step 31 is zero. Running the computation shows that 11.24 is close enough.

Optimum average $=g=K\div$step $24=\frac{100}{13.02924}=7.625$.

Using a 10 per cent P_2O_5 fertilizer, 613 pounds will be needed (=11.24 units of 5.452 pounds).

Associated ammonia $=a'=a+q'$,=4.49 units of 6.203 pounds=27.85 pounds which is 4.55 per cent of 613 pounds.

Associated potash$=c'=c+q'''$=9.68 units of 5.064 pounds=49.02 pounds, or 8 per cent of 613 pounds.

The problem has been worked out for a case in which the fixed amount spent for fertilizer is $100, and on the assumption that there is no plant-food absorption; that is, that all three values of q are zero. Other than v, taken as $0.14, and M=1,414.53, the values of the quantities are those previously reached. The answer in this case is: 613 pounds per acre of a 4.5–10–8 fertilizer applied to 7.675 acres will give the maximum profit when only $100 is to be spent for fertilizer.

Note that several trials more than those shown in Table 9 may be needed, but the trials may best begin with a value of b about that giving the best results in the previous work. Whole units may be used in establishing the approximate value of b, then tenths. The second decimal place may be determined by graphing. Two decimal places in the value of b will give sufficiently close results.

Case 2. Maximum profit per dollar's worth of fertilizer.

The equation expressing the profit on the entire amount of fertilizer is

$$P = gv(y - y_0) - K \tag{A}$$

in which the values of g and of y are those used in case 1. When these values are substituted in equation (A) and the resulting equation is differentiated with respect to a, to b, and to c, the derivatives placed equal to zero give values of R' and of R''' the same as found in case 1. When these values are substituted in the equation formed by placing the derivative with respect to c equal to zero, the equation reduced to its simplest form becomes

$$M \log_e R s r''\ R''\ S''^2 + M r''\ S''^3 - y_0 s'\ s''' = 0$$

The solution of this equation for the tobacco experimental results is given in Table 10.

TABLE 10.—*Computation of the application that will yield the maximum profit per dollar's worth of fertilizer*

[Data are those of the tobacco experimental results. In addition to the evaluations reached in case 1 the following are needed: $M \log_e R = -315.644$; $Mr''^2 = 218.472744$; and $y_0 = 237.373$]

	Step in computation	No plant food absorption. ($q=0$)					1.32 units of P_2O_5 absorbed. ($q''=1.32$)		
		First trial	Second trial	Third trial	Fourth trial	Fifth trial[1]	First trial	Second trial	Third trial[1]
1	b, in units[1]	10.	5.	4.	3.9	3.87	3.87	4.4	4.47
2	$p+b$, in units	12.43	7.43	6.43	6.33	6.30	6.30	6.83	6.90
3	$1-R''$	.93757	.80947	.76184	.75647	.75483	.75483	.78218	.78556
4	R''	.06243	.19053	.23816	.24353	.24517	.24517	.21782	.21444
5	$(1-R'')^2$	.879038	.655242	.580400	.572247	.569775	.569775	.611806	.617105
6	$(1-R'')^3$	.824159	.530398	.442172	.432888	.430083	.430083	.478542	.484773
7	$r' R''$	.067924	.207207	.259118	.264961	.266745	.266745	.236988	.233311
8	$r'' (1-R'')$	.368465	.318122	.299403	.297293	.296648	.296648	.307397	.308725
9	(Step 7)+(Step 8)	.436389	.525419	.558521	.562254	.563393	.563393	.544385	.542036
10	(Step 7)÷(Step 9)=R'	.155650	.394537	.463936	.471248	.473462	.473462	.435332	.430435
11	$1-R'$	.844350	.605463	.536064	.528752	.526538	.526538	.564668	.569565
12	$n+a$, in units	8.34	4.17						
13	a, in units	3.36	−.81				3.35	3.73	3.78
14	a', $(=a+q')$	3.36	−.81						
15	$r''' R''$	.024036	.073354	.091692	.093759	.094390	.094390	.083861	.082559
16	(Step 15)+(step 8)	.392501	.391476	.391095	.391052	.391038	.391038	.391258	.391284
17	(Step 15)÷(step 16)=R'''	.061238	.187378	.234449	.239761	.241383	.241383	.214337	.210995
18	$1-R'''$	.938762	.812622	.765551	.760239	.758617	.758617	.785663	.789005
19	$k+c$, in units	12.52	7.505	6.50	6.40	6.37	6.37	6.90	6.97
20	c, in units	8.44	3.425	2.42	2.32	2.28	2.28	2.82	2.80
21	c', $(=c+q''')$	8.44	3.425	2.42	2.32	2.28	2.28	2.82	2.89
22	b', $(=b+q'')$	10.0	5.0	4.0	3.9	3.87	5.19	5.72	5.79
23	$r' a'$	3.66							
24	$r'' b'$	3.93	1.965	1.572	1.5327	1.521	2.040	2.247960	2.275470
25	$r''' c'$	3.25	1.319	.932	.8932	.878	.878	1.085700	1.112650
26	(Step 23)+(step 24)+(step 25)	10.84	3.284	2.504	2.4259	2.399	2.918	3.333660	3.388120
27	(Step 26) $M \log_e R$	−3421.581	−1036.575	−790.278	−765.721	−757.138	−921.049192	−1052.249777	−1069.430749
28	(Step 5) $r'' R''$	.021567	.049063	.054324	.054768	.054899	.054899	.052373	.052006
29	(Step 27) (step 28)	−73.793	−50.857	−42.931	−41.937	−41.566	−50.564680	−55.109478	−55.617284
30	(Step 6) Mr''^2	180.056	115.878	96.603	94.574	93.961	93.961	104.548348	105.909688
31	(Step 9) (step 16) $(-y_0)$	−40.658	−48.825	−51.851	−52.191	−52.295	−52.295	−50.559259	−50.344443
32	(Step 29)+(step 31)	−114.451	−99.682	−94.782	−94.128	−93.861	−102.859680	−105.668737	−105.961727
33	(Step 30)+(step 32)	65.605	16.196	1.890	.446	.100	−8.898680	−1.120353	−.052030

[1] When the correct value of b is reached, step 33 is zero. The values of b, =3.87, and =4.47 were approximations reached by graphing and prove to be close enough. The computations are carried out in full, as steps 14, 21, and 22 show the quantities of plant foods to be used in making up the fertilizer and steps 23, 24, and 25 show the cost per acre.

When plant-food absorption is not involved, the application that will give most profit per dollar's worth of fertilizer is one made up with 3.87 units, or 21.1 pounds, of phosphoric acid, and 2.28 units, or 11.55 pounds, of potash, worth together \$2.40 per acre. No nitrogen is to be used in this fertilizer. Using 10 per cent of phosphoric acid, the analysis works out as 0–10–5.5, worth \$22.75 per ton.

One might fertilize 41.67 acres with \$100 worth of fertilizer at this rate. This does not mean that it would be profitable to spread the fertilizer so thinly; there might, in fact, be a heavy loss on the entire acreage. Still the increase in yield attributable to the fertilizer would be greater from such an application than that from any other.

When plant food is absorbed the computations are the same except for the values of some of the items. The three columns on the right side of Table 10 illustrate the work when it is known that phosphoric acid is absorbed by the soil, assuming a rate amounting to 1.32 units. The application which gave most profit per dollar's worth of fertilizer when there was no absorption proves to be too small (first trial). The second trial shows that 4.4 units is also too small, but the two trials provide means of approximating the figure to use in the third trial—graphing on a large scale indicates a value slightly larger than 4.47 units of b, (actually 4.4736), but computation proves that 4.47 is close enough. The results in this case call for an application of about 316 pounds of 0–10–4.65 fertilizer on 29.5 acres. The \$100 available would buy 4.656 tons of this fertilizer, and the cost would be \$21.48 per ton.

Case 3. Fixed acreage, H; fixed expenditure for fertilizer, K; optimum acreage, g, to which to apply the fertilizer.

In this case the profit from the H acres is

$$\begin{aligned} P &= gvy + (H-g)vy_0 - HC - K \\ &= gv(y-y_0) + Hvy - HC - K \qquad \text{(A)} \end{aligned}$$

Note that the variable term of equation (A) is identical with the variable term of the corresponding equation in case 2. This means that the three derivatives of the equation are identical with those of the preceding case. The solution of the problem is also identical, so that this case is in reality merely a matter of finding the fertilizer formula, the application per acre, and the number of acres required to yield the greatest profit per dollar invested in fertilizer.

But in this case the actual acres planted may be larger than the number required to give maximum profit per dollar's worth of fertilizer, so that the profit or loss from the entire crop may not be the same as in case 2.

In general the solution of the problem of greatest profit per dollar invested in fertilizer is of limited importance. The more important problem in cases where the value of fertilizer available is fixed is to find the optimum acreage to grow (and fertilize) in order to obtain maximum profit from the entire crop.

ABSORPTION, OR OCCLUSION, OF PLANT FOOD

As stated in the introduction, on some soils applications of a growth factor up to a certain (usually small) quantity appear to have no effect on yields. A number of such cases are cited below. For some reason not yet definitely understood, these small applications are not available to the growing crop. Soil chemists who have recognized the phenomenon refer to it as "absorption." In the absence of a better term based on more complete understanding, it will here be referred to as plant-food absorption by the soil. As suggested in the introduction, perhaps a better term would be plant-food occlusion by the soil.

When the amount of the growth factor applied exceeds the quantity thus held unavailable in the soil, the yield begins to increase, and appears to follow quite accurately the exponential yield curve.

According to O. W. Willcox, in a letter to the author, there are soils in Hawaii which absorb some 60 per cent of all the potash applied to them, irrespective of the quantity applied. Such cases would require different treatment from that outlined below, but may be brought within the scope of the yield curve when more knowledge is available of the "effect factors" discussed by Willcox (*11*).

When plant-food absorption of the first type above described occurs, the yield of a plot receiving none of the variable growth factor, that is, the yield of an unfertilized check plot, does not lie on the yield curve; it therefore can not be properly used in determining the constants of the yield equation. That this difficulty may be obviated by disregarding the yield on plots receiving none of the variable factor is shown by the data presented below.

Figure 9 shows the yields of corn on four plots fertilized with different quantities of phosphoric acid and a check plot receiving no phosphoric acid at the Snowshoe branch of the Pennsylvania State station (*9*). In this figure it is seen that the yield on the plot receiving no phosphoric acid is not in line with the remaining yields. The constants in the yield equation in this case were calculated from the yields of plots receiving, respectively, 1, 2, 3, and 4 units of phosphoric acid, a unit being 24 pounds.

The yield curve appears to offer a means of calculating the amount of the plant-food element absorbed by the soil. The method of finding this amount is to find the abscissa of the curve at the point at which the yield as indicated by the curve is equal to the yield without fertilizer. In Figure 9 this point is seen to lie at 0.3 unit, or 7.2 pounds of phosphoric acid, to the right of the origin.

If this interpretation is correct, then the yield should have been the same for any quantity of phosphoric acid from 0 up to 7.2 pounds per acre.

The corn for which the curve of Figure 9 was constructed was grown in rotation with oats, wheat, and hay. The oats and wheat exhibit the same absorption phenomenon, the amount of absorption for oats being approximately the same as for corn, while for wheat it was considerably larger (*6*). This is consistent with the known fact that corn can extract from the soil considerably larger quantities of phosphoric acid than can wheat.

On the hay plot receiving no phosphoric acid there was a large growth of weeds, which vitiated the yield for that plot, so that it can not be included here.

The constants in the yield curve for the corn, oats, and wheat in this Pennsylvania experiment were first calculated from the yields of all five of the plots, including the check plot which received no phosphoric acid. They were then recalculated omitting the yield on the check plot. A comparison of the departures of the curve from the observed yields for the two cases is shown in Table 11.

TABLE 11.—*Improvement in fit of yield curve when phosphoric acid absorption is taken into account*

x	Errors in calculated yields of—					
	Corn		Oats		Wheat	
	Y_0 in	Y_0 out	Y_0 in	Y_0 out	Y_0 in	Y_0 out
0	−0.28		−0.13		−0.13	
1	.07	−0.12	.45	−0.07	.44	−0.01
2	.02	.47	−.30	.28	−.30	.10
3	−1.02	−.64	−.35	−.38	−.36	−.19
4	.60	.28	.34	.17	.34	.11
Average of squares	.38	.18	.11	.06	.11	.01

The first column for each crop shows the residuals when the yield on the check plot is included, the second, when the check plot is

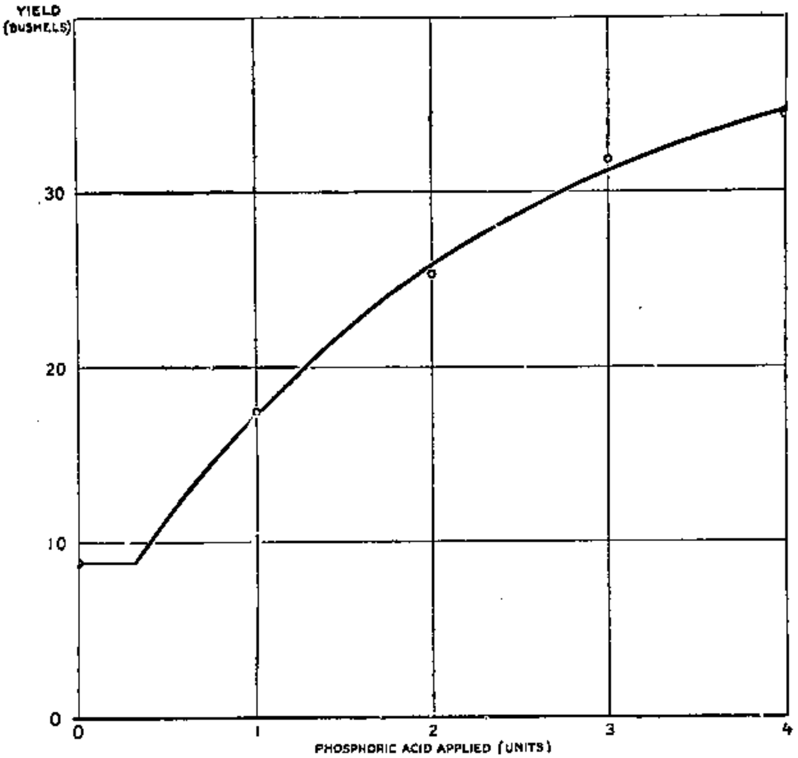

FIGURE 9.—MEASURING ABSORPTION OF PHOSPHORIC ACID

The observed yields (bushels of corn per acre) on the plots receiving phosphoric acid lie close to the exponential yield curve. The yield on the check plot, which received no phosphoric acid, 8.8 bushels does not lie on the curve. The amount of absorption is shown by the abscissa of the break in the curve, which occurs at the point $x=0.3$, in this case 7.2 pounds of phosphoric acid per acre. (The data were obtained from Bulletin 166 of the Pennsylvania Agricultural Experiment Station (9).)

omitted. In the case of corn, omitting the check plot more than doubled the accuracy of the fit, the measure in the one case being 0.38 and in the other 0.18; that is, the fit of the curve in the second column is more than twice as good as in the first. A nearly similar increase in closeness of fit is seen in the case of oats. In the case of wheat, improvement in fit by omitting the check plot is still more marked. These results are consistent with the fact that the amount of absorption, when measured by the method of Figure 9, in the case of oats is slightly less, in the case of wheat markedly greater, than in the case of corn. The check-plot yield in the oats series is therefore slightly less distant, that in the wheat series considerably more distant, from the yield curve than that in the corn series.

Part of the improvement in fit observed in these cases may be due to the fact that the yield curve can be made to fit four observations

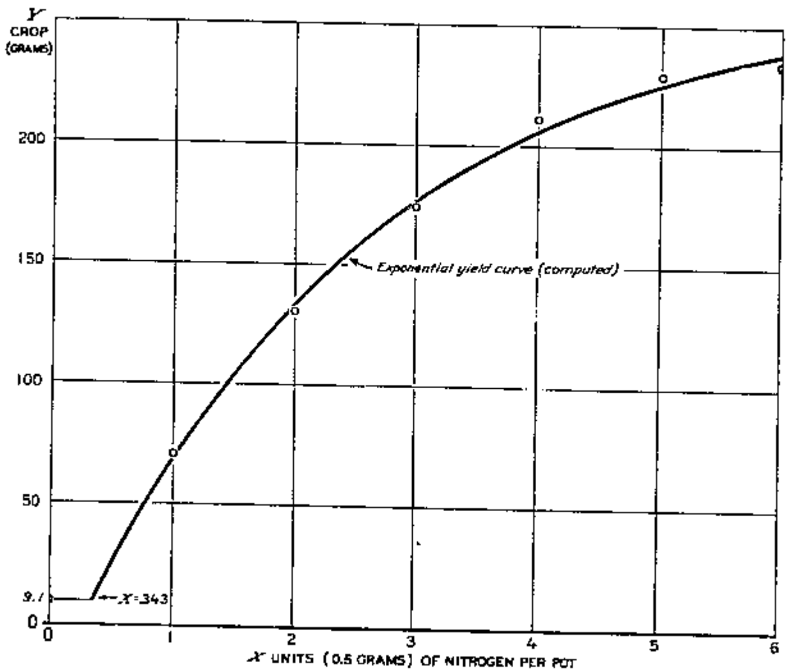

FIGURE 10.—MEASURING THE ABSORPTION OF NITROGEN

In this series, also, the observed yields lie close to the exponential yield curve computed from them. Again, the pot receiving no nitrogen produced 9.1 grams of crop, which would lie on the yield curve at the point where $x=0.343$ unit of nitrogen per pot, the measure of nitrogen absorption of the soils used in the experiment. Data from Niklas and Miller (*3*).

more accurately than it can five, assuming the errors of experiment to be of similar magnitude in the two cases. That it is not all due to this cause can easily be shown by omitting the plot receiving four units of phosphoric acid and calculating the constants from the remaining four (including the check plot). When this is done, it will be found that the fit is even poorer than when the five plots are included.

Figure 10 shows a similar situation in the case of nitrogen. Niklas and Miller (*3*) have assembled nine series of experiments in which nitrogen was the variable fertilizer element, each of which clearly exhibits the phenomenon of nitrogen absorption. Figure 10 is a graphic presentation of one of these nine series. The constants in the yield equation were calculated first with the check plot receiving no nitrogen included, and second with this plot omitted. Both the exponential curve of Mitscherlich and the parabolic curve advocated by Niklas and Miller were applied to each of the nine series.

Table 12 gives a comparison of the average squared residuals for both curves, with and without the check plot. In the case of the parabolic curve the fit is improved in six of the nine cases by omitting the unfertilized plot. In the case of the exponential curve the fit is improved in each of the nine cases. This indicates strongly that the check plot does not belong in the series. Figure 10 indicates that 0.343 unit of nitrogen, each unit being 0.5 gram, was absorbed, and had no effect on the yield.

TABLE 12.—*Improvement in fit of curves when nitrogen absorption is taken in account*

Series [1]	Average squared residuals from the—			
	Parabolic curve		Exponential curve	
	y_0 included	y_0 omitted	y_0 included	y_0 omitted
4	13.93	8.20	17.78	13.06
5	8.52	1.06	33.61	13.25
6	20.69	14.22	24.74	5.77
7	12.78	5.46	13.51	6.09
8	5.47	5.13	5.29	3.20
9	10.87	12.73	8.67	3.00
10	3.88	4.02	6.20	2.17
11	13.50	8.45	15.97	7.18
12	.80	1.48	5.82	2.18

[1] Nine experiments assembled by Niklas and Miller (*3*), in which nitrogen was the variable fertilizer element.

The data for Figure 10 and Table 12 were obtained from pot experiments.

The presence of this absorption phenomenon is indicated by the large positive residuals for the plot receiving one unit of fertilizer when the check plot is included, as seen in Table 11. In the case of corn, this residual is 0.67, the largest positive residual in the series. In the case of oats, it is 0.45, and in the case of wheat, 0.44, in each case being the largest positive residual in the series. Similar remarks are applicable to the nitrogen series dealt with by Niklas and Miller (*3*).

It is readily seen that the yields of check plots receiving no fertilizer should not be used in calculating the constants of the yield equation in cases where plant-food absorption by the soil occurs. In conducting fertilizer experiments on such soils the standard check plots should receive at least as much of each growth factor as the soil is capable of rendering unavailable.

But plots receiving no fertilizer have certain economic importance, as pointed out in the introduction. The knowledge they give is of value in cases in which it seems necessary to spread a limited quantity of fertilizer over a large acreage. Where it is feasible to do so, therefore, plots receiving no fertilizer may be included along with the standard check plots receiving fixed amounts of each factor (p. 58).

FORM OF THE YIELD CURVE

Amongst European soil scientists there has been much discussion of the form of the yield curve. Mitscherlich and those who support his contentions claim that an exponential curve represents the relation between yield and amount of a growth factor made available to the growing plant. A considerable number of other scientists, including Niklas and Miller, (*3*) contend that a parabolic curve is the true yield curve. In the article referred to, Niklas and Miller bring together 12 series of experiments by various soil scientists, and make a comparison of the fit of the two curves in each of the 12 series. They had not recognized the presence of the phenomenon of nitrogen absorption which appears to be present in nine of their series, so in their calculations they included the yield of plots receiving no fertilizer. They determined the constants of the parabolic equation for each series by the method of least squares, and calculated the resulting residuals. They compared these residuals with those arrived at by Mitscherlich, who used some method of approximation, and not the method of least squares, in arriving at the constants in his equation. In general, the fit of the parabolic curve was better than that of the exponential curve when the comparison was made on this basis.

The present writer has recalculated by the method of least squares, the constants of both curves for each of the 12 series, omitting the check plot in the case of the nitrogen series, and finds that in 6 of the 12 cases the exponential curve, and in the remaining 6 the parabolic curve, gives the best fit. These results therefore offer no basis for determining which of the curves comes nearest to expressing the true relation between yield and fertilizer applied.

The exponential equation, however, contains only two constants, whereas the parabolic equation contains three constants. It is well known that the larger the number of constants in an equation the wider the range of observations it can be made to fit. If the number of constants equals the number of observations, the fit can be made exact. The fact, therefore, that a 2-constant curve fits the observed results in 12 series (of 4 to 8 observations each) as well as a 3-constant curve is an indication that the 2-constant curve may more nearly express the existing relation than does the 3-constant curve.

A comparison of much greater significance is made in Tables 13 and 14. The assumption appears to be justified that a curve which really expresses the relation between yield and fertilizer application should give accurate results when it is used to extrapolate yields beyond the range of yields used in determining the constants in the equation, provided the yields used are accurate. A comparison of extrapolation with the two curves is given in Table 13 for the phosphoric acid series, and in Table 14 for seven of the nitrogen series used by Niklas and Miller in their article (*3*). The two remaining nitrogen series were short, with their terms equally spaced, and did not lend themselves well to a comparison of this kind.

In the case of the phosphoric acid series (Table 13) there was no indication of phosphoric acid absorption. It also happened that the last plot received four times as much fertilizer as the preceding plot. The constants in each equation were therefore calculated from the yields on the first three plots, the residuals in each case all being zero. The equations thus obtained were then used for extrapolating the yield of the fourth plot. The actual yield of the fourth plot in the first series was 50.6. The yield calculated by the exponential curve was 51.7, the residual, that is, the difference between the observed and the calculated yield, being 1.1. This must be considered highly accurate extrapolation. In the case of the parabolic curve the calculated yield of the fourth plot was 5.0, the residual being −45.6, a very poor result. In the case of series 2 neither curve gave very good results, presumably because of inaccuracy in the yields of one or more of the three preceding plots. It will be observed, however, that while the observed yield was 52.5 on the fourth plot, the yield calculated by the parabolic curve was −129.8, a result extraordinarily poor.

In the case of series 3 the comparison gives little advantage to either curve, both results being poor.

It is obvious that a comparison of this kind, to be decisive, must be based on yields ascertained with a very high degree of accuracy. A small experimental error in the case of any one plot, where the constants are calculated from only three plots, makes a marked difference in the results of extrapolation. Though the result of this comparison is distinctly more favorable to the exponential curve than to the parabolic, the data on which the comparison is based are too limited to be definitely conclusive.

In Table 14 the comparison is more nearly conclusive. It shows the residuals for seven experimental series in which nitrogen was the variable growth factor. In each series the constants of both yield equations were calculated (by the method of least squares) from the yields of the first five plots, and the yields of the remaining three plots were then calculated by the equation thus obtained.

TABLE 13.—*Results of extrapolation by means of the parabolic curve (P) and the exponential curve (E), the constants of each being computed from three observations*

[Curves fitted to $x=0$, $x=0.10$, and $x=0.25$; errors at these points are 0.]

Series [1]	Yield at $x=1.00$			Errors, using—	
	Actual	Extrapolated, using—			
		P	E	P	E
1	50.6	5.0	51.7	−45.6	1.1
2	52.5	−129.8	38.2	−182.3	−14.3
3	44.9	87.2	92.3	42.3	47.4

[1] Phosphoric acid series of Niklas and Miller (3).

It will be observed that in series 7, 9, 10, and 11, the fit of the parabolic curve within the range used in determining the constants is better than that of the exponential curve. This may be interpreted as resulting from the larger number of constants in the parabolic curve. Yet in these series the extrapolation by the exponential curve is very much better in three and considerably better in the fourth than by the parabolic curve. In series 6 the exponential curve gives a better fit in the case of the first five plots than the parabolic, and the fit of the extrapolated yields is more than eight times as accurate with this curve as with the parabolic. In series 8 the fit of the two curves to the first five yields is approximately the same, but the extrapolation is slightly in favor of the exponential curve. In series 12 the fit of the exponential curve is better than that of the parabolic although the fit of the extrapolated yields is not so good.

Of the entire seven series the results with the exponential curve are best in six, and with the parabolic curve in one. Here, again, it must be noted that great accuracy in the experimental yields is necessary in a comparison of this kind.

TABLE 14.—*Fit of the parabolic curve (P) compared with that of the exponential curve (E) in seven series,*[1] *within the range of the first five observations used in computing the constants of the curves, and extrapolations*

ERRORS WITHIN THE RANGE USED IN COMPUTING THE CONSTANTS

Observation point	Series 6		Series 7		Series 8		Series 9		Series 10		Series 11		Series 12	
	P	E	P	E	P	E	P	E	P	E	P	E	P	E
$x=0.25$	2.11	1.07	0.04	−0.47	0.08	−0.03	0.09	−0.25	−0.28	−0.56	−1.00	−1.03	0.27	0.01
$x=0.50$	−4.39	−3.02	.23	.68	.21	.28	−.47	.47	1.19	1.95	2.32	3.00	−.28	.21
$x=0.75$	.88	1.26	−.94	1.35	−1.04	−.86	.85	.58	−1.88	−2.09	−.96	−1.50	−.84	−.80
$x=1.00$	3.33	2.45	1.03	−2.96	1.01	1.07	−.67	−1.71	1.31	.47	−1.04	−2.11	1.36	.94
$x=1.25$	−1.75	−1.79	−.36	1.39	−.32	−.45	.19	.89	−.34	.22	.68	1.66	−.55	−.37
Average of squares	7.33	4.26	.43	2.64	.45	.43	.29	.87	1.37	1.75	1.77	3.90	.60	.34

ERRORS OF THE EXTRAPOLATED YIELDS

Observation point	Series 6 P	Series 6 E	Series 7 P	Series 7 E	Series 8 P	Series 8 E	Series 9 P	Series 9 E	Series 10 P	Series 10 E	Series 11 P	Series 11 E	Series 12 P	Series 12 E
$x=1.50$	−0.36	−4.74	4.60	−7.05	4.73	4.29	−2.58	3.85	−0.24	−3.06	3.20	10.10	−.05	2.58
$x=1.75$	−23.40	−9.02	14.20	−11.58	14.46	13.59	−16.38	.88	−15.48	−1.25	−9.18	8.81	−.93	6.62
$x=2.00$	−45.57	−15.30	22.84	−8.32	23.26	21.84	−36.31	−2.38	−29.56	−1.47	−25.66	9.05	−4.30	11.16
Average of squares	903.93	113.56	248.15	88.83	257.40	226.69	531.12	7.09	309.60	6.47	165.45	87.17	6.45	68.34

[1] Of the nitrogen series of Niklas and Miller (*3*) as recomputed.

In Tables 15 and 16, relating to series 10 of the Niklas and Miller paper, the comparison is made in a different manner. Table 15 gives the residuals for all values of x obtained from each curve when values of x from 1 to 8 were used in determining the constants in both equations.

TABLE 15.—*Residuals in series 10 when Y_1 to Y_8 are used in calculating the constants in the parabolic curve (P) and the exponential curve (E)*

Value of x	Residuals		Value of x	Residuals	
	P	E		P	E
1	1.61	−.59	5	2.50	1.16
2	−.15	1.85	6	−.36	−2.60
3	−4.14	−1.95	7	1.75	.52
4	.44	.98	8	−1.66	.66

TABLE 16.—*Relative fit and correctness of extrapolations of the parabolic curve (P) and the exponential curve (E) when the values of Y used in calculating the constants are as shown in first column*

[Data of series 10]

Values of Y used in calculating the constants	Average of squared residuals—			
	Within range of Y values used		For extrapolated values of Y	
	P	E	P	E
Y_1 to Y_4	1.46	0.51	668.33	2.25
Y_1 to Y_5	1.38	1.75	309.60	6.47
Y_1 to Y_6	3.09	2.23	31.58	13.96
Y_1 to Y_7	4.76	4.10	99.40	.50

In Table 16 the average of the squared residuals is given for each curve when the constants are calculated (1) from yields Y_1 to Y_4, (2) from yields Y_1 to Y_5, (3) from yields Y_1 to Y_6, and (4) from yields Y_1 to Y_7. With the constants thus determined, the yields are then calculated (1) within the range of the Y values used in calculating the constants, and (2) beyond this range.

Within the range of Y values used in calculating the constants, both curves give fair agreement with observed yields, the exponential curve giving the best fit in three cases, the parabolic curve in one.

When the two curves are used in calculating the extrapolated yields, the exponential curve gives fair results in all cases, but the parabolic curve falls down badly.

It will be observed that the case in which the parabolic curve gives moderately good results is the one in which the last yield used in calculating the constants is very high. This condition tends to bend the curve upward, and thus to increase its radius of curvature.

The question which of these curves more nearly expresses the relation between yield and fertilizer application must be settled on the basis of comparisons similar to those above.

Note that in series 12 the fifth residual is negative in the case of both curves, also the third residual, whereas the fourth residual is positive. This distribution of experimental errors tends to increase the radius of curvature of both curves. A little study of the two curves will show that this effect in the case of the parabolic curve is greater than in the case of the exponential curve. In general, the parabolic curve appears to give good extrapolations only in cases in which the experimental errors are such as to increase considerably its radius of curvature.

Though a decision of the question between these two curves can not be made definitely on the basis of the preceding results, the fact remains that the data at hand are strongly in favor of the exponential curve and against the parabolic curve. Because of the greater number of constants in the parabolic curve, it can be made to fit a wider range of experimental results than can the exponential curve, but the real test comes when the two curves are used for extrapolation. In the comparisons it has been possible to make here the evidence is strongly in favor of the exponential curve.

It will probably be conceded that the exponential curve gives at least as good a fit to experimental data as the parabolic curve within the range of the yields used in computing the constants of the two curves.

Assuming, then, that within the usual range of fertilizer applications in practical farming the exponential curve gives as good results as the parabolic, there is another important reason for preferring the exponential curve. It is the fact that the exponential equation may be so written as to apply to cases in which two or more growth factors vary. At present, at least, this is not the case with the parabolic curve.

Some of the advantages of this more general type of equation are:

(1) When the constants have been evaluated, the equation may be used in calculating the yield from any quantity of fertilizer of any composition, within the toxic limit of the fertilizer.

(2) It may be used in determining the most profitable amounts of nitrogen and potash to use with any desired quantity of phosphoric acid.

(3) It may be used in determining the most profitable quantity of fertilizer having the optimum analysis to use in any case.

For these reasons, it would appear to be justifiable to use the exponential curve until something better presents itself.

DERIVATION OF THE EXPONENTIAL YIELD CURVE

ONE-VARIABLE FORM

The exponential yield curve for a single variable growth factor is illustrated in Figure 1.

In the figure, horizontal distances along the X axis represent amounts of a causal factor, vertical distances the amount of the resulting effect. Thus, if 50 pounds of potash per acre be taken as a unit of the causal factor, then the figure shows the effect of four such units on yield of a crop. The first unit produces an increase in yield represented by a; the second unit produces a further increase, b; the third unit produces increase c, and so on.

An important property of the curve of Figure 1 is that the quantities a, b, c, d, etc., are the terms of a decreasing geometric series, having a constant ratio. Thus, if b is 60 per cent of a, then c tends to be 60 per cent of b; d, 60 per cent of c; and so on. When the quantity of the growth factor becomes large enough to become injurious to the crop, the curve no longer applies.

In the figure, the ratio of the series is 0.6, each term of the series (after the first term) being 0.6 of the preceding term.

The curve of Figure 1 is seen to approach a horizontal line at distance A above the X axis. A is thus the limit toward which the value of the ordinate of the curve approaches as the quantity of the growth factor, potash, increases.

Observe that in Figure 1, if we let y_1, y_2, y_3, etc., represent, respectively, the increase in yield due to 1, 2, 3, etc., units of the growth factor then

$$\begin{aligned} y_1 &= a \\ y_2 &= a+b \\ y_3 &= a+b+c \\ y_4 &= a+b+c+d \\ y_x &= a+b+c+\ .\ .\ .\ +x \end{aligned}$$

If, now, R represent the ratio of the decreasing geometric series, a, b, c, d, etc., then

$$\begin{aligned} b &= aR \\ c &= bR = aR^2 \\ d &= cR = aR^3 \end{aligned}$$

The equation for y_x may therefore be written:

$$y_x = a + aR + aR^2 + aR^3 + \ .\ .\ .\ + aR^{x-1}$$

Multiplying through by R,

$$Ry_x = aR + aR^2 + aR^3 + \ .\ .\ .\ + aR^{x-1} + aR^x$$

Subtracting this last from the preceding equation,

$$y_x(1-R) = a - aR^x = a(1-R^x)$$

whence

$$y_x = \frac{a}{1-R}(1-R^x),$$

which shows the increase in yield for x units of the growth factor.

Since R is less than 1, as x increases, R^x decreases; and as x approaches infinity, R^x approaches zero. Hence, if A represent the value of y_x when x is infinite, we have

$$A = \frac{a}{1-R}$$

Substituting A for $\frac{a}{1-R}$, and dropping the x subscript, the yield equation may be written

$$y = A(1-R^x) \qquad (13)$$

This equation expresses the relation between increase in yield and increase in a growth factor when all other growth factors are held constant.

The writer discovered this equation in 1920 (*4*) and later learned that it had also been discovered in 1912 by the German experimenter Mitscherlich (*2*). Mitscherlich has shown that the formula has wide applicability to the soils of eastern Germany, while the present writer has shown its applicability to many soils in the United States, especially in eastern humid regions, as far west as Indiana and Michigan (*7*).

In the writer's earlier work, equation (13), or rather, a modified form of it seen in the equation $Y = M - AR^x$, the derivation of which is given later, was applied to experiments in which the unit of x was a given quantity of a mixed fertilizer, so that all three constituents varied together (in the same ratio). That this procedure is ordinarily justified is seen in the curve of Figure 11, in which the yield for different quantities of fertilizer is shown, the unit being 100 pounds of 10–10–10 fertilizer.

This curve shows results such as should be obtained in water or sand cultures, in which the total quantity of each plant-food element available to the plant is known.

The curve is at first (i. e., for small values of x) convex downward; for higher values of x it is concave. It therefore has a point of inflection.

In ordinary fertilizer practice, especially on fairly good soils, the lower part of this curve represents growth due to plant food in the soil. Where the supply of such material in the soil is equivalent to

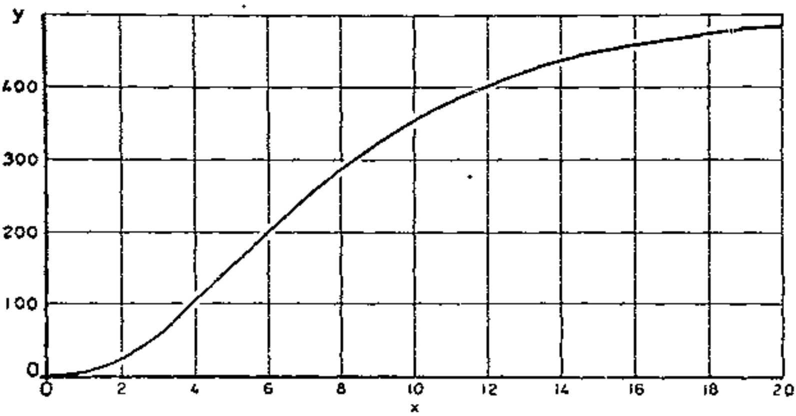

FIGURE 11.—YIELD CURVE FOR THREE VARIABLE GROWTH FACTORS

Abscissas represent units of fertilizer, a unit being 100 pounds of 10–10–10 fertilizer. Ordinates represent yields. The curve is a so-called S curve.

four of the units of fertilizer applied, then the part of the curve obtained by experiment would be that part to the right of the vertical line at $x=4$. This part is seen to have about the same form as the curve of Figure 1, which applies to a single plant-food element, and the experimental results with the mixed fertilizer could be fitted to observations based on equation (1) very satisfactorily. In any case in practice, it is the upper part of the curve that is important, and this part always has a form similar to that of Figure 1.

The curve of Figure 1 represents increases in yields as the quantity of a growth factor made available to the crop increases. If Y_0 represent yield when the quantity applied is zero, and if this yield be added to each member of equation (13), then

$$y+Y_0=A+Y_0-AR^x \qquad \text{(A)}$$

Since Y_0 is the yield for $x=0$ and y the increase in yield for x units of the factor, then $y+Y_0$ is the actual yield, Y, for x units. Also, since A is the maximum increase in yield from $x=0$ to $x=\infty$, then $A+Y_0$ is the theoretical maximum obtainable yield, M, with any

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quantity of the factor. Equation (A) above may therefore be written

$$Y = M - AR^x$$

which is equation (1) of the preceding discussions.

GENERAL FORM

The German mathematician Baule (*1*) was the first to point out that when two or more growth factors are varied at the same time, the exponential yield equation has the form

$$y = A(1 - R_1^{x_1})\ (1 - R_2^{x_2})\ (1 - R_3^{x_3}) \tag{14}$$

in which x_1, x_2, x_3, etc., are the respective total quantities of the first, second, third, etc., growth factors available to the plant, while R_1, R_2, R_3, etc., are the ratios of the respective series of increments in yield due to successive equal increases in the respective growth factors. Mitscherlich's experiments conform to equation (14). The author has applied equation (14) to the results of some American field experiments, with very satisfactory results.

Formula (14) is adapted to water or sand cultures, in which the total quantity of nitrogen, phosphoric acid, and potash available to the growing crop is accurately known. It may be adapted to field experiments by writing it in the form

$$y = A(1 - R_1^{n+a})\ (1 - R_2^{p+b})\ (1 - R_3^{k+c}) \tag{15}$$

in which n, p, and k are the respective quantities of available nitrogen, phosphoric acid, and potash in the unfertilized soil, while a, b, c are the respective quantities in the fertilizer applied.

This equation is easily converted into equation (7), already given, by suitable changes in the size of units in which n, a, p, b, k, and c are measured.

By means of a simple series of experimental plots, suitably arranged and fertilized, the constants A, n, p, and k, R_1, R_2, and R_3 can be determined for any crop on any soil which responds to fertilizers in accordance with equation (1) when a single plant-food element is varied. The yields of the plots will also reveal whether there is such response in any particular case.

Since equation (5) makes it possible to determine the available nitrogen, phosphoric acid, and potash in a given soil, it is capable of measuring the effect on available plant-food elements of such soil amendments as lime, manure, a green-manure crop, etc.; also the effect on available plant food of different methods of tillage. This may be done by running one series of experimental plots on land receiving lime, manure, a green-manure crop turned under, a given type of tillage, etc., and another similar series on similar land not receiving the treatment. Such series, if planned as suggested later herein, would give the values of n, p, and k for the soil on which each series is conducted. Comparison of these values for the two series would then show the effect of the treatment in rendering plant food available.

Again, plants do not use all the fertilizer elements applied to the soil, especially in the case of large applications. There may, therefore, be accumulations of fertilizer residues in heavily fertilized soils that become injurious to the succeeding crops. Such cases have occurred in practice, particularly amongst growers of truck crops in certain localities. Equation (5) makes it possible to measure these residue accumulations by running a series of plots year after year, changing the plots to a new location each year. Each year the available nitrogen, phosphoric acid, and potash in the soil is measured. The rate of accumulation is thus made known, and the rate at which fertilizers are applied may be so adjusted as to take account of the changing quantities of plant food in the soil.

The above-mentioned advantages of a usable yield equation may be summarized as follows: When the constants of equation (7) have been determined then this equation and equation (5) may be used to determine—

1) The yield from any quantity of the three ingredients.
(2) The most profitable fertilizer analysis.
(3) The most profitable quantity of this optimum analysis.

Equation (7) does not apply to the average results over a series of years on permanent or semipermanent experimental plots. The reason is that the more heavily fertilized plots change from year to year in content of available plant-food elements; hence the difference in yield between a lightly and a heavily fertilized plot, especially after the test has run several years, is due partly to difference in current applications and partly to unequal accumulations of plant-food elements in the soil.

(4) The effect of lime on the availability of plant food in the soil.
(5) The effect of a legume crop on the amount of available nitrogen in the soil.
(6) The change in amount of available plant food in the soil as a result of any system of tillage or of fertilizer application.

It is therefore essential to move the test plots frequently; that is, as soon as cumulative effects appear, to a new location. Replication of the plots will, of course, add to the reliability of the experimental results, and hence to the deductions to be made from them. The use of several different quantities of each growth factor is equivalent to replication of the fundamental series, and requires fewer plots, as will be pointed out later.

Fertilizers also affect the quality of certain crops, especially tobacco, of which effect the equation gives no hint; it deals with yields, not with quality of the product.

Many other conditions, such, for instance, as the length of day, the acidity of the soil, temperature, character of seed, all affect the yield of crops. Some of these may ultimately be brought in line with equation (7), but this is work for the future.

It may be remarked here that if other growth factors are found that conform to equation (1) above, they may be brought into the picture by adding a factor of the form $(1-R^z)$ to equation (7), and an additional series of plots in which the new factor is varied. Amount of irrigation water and intensity of sunlight have been shown to follow equation (1).

PLAN FOR OBTAINING DATA

Since the most general form of the yield equation contains seven constants, A, n, p, k, R_1, R_2 and R_3, the yields from at least seven plots, suitably fertilized, are necessary to determine the value of these constants. A larger number of plots, with suitable check plots, will add materially to the accuracy of the determinations.

SERIES OF PLOTS

In Table 17 of alternative series several series are outlined. The amount of nitrogen, phosphoric acid, or potash to use as a unit in each case is more or less an empirical matter. The units should be such that the largest application of each fertilizer constituent will produce 80 to 90 per cent of the total possible effect of that constituent. In any given case, unless something is already known of the crop response to fertilizers, it may be necessary to run a preliminary test the first year before deciding how many pounds per acre shall be taken as the unit of nitrogen, how many as the unit of phosphoric acid, and how many as the unit of potash. It is not at all essential that the unit of each constituent should be the same.

The number of units of each fertilizer constituent to use on the various plots in the series is also more or less an arbitrary matter. The numbers suggested in the table greatly simplify the mathematical work in computing the value of the constants.

TABLE 17.—*Alternative series of fertilizer plots*

[a=nitrogen (N); b=phosphoric acid (P_2O_5); c=potash (K_2O)]

Plot No.	Units of—		
	a	b	c
SERIES 1			
1	1	1	1
2	2	1	1
3	3	1	1
4	1	2	1
5	1	3	1
6	1	1	2
7	1	1	3
SERIES 2			
1	1	1	1
2	2	1	1
3	3	1	1
4	1	1	1
5	1	2	1
6	1	3	1
7	1	1	1
8	1	1	2
9	1	1	3
10	1	1	1
SERIES 3 [1]			
1	1	1	1
2	2	1	1
3	3	1	1
4	1	1	1
5	4	1	1
6	5	1	1
7	1	1	1
8	1	2	1
9	1	3	1
10	1	1	1
11	1	4	1
12	1	5	1
13	1	1	1
14	1	1	2
15	1	1	3
16	1	1	1
17	1	1	4
18	1	1	5
19	1	1	1
SERIES 4 [2]			
1	1	1	1
2	2	1	1
3	3	1	1
4	1	1	1
5	4	1	1
6	5	1	1
7	1	1	1
8	6	1	1
9	7	1	1
10	1	1	1
11	1	2	1
12	1	3	1
13	1	1	1
14	1	4	1
15	1	5	1
16	1	1	1
17	1	6	1
18	1	7	1
19	1	1	1
20	1	1	2
21	1	1	3
22	1	1	1
23	1	1	4
24	1	1	5
25	1	1	1
26	1	1	6
27	1	1	7
28	1	1	1
SERIES 5 [3]			
1	1	1	1
2	2	1	1
3	3	1	1
4	1	1	1
5	4	1	1
6	5	1	1
7	1	1	1
8	6	1	1
9	7	1	1
10	1	1	1
11	8	1	1
12	9	1	1
13	1	1	1
14	1	2	1
15	1	3	1
16	1	1	1
17	1	4	1
18	1	5	1
19	1	1	1
20	1	6	1
21	1	7	1
22	1	1	1
23	1	8	1
24	1	9	1
25	1	1	1
26	1	1	2
27	1	1	3
28	1	1	1
29	1	1	4
30	1	1	5
31	1	1	1
32	1	1	6
33	1	1	7
34	1	1	1
35	1	1	8
36	1	1	9
37	1	1	1

[1] Equivalent to 3 sets of series 2. [2] Equivalent to 5 sets of series 2. [3] Equivalent to 7 sets of series 2.

Series 1 of Table 17 may be called the fundamental series. It represents the absolute minimum of data required. In a preliminary note by the writer (*5*) a smaller number of plots was given as the minimum number required. At the time that note was written it was supposed that the values of R_1, R_2, and R_3, published by Mitscherlich, as indicated by Willcox (*10*), were generally applicable. It develops that they are not always applicable to the results of field experiments in the United States, so that it is necessary to determine the three R's of the equation for each case, along with the four other constants. But this series is devoid of check plots, and would have to be replicated many times to make the yields reliable.

Series 2 is similar to series 1, but in it every third plot is a check plot. These two series provide for the use of three different quantities of each growth factor. In use, it would require several replications of series 2 to give results sufficiently accurate.

Series 3, in which five different quantities of each variable element are used, is, from the standpoint of reliability of results, equivalent to three replications of series 2. Series 4 and 5 are, respectively, equivalent to five and seven replications of series 2. In careful experimental work series 4 (28 plots) would be about the minimum limit, and series 5 (37 plots) would give results considerably more reliable and is to be recommended. The discussion that follows is based on series 3 (19 plots) merely on grounds of economy of space. This series illustrates the principles involved as well as any other.

The plots in series 2 to 5 that receive 1 unit each of a, b, and c are the standard check plots, and constitute every third plot in each series. A check plot begins and ends each series. Experimenters often make every fourth plot a check plot. This reduces materially the usefulness of check plots. The added accuracy obtained by making every third plot a check is well worth while in most cases. In cases in which the soil is quite variable, and thus not well adapted to experimental-plot work, every second plot might well be a check.

The check plots in the series outlined are all fertilized, and the application is the same on all of them. The reason for this is that if the phenomenon of absorption of plant food by the soil is present, irregularities due to this phenomenon are thus avoided.

In cases in which it is desirable to measure the amount of this absorption, a separate group of plots for this purpose may be used. An outline of procedure in such cases is given later (p. 58).

USE OF CHECK PLOTS

Check plots are used in eliminating from the computations so far as possible, the effect of unevenness in the yielding power of the soil in different parts of the experimental field. These differences are adjusted in the following manner.

"Check yields" are computed for each plot. The check yield of a plot is the yield it would presumably have produced if it had had the same application of fertilizers as the check plots. The check yield of each check plot is, of course, the yield of the plot as recorded. The check yield of the series is the average of the check yields of the check plots. The check yield of the plots between check plots is the yield of the plot plus part of the difference between the adjacent check plots. In the 19-plot series, where every third plot is a check plot, the fractions are one-third for the plot next to the lower numbered check plot and two-thirds for the plot next to the higher numbered check plot. If three plots intervene between check plots, the fractions are one-fourth, one-half, and three-fourths.

This procedure assumes that the soil changes uniformly from one plot to the next. This is not strictly true, so that there is some error in the check yields. The error is greater the greater the number of plots between checks. Without check plots the experimental errors are often so great as to vitiate the results for any careful study.

In the 19-plot series the check yields are of the following description, plots 1 and 4 being check plots:

On plot 1, check yield (C_1) is the actual yield (y_1);
on plot 2, $C_2=y_2+\frac{1}{3}(y_4-y_1)$;
on plot 3, $C_3=y_3+\frac{2}{3}(y_4-y_1)$;
on plot 4, $C_4=y_4$;
on plot 5, $C_5=y_5+\frac{1}{3}(y_7-y_4)$; and so on through the series.

The check yields having been found, the relative yields of the plots are computed by dividing the actual yield of each plot by its check yield. For the check plots the relative yields are of course 1. Thus the relative yield (U_2) of plot 2 is U_2/C_2; of plot 3 it is U_3/C_3, and so on.

The "adjusted yields" of the several plots are computed by multiplying the check yield of the series (average of the yields on the check plots) by the relative yield of the plot. The adjusted yield of each check plot is the average of all of them taken together. These adjusted yields are the yields used in all subsequent calculations.

When some of the plant food applied is occluded by the soil, and it is desired to measure the amount of this occlusion, the data necessary to do this may be obtained by inserting plots in the series listed on page 56, as shown below.

If part of each of the three elements is occluded, insert in the series at the places indicated below, groups of 4 plots each, fertilized thus:

a	b	c	
1	1	1	check plot.
0	1	1	zero plot for N.
1	0	1	zero plot for P_2O_5.
1	1	0	zero plot for K_2O.

This group is to be inserted as follows: In series 2, (1) preceding plot 1, (2) between plots 3 and 4, (3) between plots 6 and 7. In series 3, (1) preceding plot 1; (2) between plots 6 and 7; and (3) between plots 12 and 13. In series 4, (1) preceding plot 1, (2) between plots 9 and 10, and (3) between plots 18 and 19; and so on.

If the occlusion does not affect any one of the plant-food elements, the zero plots for that element may be omitted.

These insertions will, of course, change the plot numbers throughout the series.

The average adjusted yields of the three zero plots for each element are taken as the yield when that element is omitted from the fertilizer. Insertion of the group of plots at three points in the series gives three replications for each zero plot.

OTHER FACTORS TO BE CONSIDERED

This bulletin deals only with the relation between plant growth and the quantity of nitrogen, phosphoric acid, and potash available to the growing crop. Many other factors affect the growth of plants, such as soil acidity, magnesium, sulphur, the organic matter of the soil, tilth, date of planting, amount of sunlight, length of day, availability of soil moisture, texture of the soil, and temperature.

Some of these factors have been shown to produce yield curves similar to those discussed herein. This is the case with sunlight and soil moisture. Whether others behave in a similar manner remains for further research to reveal.

Additional factors that do produce yield curves similar to those dealt with here may be taken into account by adding to the generalized yield equation a factor of the form $(1-R^x)$ for each additional growth factor, and by adding to the list of experimental plots for each new growth factor a series of plots in which the new growth factor is varied in the same way as nitrogen, phosphoric acid, and potash are varied in the series listed.

SUMMARY

Three methods are given for determining the constants of the exponential yield equation for a single variable growth factor. One of these methods makes possible the determination of the most probable value of the constants by the method of least squares.

A method of determining the amount of available plant food in the soil is outlined.

The application of the exponential yield curve to a specific case in which each of the three common plant-food elements is varied is given, including the method of passing from the 1-variable form to the general form of the equation.

A method is presented for determining the amount, if any, of each of the three common plant-food elements absorbed by the soil and held in a condition unavailable to the growing crop.

The form of the yield curve is discussed, and reasons given for preferring the exponential to the parabolic form.

The derivation of all the commonly employed forms of the exponential yield curve is presented.

Formulas are worked out for determining for specific cases the optimum fertilizer formula to use, the optimum quantity of fertilizer to apply for greatest profit per acre, and for determining the optimum formula to use and the optimum acreage to which to apply a fixed amount (value) of fertilizer for greatest profit per dollar invested in fertilizer, in both presence and absence of the phenomenon of plant-food occlusion by the soil.

Plans are outlined for obtaining, from a relatively small number of experimental plots, the data necessary for determining the constants in the exponential yield equation, and the manner of utilizing check plots as a means of eliminating, in so far as this can be done, unevenness in the yielding power of the soil of the experimental field is discussed.

APPENDIX

TABLES OF VALUES OF R^x AND OF $1-R^x$

Tables 18 and 19 are appended because of their great usefulness in computations of the character required in work of the type outlined in this bulletin.

Table 18 gives the values of R^x for all values of R from 0.01 to 0.99 and of x from zero to 20. A given power of a given value of R stands on line with the value of R in the left-hand column of the page, in the column headed by the index of the power.

This table was computed by Y. Kutsunai, of the Hawaiian Sugar Experiment Station.

For convenience in computation a table of values of $1-R^x$ has been prepared on the basis of $R=0.8$, between $x=0$ and $x=29.99$. (Table 19.) Higher values of x rarely occur in connection with the yield equation.

As explained in the text, conversion of the actual ratio of the problem in hand to the ratio 0.8 is readily made.

To find $1-R^x$ when $x=3.27$, for example, run down the left-hand column of page 63 to the line showing the integral and first decimal place of x (3.2) and read the value in the column headed by the figure in the second decimal place (7). This is stated as 51794 and is a decimal fraction, all decimal points having been omitted in the printing.

To find x when $1-R^x$ is obtained in the computations, find the $1-R^x$ in the table nearest to the figure obtained and read the value of x. Thus x nearest corresponding to $1-R^x=0.57266$ is found in the column headed 1 on line 3.8, so that x in this case is 3.81. Ordinarily it is not necessary to carry the value of x beyond the second decimal place. Additional decimal places are found by interpolation; the further decimal places are the quotient of the excess above the lower tabular number divided by the tabular difference.

R^x (when $R=0.8$) may be found by subtracting the table figure for $1-R^x$ from 1.00000.

TABLE 18.—*Values of R^x*

[Value of $x=0$ is 1 in all cases; of $x=1$, same as R]

R	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$	$x=7$	$x=8$	$x=9$	$x=10$	$x=11$
0.01	0.0001	0.000001								
.02	.0004	.000008								
.03	.0009	.000027	0.000001							
.04	.0016	.000064	.000003							
.05	.0025	.000125	.000006							
.06	.0036	.000216	.000013	0.000001						
.07	.0049	.000343	.000024	.000002						
.08	.0064	.000512	.000041	.000003						
.09	.0081	.000729	.000066	.000006	0.000001					
.10	.0100	.001000	.000100	.000010	.000001					
.11	.0121	.001331	.000146	.000016	.000002					
.12	.0144	.001728	.000207	.000025	.000003					
.13	.0169	.002197	.000286	.000037	.000005	0.000001				
.14	.0196	.002744	.000384	.000054	.000008	.000001				
.15	.0225	.003375	.000506	.000076	.000011	.000002				
.16	.0256	.004096	.000655	.000105	.000017	.000003				
.17	.0289	.004913	.000835	.000142	.000024	.000004	0.000001			
.18	.0324	.005832	.001050	.000189	.000034	.000006	.000001			
.19	.0361	.006859	.001303	.000248	.000047	.000009	.000002			
.20	.0400	.008000	.001600	.000320	.000064	.000013	.000003	0.000001		
.21	.0441	.009261	.001945	.000408	.000086	.000018	.000004	.000001		
.22	.0484	.010648	.002343	.000515	.000113	.000025	.000005	.000001		
.23	.0529	.012167	.002798	.000644	.000148	.000034	.000008	.000002		
.24	.0576	.013824	.003318	.000796	.000191	.000046	.000011	.000003	0.000001	
.25	.0625	.015625	.003906	.000977	.000244	.000061	.000015	.000004	.000001	
.26	.0676	.017576	.004570	.001188	.000309	.000080	.000021	.000005	.000001	
.27	.0729	.019683	.005314	.001435	.000387	.000105	.000028	.000008	.000002	0.000001
.28	.0784	.021952	.006147	.001721	.000482	.000135	.000038	.000011	.000003	.000001
.29	.0841	.024389	.007073	.002051	.000595	.000172	.000050	.000015	.000004	.000001
.30	.0900	.027000	.008100	.002430	.000729	.000219	.000066	.000020	.000006	.000002
.31	.0961	.029791	.009235	.002863	.000888	.000275	.000085	.000026	.000008	.000003
.32	.1024	.032768	.010486	.003355	.001074	.000344	.000110	.000035	.000011	.000004
.33	.1089	.035937	.011859	.003914	.001291	.000426	.000141	.000046	.000015	.000005
.34	.1156	.039304	.013363	.004544	.001545	.000525	.000179	.000061	.000021	.000007
.35	.1225	.042875	.015006	.005252	.001838	.000643	.000225	.000079	.000028	.000010
.36	.1296	.046656	.016796	.006047	.002177	.000784	.000282	.000102	.000037	.000013
.37	.1369	.050653	.018742	.006934	.002566	.000949	.000351	.000130	.000048	.000018
.38	.1444	.054872	.020851	.007924	.003011	.001144	.000435	.000165	.000063	.000024
.39	.1521	.059319	.023134	.009022	.003519	.001372	.000535	.000209	.000081	.000032
.40	.1600	.064000	.025600	.010240	.004096	.001638	.000655	.000262	.000105	.000042
.41	.1681	.068921	.028258	.011586	.004750	.001948	.000798	.000327	.000134	.000055
.42	.1764	.074088	.031117	.013069	.005489	.002305	.000968	.000407	.000171	.000072
.43	.1849	.079507	.034188	.014701	.006321	.002718	.001169	.000503	.000216	.000093
.44	.1936	.085184	.037481	.016492	.007256	.003193	.001405	.000618	.000272	.000120
.45	.2025	.091125	.041006	.018453	.008304	.003737	.001682	.000757	.000341	.000153
.46	.2116	.097336	.044775	.020596	.009474	.004358	.002005	.000922	.000424	.000195
.47	.2209	.103823	.048797	.022935	.010779	.005066	.002381	.001119	.000526	.000247
.48	.2304	.110592	.053084	.025480	.012231	.005871	.002818	.001353	.000649	.000312
.49	.2401	.117649	.057648	.028248	.013841	.006782	.003323	.001628	.000798	.000391
.50	.2500	.125000	.062500	.031250	.015625	.007813	.003906	.001953	.000977	.000488
.51	.2601	.132651	.067652	.034503	.017596	.008974	.004577	.002334	.001190	.000607
.52	.2704	.140608	.073116	.038020	[illegible]9771	.010251	.005346	.002780	.001446	.000752
.53	.2809	.148877	.078905	.041820	.022164	.011747	.006226	.003300	.001749	.000927

TABLE 18.—*Values of R^x*—Continued

R	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$	$x=7$	$x=8$	$x=9$	$x=10$	$x=11$
0.54	0.2916	0.157464	0.085031	0.045917	0.024795	0.013389	0.007230	0.003904	0.002108	0.001138
.55	.3025	.166375	.091506	.050328	.027681	.015224	.008373	.004605	.002533	.001393
.56	.3136	.175616	.098345	.055073	.030841	.017271	.009672	.005416	.003033	.001699
.57	.3249	.185193	.105560	.060169	.034296	.019549	.011143	.006351	.003620	.002064
.58	.3364	.195112	.113165	.065636	.038069	.022080	.012806	.007428	.004308	.002499
.59	.3481	.205379	.121174	.071492	.042181	.024887	.014683	.008663	.005111	.003016
.60	.3600	.216000	.129600	.077760	.046656	.027994	.016796	.010078	.006047	.003628
.61	.3721	.226981	.138458	.084460	.051520	.031427	.019171	.011694	.007133	.004351
.62	.3844	.238328	.147763	.091613	.056800	.035216	.021834	.013537	.008393	.005204
.63	.3969	.250047	.157530	.099244	.062524	.039390	.024816	.015634	.009849	.006205
.64	.4096	.262144	.167772	.107374	.068719	.043980	.028147	.018014	.011529	.007379
.65	.4225	.274625	.178506	.116029	.075419	.049022	.031864	.020712	.013463	.008751
.66	.4356	.287496	.189747	.125233	.082654	.054552	.036004	.023763	.015683	.010351
.67	.4489	.300763	.201511	.135013	.090458	.060607	.040607	.027207	.018228	.012213
.68	.4624	.314432	.213814	.145393	.098867	.067230	.045716	.031087	.021139	.014375
.69	.4761	.328509	.226671	.156403	.107918	.074464	.051380	.035452	.024462	.016879
.70	.4900	.343000	.240100	.168070	.117649	.082354	.057648	.040354	.028248	.019773
.71	.5041	.357911	.254117	.180423	.128100	.090951	.064575	.045849	.032552	.023112
.72	.5184	.373248	.268739	.193492	.139314	.100306	.072220	.051999	.037439	.026956
.73	.5329	.389017	.283982	.207307	.151334	.110474	.080646	.058872	.042976	.031373
.74	.5476	.405224	.299866	.221901	.164206	.121513	.089919	.066540	.049240	.036438
.75	.5625	.421875	.316406	.237305	.177979	.133484	.100113	.075085	.056314	.042235
.76	.5776	.438976	.333622	.253553	.192700	.146452	.111303	.084591	.064289	.048860
.77	.5929	.456533	.351530	.270678	.208422	.160485	.123574	.095152	.073267	.056415
.78	.6084	.474552	.370151	.288717	.225200	.175656	.137011	.106869	.083358	.065019
.79	.6241	.493039	.389501	.307706	.243087	.192039	.151711	.119852	.094683	.074799
.80	.6400	.512000	.409600	.327680	.262144	.209715	.167772	.134218	.107374	.085899
.81	.6561	.531441	.430467	.348678	.282430	.228768	.185302	.150095	.121577	.098477
.82	.6724	.551368	.452122	.370740	.304007	.249285	.204414	.167620	.137448	.112707
.83	.6889	.571787	.474583	.393904	.326940	.271361	.225229	.186940	.155160	.128783
.84	.7056	.592704	.497871	.418212	.351298	.295090	.247876	.208216	.174901	.146917
.85	.7225	.614125	.522006	.443705	.377150	.320577	.272491	.231617	.196874	.167343
.86	.7396	.636056	.547008	.470427	.404567	.347928	.299218	.257327	.221302	.190319
.87	.7569	.658503	.572898	.498421	.433626	.377255	.328212	.285544	.248423	.216128
.88	.7744	.681472	.599695	.527732	.464404	.408676	.359635	.316478	.278501	.245081
.89	.7921	.704969	.627422	.558406	.496981	.442313	.393659	.350356	.311817	.277517
.90	.8100	.729000	.656100	.590490	.531441	.478297	.430467	.387420	.348678	.313811
.91	.8281	.753571	.685750	.624032	.567869	.516761	.470253	.427930	.389416	.354369
.92	.8464	.778688	.716393	.659082	.606355	.557847	.513219	.472161	.434388	.399637
.93	.8649	.804357	.748052	.695688	.646990	.601701	.559582	.520411	.483982	.450104
.94	.8836	.830584	.780749	.733904	.689870	.648478	.609569	.572995	.538615	.506298
.95	.9025	.857375	.814506	.773781	.735092	.698337	.663420	.630249	.598737	.568800
.96	.9216	.884736	.849347	.815373	.782758	.751447	.721390	.692534	.664833	.638239
.97	.9409	.912673	.885293	.858734	.832972	.807983	.783743	.760231	.737424	.715301
.98	.9604	.941192	.922368	.903921	.885842	.868126	.850763	.833748	.817073	.800731
.99	.9801	.970299	.960596	.950990	.941480	.932065	.922745	.913517	.904382	.895338

R	$x=12$	$x=13$	$x=14$	$x=15$	$x=16$	$x=17$	$x=18$	$x=19$	$x=20$
0.30	0.000001								
.31	.000001								
.32	.000001								
.33	.000002	0.000001							
.34	.000002	.000001							
.35	.000003	.000001							
.36	.000005	.000002	0.000001						
.37	.000007	.000002	.000001						
.38	.000009	.000003	.000001						
.39	.000012	.000005	.000002	0.000001					
.40	.000017	.000007	.000003	.000001					
.41	.000023	.000009	.000004	.000002	0.000001				
.42	.000030	.000013	.000005	.000002	.000001				
.43	.000040	.000017	.000007	.000003	.000001	0.000001			
.44	.000053	.000023	.000010	.000004	.000002	.000001			
.45	.000069	.000031	.000014	.000006	.000003	.000001	0.000001		
.46	.000090	.000041	.000019	.000009	.000004	.000002	.000001		
.47	.000116	.000055	.000026	.000012	.000006	.000002	.000001	0.000001	
.48	.000150	.000072	.000034	.000017	.000008	.000004	.000002	.000001	
.49	.000192	.000094	.000046	.000023	.000011	.000005	.000003	.000001	0.000001
.50	.000244	.000122	.000061	.000031	.000015	.000008	.000004	.000002	.000001
.51	.000310	.000158	.000081	.000041	.000021	.000011	.000005	.000003	.000001
.52	.000391	.000203	.000106	.000055	.000029	.000015	.000008	.000004	.000002
.53	.000491	.000260	.000138	.000073	.000039	.000021	.000011	.000006	.000003
.54	.000615	.000332	.000179	.000097	.000052	.000028	.000015	.000008	.000004
.55	.000766	.000421	.000232	.000127	.000070	.000039	.000021	.000012	.000006
.56	.000951	.000533	.000298	.000167	.000094	.000052	.000029	.000016	.000009
.57	.001176	.000670	.000382	.000218	.000124	.000071	.000040	.000023	.000013
.58	.001449	.000841	.000488	.000283	.000164	.000095	.000055	.000032	.000019
.59	.001779	.001050	.000619	.000365	.000216	.000127	.000075	.000044	.000026
.60	.002177	.001306	.000784	.000470	.000282	.000169	.000102	.000061	.000037

TABLE 18.—*Values of R^x*—Continued

R	x=12	x=13	x=14	x=15	x=16	x=17	x=18	x=19	x=20
0.61	0.002654	0.001619	0.000988	0.000602	0.000368	0.000224	0.000137	0.000083	0.000051
.62	.003226	.002000	.001240	.000769	.000477	.000296	.000183	.000114	.000070
.63	.003909	.002463	.001552	.000977	.000616	.000388	.000244	.000154	.000099
.64	.004722	.003022	.001934	.001238	.000792	.000507	.000325	.000208	.000133
.65	.005688	.003697	.002403	.001562	.001015	.000660	.000429	.000279	.000181
.66	.006852	.004509	.002976	.001964	.001296	.000856	.000565	.000373	.000246
.67	.008183	.005482	.003673	.002461	.001640	.001105	.000740	.000496	.000332
.68	.009775	.006647	.004520	.003074	.002090	.001421	.000966	.000657	.000447
.69	.011646	.008036	.005545	.003826	.002640	.001822	.001257	.000867	.000598
.70	.013841	.009689	.006782	.004748	.003323	.002326	.001628	.001140	.000798
.71	.016410	.011651	.008272	.005873	.004170	.002961	.002102	.001492	.001060
.72	.019408	.013974	.010061	.007244	.005216	.003755	.002704	.001947	.001402
.73	.022902	.016718	.012205	.008909	.006504	.004748	.003466	.002530	.001847
.74	.026964	.019953	.014765	.010926	.008086	.005983	.004428	.003276	.002425
.75	.031676	.023757	.017818	.013363	.010023	.007517	.005638	.004228	.003171
.76	.037133	.028221	.021448	.016301	.012388	.009415	.007156	.005438	.004133
.77	.043440	.033449	.025756	.019832	.015270	.011758	.009054	.006971	.005368
.78	.050715	.039558	.030855	.024067	.018772	.014642	.011421	.008908	.006949
.79	.059092	.046682	.036879	.029134	.023016	.018183	.014364	.011348	.008965
.80	.068719	.054976	.043980	.035184	.028147	.022518	.018014	.014412	.011529
.81	.079766	.064611	.052335	.042391	.034337	.027813	.022528	.018248	.014781
.82	.092420	.075784	.062143	.050957	.041785	.034264	.028096	.023039	.018892
.83	.106890	.088719	.073637	.061118	.050728	.042104	.034947	.029006	.024075
.84	.123410	.103665	.087078	.073146	.061442	.051612	.043354	.036417	.030590
.85	.142242	.120905	.102770	.087354	.074251	.063113	.053646	.045599	.038760
.86	.163675	.140760	.121054	.104106	.089531	.076997	.066217	.056947	.048974
.87	.188032	.163588	.142321	.123819	.107723	.093719	.081535	.070936	.061714
.88	.215671	.189791	.167016	.146974	.129337	.113817	.100159	.088140	.077563
.89	.246990	.219821	.195641	.174121	.154967	.137921	.122750	.109247	.097230
.90	.282430	.254187	.228768	.205891	.185302	.166772	.150095	.135085	.121577
.91	.322475	.293453	.267042	.243008	.221137	.201235	.183124	.166643	.151645
.92	.367666	.338253	.311193	.286297	.263394	.242322	.222936	.205101	.188693
.93	.418596	.389295	.362044	.336701	.313132	.291213	.270828	.251870	.234239
.94	.475920	.447365	.420523	.395292	.371574	.349280	.328323	.308624	.290106
.95	.540360	.513342	.487675	.463291	.440127	.418120	.397214	.377354	.358486
.96	.612710	.588201	.564673	.542086	.520403	.499587	.479603	.460419	.442002
.97	.693842	.673027	.652836	.633251	.614254	.595826	.577951	.560613	.543794
.98	.784717	.769022	.753642	.738569	.723798	.709322	.695135	.681233	.667608
.99	.886385	.877521	.868746	.860058	.851458	.842943	.834514	.826169	.817907

TABLE 19.—*Values of $1-R^x$ when $R=0.8$*

[All values are decimal fractions; that is, decimal points to be added]

x	0	1	2	3	4	5	6	7	8	9
0.0	00000	00223	00445	00667	00889	01110	01330	01550	01769	01988
.1	02207	02425	02642	02859	03076	03292	03507	03722	03937	04151
.2	04365	04578	04791	05003	05215	05426	05637	05847	06057	06266
.3	06475	06684	06892	07099	07306	07513	07719	07925	08130	08335
.4	08539	08743	08946	09149	09352	09554	09755	09956	10157	10357
.5	10557	10757	10956	11154	11352	11550	11747	11944	12140	12336
.6	12531	12726	12920	13114	13308	13501	13694	13887	14079	14270
.7	14461	14652	14842	15032	15221	15410	15599	15787	15975	16162
.8	16349	16535	16721	16907	17092	17277	17461	17645	17829	18012
.9	18195	18377	18559	18741	18922	19102	19283	19463	19642	19821
1.0	20000	20178	20356	20534	20711	20888	21064	21240	21415	21590
1.1	21765	21940	22114	22287	22461	22634	22806	22978	23150	23321
1.2	23492	23662	23832	24002	24172	24341	24509	24677	24845	25013
1.3	25180	25347	25513	25679	25845	26010	26175	26340	26504	26668
1.4	26831	26994	27157	27319	27481	27643	27804	27965	28126	28286
1.5	28446	28605	28764	28923	29082	29240	29397	29555	29712	29869
1.6	30025	30181	30336	30492	30647	30801	30955	31109	31263	31416
1.7	31569	31722	31874	32026	32177	32328	32479	32630	32780	32930
1.8	33079	33228	33377	33526	33674	33822	33969	34116	34263	34410
1.9	34556	34702	34847	34992	35137	35282	35426	35570	35714	35857
2.0	36000	36143	36285	36427	36569	36710	36851	36992	37132	37272
2.1	37412	37552	37691	37830	37968	38106	38244	38382	38520	38657
2.2	38793	38930	39066	39202	39337	39472	39607	39742	39876	40010
2.3	40144	40277	40410	40543	40676	40808	40940	41072	41203	41334
2.4	41465	41595	41725	41855	41985	42114	42243	42372	42501	42629
2.5	42757	42884	43012	43139	43265	43392	43518	43644	43769	43895
2.6	44020	44145	44269	44393	44517	44641	44764	44887	45010	45133
2.7	45255	45377	45499	45620	45742	45863	45983	46104	46224	46344
2.8	46463	46583	46702	46820	46939	47057	47175	47293	47410	47528
2.9	47645	47761	47878	47994	48110	48226	48341	48456	48571	48686
3.0	48800	48914	49028	49141	49255	49368	49481	49594	49706	49818
3.1	49930	50042	50153	50264	50375	50485	50596	50706	50816	50925

TABLE 19.—*Values of* $1-R^x$ *when* $R=0.8$—Continued

x	0	1	2	3	4	5	6	7	8	9
3.2	51035	51144	51253	51361	51470	51578	51686	51794	51901	52008
3.3	52115	52222	52329	52435	52541	52647	52752	52857	52962	53067
3.4	53172	53276	53380	53484	53588	53692	53795	53898	54000	54103
3.5	54205	54307	54409	54511	54612	54713	54814	54915	55016	55116
3.6	55216	55316	55415	55515	55614	55713	55811	55910	56008	56106
3.7	56204	56302	56399	56496	56593	56690	56787	56883	56979	57075
3.8	57171	57266	57361	57456	57551	57646	57740	57834	57928	58022
3.9	58116	58209	58302	58395	58488	58580	58673	58765	58857	58948
4.0	59040	59131	59222	59313	59404	59494	59585	59675	59765	59854
4.1	59944	60033	60122	60211	60300	60388	60477	60565	60653	60740
4.2	60828	60915	61002	61089	61176	61262	61349	61435	61521	61607
4.3	61692	61778	61863	61948	62033	62117	62202	62286	62370	62454
4.4	62538	62621	62704	62788	62870	62953	63036	63118	63200	63282
4.5	63364	63446	63527	63609	63690	63771	63851	63932	64012	64093
4.6	64173	64253	64332	64412	64491	64570	64649	64728	64807	64885
4.7	64963	65041	65119	65197	65275	65352	65429	65506	65583	65660
4.8	65736	65813	65889	65965	66041	66117	66192	66268	66343	66418
4.9	66493	66567	66642	66716	66790	66864	66938	67012	67085	67159
5.0	67232	67305	67378	67451	67523	67596	67668	67740	67812	67883
5.1	67955	68027	68098	68169	68240	68311	68381	68451	68522	68592
5.2	68662	68732	68802	68871	68941	69010	69079	69148	69217	69285
5.3	69354	69422	69490	69558	69626	69694	69761	69829	69896	69963
5.4	70030	70097	70164	70230	70296	70363	70429	70495	70560	70626
5.5	70691	70757	70822	70887	70952	71016	71081	71146	71210	71274
5.6	71338	71402	71466	71529	71592	71656	71719	71782	71845	71908
5.7	71971	72033	72095	72158	72220	72282	72344	72406	72467	72528
5.8	72589	72650	72711	72772	72833	72893	72954	73014	73074	73134
5.9	73194	73254	73313	73373	73432	73491	73551	73610	73668	73727
6.0	73786	73844	73902	73960	74018	74076	74134	74192	74249	74307
6.1	74364	74421	74478	74535	74592	74649	74705	74761	74818	74874
6.2	74930	74986	75041	75097	75153	75208	75263	75318	75373	75428
6.3	75483	75538	75592	75647	75701	75755	75809	75863	75917	75970
6.4	76024	76077	76131	76184	76237	76290	76343	76396	76448	76501
6.5	76553	76605	76658	76710	76761	76813	76865	76917	76968	77019
6.6	77071	77122	77173	77224	77274	77325	77375	77426	77476	77526
6.7	77576	77626	77676	77726	77776	77825	77875	77924	77973	78022
6.8	78071	78120	78169	78218	78266	78315	78363	78411	78459	78507
6.9	78556	78603	78651	78698	78746	78793	78840	78888	78935	78982
7.0	79028	79075	79122	79168	79215	79261	79307	79354	79400	79445
7.1	79491	79537	79583	79628	79674	79719	79764	79809	79854	79899
7.2	79944	79989	80033	80078	80122	80166	80211	80255	80299	80343
7.3	80386	80430	80474	80517	80561	80604	80647	80690	80733	80776
7.4	80819	80862	80905	80947	80990	81032	81074	81117	81159	81201
7.5	81242	81284	81326	81367	81409	81451	81492	81533	81574	81615
7.6	81656	81697	81738	81779	81819	81860	81900	81941	81981	82021
7.7	82061	82101	82141	82181	82221	82260	82300	82339	82379	82418
7.8	82457	82496	82535	82574	82613	82652	82690	82729	82768	82806
7.9	82844	82882	82920	82959	82997	83035	83072	83110	83148	83185
8.0	83223	83260	83297	83335	83372	83409	83446	83483	83520	83557
8.1	83593	83630	83666	83702	83739	83775	83811	83847	83883	83919
8.2	83955	83991	84027	84062	84098	84133	84168	84204	84239	84274
8.3	84309	84344	84379	84414	84449	84483	84518	84552	84587	84621
8.4	84655	84690	84724	84758	84792	84826	84859	84893	84927	84960
8.5	84994	85027	85061	85094	85127	85161	85194	85227	85259	85292
8.6	85325	85358	85391	85423	85456	85488	85520	85552	85585	85617
8.7	85649	85681	85713	85745	85776	85808	85840	85872	85903	85934
8.8	85966	85997	86028	86059	86090	86121	86152	86183	86214	86245
8.9	86274	86306	86336	86367	86397	86428	86458	86488	86518	86548
9.0	86578	86608	86638	86668	86697	86727	86757	86786	86816	86845
9.1	86875	86904	86933	86962	86991	87018	87049	87078	87107	87136
9.2	87164	87193	87221	87250	87278	87306	87335	87363	87391	87419
9.3	87447	87475	87503	87531	87559	87587	87614	87642	87669	87697
9.4	87724	87752	87779	87806	87833	87860	87887	87915	87942	87968
9.5	87995	88022	88049	88075	88102	88128	88155	88181	88208	88234
9.6	88260	88286	88312	88338	88364	88390	88416	88442	88468	88494
9.7	88519	88545	88570	88596	88621	88647	88672	88697	88722	88747
9.8	88773	88798	88823	88847	88872	88897	88922	88947	88971	88996
9.9	89020	89045	89069	89094	89118	89142	89166	89190	89215	89239
10.0	89263	89287	89310	89334	89358	89382	89405	89429	89453	89476
10.1	89500	89523	89546	89570	89593	89616	89639	89662	89685	89708
10.2	89731	89754	89777	89800	89822	89845	89868	89890	89913	89936
10.3	89958	89980	90003	90025	90047	90069	90091	90113	90135	90157
10.4	90179	90201	90223	90245	90267	90288	90310	90332	90353	90375
10.5	90396	90418	90439	90460	90482	90503	90524	90545	90566	90587
10.6	90608	90629	90650	90671	90692	90712	90733	90754	90774	90795
10.7	90815	90836	90856	90877	90897	90917	90937	90958	90978	90998
10.8	91018	91038	91058	91078	91098	91118	91138	91157	91177	91197
10.9	91216	91236	91255	91275	91294	91314	91333	91352	91372	91391
11.0	91410	91429	91448	91467	91486	91505	91524	91543	91562	91581
11.1	91600	91618	91637	91656	91674	91693	91711	91730	91748	91767
11.2	91785	91803	91822	91840	91858	91876	91894	91912	91930	91948
11.3	91966	91984	92002	92020	92038	92055	92073	92091	92108	92126
11.4	92144	92161	92179	92196	92213	92231	92248	92265	92283	92300

TABLE 19.—*Values of* $1-R^x$ *when* $R=0.8$—Continued

x	0	1	2	3	4	5	6	7	8	9
11.5	92317	92334	92351	92368	92385	92402	92419	92436	92453	92470
11.6	92487	92503	92520	92537	92553	92570	92586	92603	92619	92636
11.7	92652	92669	92685	92701	92718	92734	92750	92766	92782	92798
11.8	92814	92830	92846	92862	92878	92894	92910	92926	92942	92957
11.9	92973	92989	93004	93020	93035	93051	93066	93082	93097	93113
12.0	93128	93143	93159	93174	93189	93204	93219	93235	93250	93265
12.1	93280	93295	93310	93324	93339	93354	93369	93384	93399	93413
12.2	93428	93443	93457	93472	93486	93501	93515	93530	93544	93559
12.3	93573	93587	93602	93616	93630	93644	93658	93673	93687	93701
12.4	93715	93729	93743	93757	93771	93785	93798	93812	93826	93840
12.5	93853	93867	93881	93895	93908	93922	93935	93949	93962	93976
12.6	93989	94003	94016	94029	94043	94056	94069	94082	94095	94109
12.7	94122	94135	94148	94161	94174	94187	94200	94213	94226	94239
12.8	94252	94264	94277	94290	94303	94315	94328	94341	94353	94366
12.9	94378	94391	94403	94416	94428	94441	94453	94465	94478	94490
13.0	94502	94515	94527	94539	94551	94563	94575	94588	94600	94612
13.1	94624	94636	94648	94660	94671	94683	94695	94707	94719	94731
13.2	94742	94754	94766	94777	94789	94801	94812	94824	94835	94847
13.3	94858	94870	94881	94893	94904	94915	94927	94938	94949	94961
13.4	94972	94983	94994	95005	95017	95028	95039	95050	95061	95072
13.5	95083	95094	95105	95116	95126	95137	95148	95159	95170	95181
13.6	95191	95202	95213	95223	95234	95245	95255	95266	95276	95287
13.7	95297	95308	95318	95329	95339	95349	95360	95370	95381	95391
13.8	95401	95411	95422	95432	95442	95452	95462	95472	95483	95493
13.9	95503	95513	95523	95533	95543	95553	95563	95572	95582	95592
14.0	95602	95612	95621	95631	95641	95651	95660	95670	95680	95689
14.1	95699	95709	95718	95728	95737	95747	95756	95766	95775	95784
14.2	95794	95803	95813	95822	95831	95841	95850	95859	95868	95878
14.3	95887	95896	95905	95914	95923	95932	95941	95951	95960	95969
14.4	95978	95987	95995	96004	96013	96022	96031	96040	96049	96058
14.5	96066	96075	96084	96092	96101	96110	96119	96127	96136	96144
14.6	96153	96162	96170	96179	96187	96196	96204	96213	96221	96230
14.7	96238	96246	96255	96263	96271	96280	96288	96296	96304	96313
14.8	96321	96329	96337	96346	96354	96362	96370	96378	96386	96393
14.9	96402	96410	96418	96426	96434	96442	96450	96458	96466	96474
15.0	96482	96489	96497	96505	96513	96521	96528	96536	96544	96551
15.1	96559	96567	96574	96582	96590	96597	96605	96612	96620	96628
15.2	96635	96643	96650	96658	96665	96672	96680	96687	96695	96702
15.3	96709	96717	96724	96731	96739	96746	96753	96760	96768	96775
15.4	96782	96789	96796	96803	96811	96818	96825	96832	96839	96846
15.5	96853	96860	96867	96874	96881	96888	96895	96902	96909	96916
15.6	96923	96929	96936	96943	96950	96957	96963	96970	96977	96984
15.7	96991	96998	97004	97010	97017	97024	97030	97037	97044	97050
15.8	97057	97063	97070	97076	97083	97089	97096	97102	97109	97115
15.9	97122	97128	97135	97141	97147	97154	97160	97166	97173	97179
16.0	97185	97192	97198	97204	97210	97216	97223	97229	97235	97241
16.1	97247	97253	97260	97266	97272	97278	97284	97290	97296	97302
16.2	97308	97314	97320	97326	97332	97338	97344	97350	97356	97362
16.3	97368	97373	97379	97385	97391	97397	97403	97408	97414	97420
16.4	97426	97431	97437	97443	97448	97454	97460	97465	97471	97477
16.5	97482	97488	97494	97499	97505	97510	97516	97521	97527	97532
16.6	97538	97544	97549	97554	97560	97565	97571	97576	97581	97587
16.7	97592	97598	97603	97608	97614	97619	97624	97630	97635	97640
16.8	97645	97651	97656	97661	97666	97671	97677	97682	97687	97692
16.9	97697	97702	97708	97713	97718	97723	97728	97733	97738	97743
17.0	97748	97753	97758	97763	97768	97773	97778	97783	97788	97793
17.1	97798	97803	97808	97813	97817	97822	97827	97832	97837	97842
17.2	97846	97851	97856	97861	97866	97870	97875	97880	97885	97889
17.3	97894	97899	97903	97908	97913	97917	97922	97927	97931	97936
17.4	97940	97945	97950	97954	97959	97963	97968	97972	97977	97981
17.5	97986	97990	97995	97999	98004	98008	98013	98017	98022	98026
17.6	98030	98035	98039	98043	98048	98052	98056	98061	98065	98069
17.7	98074	98078	98082	98087	98091	98095	98099	98104	98108	98112
17.8	98116	98120	98125	98129	98133	98137	98141	98146	98150	98154
17.9	98158	98162	98166	98170	98174	98178	98182	98186	98190	98194
18.0	98199	98203	98207	98211	98215	98219	98223	98226	98230	98234
18.1	98238	98242	98246	98250	98254	98258	98262	98266	98269	98273
18.2	98277	98281	98285	98289	98292	98296	98300	98304	98308	98311
18.3	98315	98319	98323	98326	98330	98334	98338	98341	98345	98349
18.4	98352	98356	98360	98363	98367	98371	98374	98378	98382	98385
18.5	98389	98392	98396	98399	98403	98406	98410	98414	98417	98421
18.6	98424	98428	98431	98435	98438	98442	98445	98449	98452	98456
18.7	98459	98462	98466	98469	98472	98476	98480	98483	98486	98490
18.8	98494	98497	98500	98503	98506	98510	98513	98516	98520	98523
18.9	98526	98530	98533	98536	98539	98543	98546	98549	98552	98556
19.0	98559	98562	98565	98568	98572	98575	98578	98581	98584	98587
19.1	98591	98594	98597	98600	98603	98606	98609	98612	98616	98619
19.2	98622	98625	98628	98631	98634	98637	98640	98643	98646	98649
19.3	98652	98655	98658	98661	98664	98667	98670	98673	98676	98679
19.4	98682	98685	98688	98691	98694	98697	98699	98702	98705	98708
19.5	98711	98714	98717	98720	98722	98725	98728	98731	98734	98737
19.6	98739	98742	98745	98748	98751	98753	98756	98759	98762	98765
19.7	98767	98770	98773	98775	98778	98781	98784	98786	98789	98792

TABLE 19.—*Values of* $1-R^x$ *when* $R=0.8$—Continued

x	0	1	2	3	4	5	6	7	8	9
19.8	98794	98797	98800	98802	98805	98808	98410	98813	98816	98819
19.9	98821	98824	98826	98829	98832	98834	98837	98839	98842	98845
20.0	98847	98850	98852	98855	98857	98860	98862	98865	98867	98870
20.1	98873	98876	98878	98880	98883	98885	98888	98890	98892	98895
20.2	98897	98900	98902	98905	98907	98910	98912	98914	98917	98919
20.3	98922	98924	98927	98929	98931	98934	98936	98938	98941	98943
20.4	98945	98948	98950	98953	98955	98957	98960	98962	98964	98966
20.5	98969	98971	98973	98976	98978	98980	98982	98985	98988	98990
20.6	98992	98994	98996	98998	99001	99003	99005	99007	99009	99012
20.7	99014	99016	99018	99020	99023	99025	99027	99029	99031	99034
20.8	99036	99038	99040	99042	99044	99046	99048	99051	99053	99055
20.9	99057	99059	99061	99063	99065	99067	99069	99071	99073	99075
21.0	99077	99080	99082	99084	99086	99088	99090	99092	99094	99096
21.1	99098	99100	99102	99104	99106	99108	99110	99112	99114	99116
21.2	99118	99120	99122	99124	99126	99128	99130	99132	99134	99136
21.3	99137	99139	99141	99143	99145	99147	99149	99151	99153	99155
21.4	99157	99158	99160	99162	99164	99166	99168	99170	99171	99173
21.5	99175	99177	99179	99181	99182	99184	99186	99188	99190	99192
21.6	99193	99195	99197	99199	99201	99202	99204	99206	99208	99209
21.7	99211	99213	99215	99216	99218	99220	99222	99223	99225	99227
21.8	99229	99230	99232	99234	99235	99237	99239	99241	99242	99244
21.9	99246	99247	99249	99251	99252	99254	99256	99257	99259	99261
22.0	99262	99264	99266	99267	99269	99270	99272	99274	99275	99277
22.1	99279	99280	99282	99283	99285	99286	99288	99290	99291	99293
22.2	99294	99296	99298	99299	99300	99302	99304	99305	99307	99308
22.3	99310	99312	99313	99315	99316	99318	99319	99321	99322	99324
22.4	99325	99327	99328	99330	99331	99333	99334	99336	99338	99339
22.5	99340	99341	99343	99344	99346	99347	99349	99350	99352	99353
22.6	99355	99356	99358	99359	99360	99362	99363	99365	99366	99368
22.7	99369	99370	99372	99373	99374	99376	99377	99379	99380	99381
22.8	99383	99384	99386	99387	99388	99390	99391	99392	99394	99395
22.9	99396	99398	99399	99401	99402	99403	99405	99406	99407	99408
23.0	99410	99411	99412	99414	99415	99416	99418	99419	99420	99421
23.1	99423	99424	99425	99427	99428	99429	99431	99432	99433	99434
23.2	99436	99437	99438	99439	99440	99442	99443	99445	99446	99447
23.3	99448	99449	99450	99452	99453	99454	99455	99456	99458	99459
23.4	99460	99461	99463	99464	99465	99466	99467	99468	99470	99471
23.5	99472	99473	99374	99476	99477	99478	99479	99480	99481	99483
23.6	99484	99485	99486	99487	99488	99490	99491	99492	99493	99494
23.7	99495	99496	99497	99498	99500	99501	99502	99503	99504	99505
23.8	99506	99507	99508	99509	99511	99512	99513	99514	99515	99516
23.9	99517	99518	99519	99520	99521	99522	99524	99525	99526	99527
24.0	99528	99529	99530	99531	99532	99533	99534	99535	99536	99537
24.1	99538	99539	99540	99541	99542	99543	99544	99545	99546	99547
24.2	99548	99549	99550	99551	99552	99553	99554	99555	99556	99557
24.3	99558	99559	99560	99561	99562	99563	99564	99565	99566	99567
24.4	99568	99569	99570	99571	99572	99573	99574	99575	99576	99577
24.5	99578	99579	99580	99581	99581	99582	99583	99584	99585	99586
24.6	99587	99588	99589	99590	99591	99592	99593	99593	99594	99595
24.7	99596	99597	99598	99599	99600	99601	99602	99602	99603	99604
24.8	99605	99606	99607	99608	99609	99609	99610	99611	99612	99613
24.9	99614	99615	99615	99616	99617	99618	99619	99620	99621	99621
25.0	99622	99623	99624	99625	99626	99626	99627	99628	99628	99629
25.1	99630	99631	99632	99633	99634	99635	99636	99636	99637	99638
25.2	99639	99640	99640	99641	99642	99643	99644	99644	99645	99646
25.3	99647	99648	99648	99649	99650	99651	99651	99652	99653	99654
25.4	99655	99655	99656	99657	99658	99659	99659	99660	99661	99661
25.5	99662	99663	99664	99664	99665	99666	99667	99668	99668	99669
25.6	99670	99670	99671	99672	99672	99673	99674	99675	99675	99676
25.7	99677	99678	99678	99679	99680	99681	99681	99682	99683	99683
25.8	99684	99685	99685	99686	99687	99687	99688	99689	99690	99690
25.9	99691	99692	99692	99693	99694	99694	99695	99696	99696	99697
26.0	99698	99699	99699	99700	99700	99701	99702	99703	99703	99704
26.1	99704	99705	99706	99706	99707	99707	99708	99708	99709	99710
26.2	99711	99712	99712	99713	99714	99714	99715	99716	99716	99717
26.3	99717	99718	99719	99719	99720	99720	99721	99722	99722	99723
26.4	99724	99724	99725	99726	99726	99727	99727	99728	99728	99729
26.5	99730	99730	99731	99731	99732	99733	99733	99734	99734	99735
26.6	99736	99736	99737	99738	99738	99739	99739	99740	99740	99741
26.7	99741	99742	99743	99743	99744	99744	99745	99746	99746	99747
26.8	99747	99748	99749	99749	99750	99750	99751	99751	99752	99752
26.9	99753	99753	99754	99755	99755	99756	99756	99757	99757	99758
27.0	99759	99759	99759	99760	99760	99761	99761	99762	99762	99763
27.1	99764	99764	99765	99765	99766	99766	99767	99767	99768	99768
27.2	99769	99769	99770	99770	99771	99771	99772	99772	99773	99773
27.3	99774	99774	99775	99775	99776	99776	99777	99777	99778	99778
27.4	99779	99779	99780	99780	99781	99781	99782	99782	99783	99783
27.5	99784	99784	99785	99785	99786	99786	99787	99787	99788	99788
27.6	99789	99789	99789	99790	99790	99791	99791	99792	99792	99793
27.7	99793	99794	99794	99795	99795	99796	99796	99796	99797	99797
27.8	99798	99798	99799	99799	99800	99800	99800	99801	99801	99802
27.9	99802	99803	99803	99804	99804	99805	99805	99805	99806	99806

TABLE 19.—*Values of* $1-R^x$ *when* $R=0.8$—Continued

x	0	1	2	3	4	5	6	7	8	9
28.0	99807	99807	99807	99808	99808	99809	99809	99810	99810	99810
28.1	99811	99811	99812	99812	99813	99813	99813	99814	99814	99815
28.2	99815	99816	99816	99816	99817	99817	99817	99818	99818	99819
28.3	99819	99820	99820	99820	99821	99821	99821	99822	99822	99823
28.4	99823	99823	99824	99824	99825	99825	99825	99826	99826	99827
28.5	99827	99827	99828	99828	99829	99829	99829	99830	99830	99830
28.6	99831	99831	99832	99832	99832	99833	99833	99833	99834	99834
28.7	99835	99835	99835	99836	99836	99836	99836	99837	99837	99838
28.8	99838	99839	99839	99839	99840	99840	99840	99841	99841	99841
28.9	99842	99842	99842	99843	99843	99844	99844	99844	99845	99845
29.0	99845	99846	99846	99846	99847	99847	99847	99848	99848	99848
29.1	99849	99849	99849	99850	99850	99850	99851	99851	99851	99852
29.2	99852	99852	99853	99853	99853	99854	99854	99854	99855	99855
29.3	99855	99856	99856	99856	99857	99857	99857	99858	99858	99858
29.4	99858	99859	99859	99859	99860	99860	99860	99861	99861	99861
29.5	99862	99862	99862	99862	99863	99863	99863	99864	99864	99864
29.6	99865	99865	99865	99866	99866	99866	99866	99867	99867	99867
29.7	99868	99868	99868	99869	99869	99869	99869	99870	99870	99870
29.8	99871	99871	99871	99871	99872	99872	99872	99873	99873	99873
29.9	99873	99874	99874	99874	99875	99875	99875	99875	99876	99876

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