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# Stochastic efficiency analysis with risk aversion bounds: a correction

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A recent paper by Hardaker *et al.* (The Australian Journal of Agricultural and Resource Economics, 48, 2004a, 253) and book by Hardaker *et al.* (Coping with Risk in Agriculture, 2004b) describe a procedure for determining an efficient set from among a set of random alternatives. This procedure, called stochastic efficiency with respect to a function (SERF), is claimed to make the same assumption concerning the risk aversion measures as does stochastic dominance with respect to a function (SDRF). This is claim is incorrect. SERF imposes an additional requirement on the risk aversion measures of the decision makers. Both procedures assume a lower and an upper bound on risk aversion, but SERF also assumes that all risk aversion measures are of the same functional form as these lower and upper bound functions. This additional strong requirement on risk preferences implies that the efficient set identified under SERF is usually smaller than that identified using SDRF.

**Key words:** stochastic dominance, stochastic dominance analysis with respect to a function, stochastic efficiency with respect to a function.

A recent paper and book, Hardaker *et al.* (2004a,b), respectively, describe the procedure for determining an efficient set from among a set of random alternatives described by probability distribution functions. This procedure is referred to as stochastic efficiency with respect to a function (SERF), and is suggested to be 'a method of stochastic dominance analysis with respect to a function' (SDRF). These authors claim that the SERF procedure makes the same assumption concerning the risk aversion measures of the decision makers as does SDRF. The purpose of this short note is to indicate that this is not correct. SERF imposes a requirement on the risk aversion measures of the decision makers in addition to the assumption imposed by SDRF, and this additional requirement is a strong one. Both procedures assume a lower and an upper bound on the risk aversion measure of the decision maker, but SERF also assumes that the decision maker's risk aversion measure is of the same functional form as those lower and upper bounds. If the lower and upper bounds are constants, for instance, SERF assumes that all decision makers are constant absolute risk averse (CARA). Imposing this strong additional restriction on risk aversion leads to smaller efficient sets. A short

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example is provided to illustrate the difference between the two procedures. In addition, the relationship between SERF and McCarl's (1988) Riskroot procedure is briefly discussed.

Stochastic dominance with respect to a function, as presented in Meyer (1977), assumes that the absolute risk aversion measures of the decision makers under consideration lie between arbitrary lower and upper bounds denoted as $r_L(w)$ and $r_U(w)$, respectively. These lower and upper bound functions can be any function of $w$, although in practice these bounds are often assumed to be constants. No other assumption on risk aversion is made. Thus, the risk aversion measures considered are all measures $r_a(w)$ satisfying $r_L(w) \leq r_a(w) \leq r_U(w)$ for all values of $w$. The SDRF procedure eliminates inefficient alternatives by determining the risk aversion measure $r_a^*(w)$ that lies between the lower and upper bounds, and which minimizes the difference in expected utility from alternatives F and G, $EU_F - EU_G$, and does this for all pairs of alternatives. When the value of $(EU_F - EU_G)$ is non-negative for this $r_a^*(w)$, then F is preferred or indifferent to G by all decision makers, and G can be eliminated from the set of alternatives. When the value for $(EU_F - EU_G)$ is negative for $r_a^*(w)$, then the decision maker with risk aversion measure $r_a^*(w)$, and perhaps other decision makers as well, prefer G to F and alternative G is not eliminated.

Recall that the minimizing risk aversion measure $r_a^*(w)$ is found as a bang-bang solution in an optimal control formulation of the decision problem. Thus, the $r_a^*(w)$ that minimizes the value of $EU_F - EU_G$ consists of alternating segments of the lower and upper bound functions $r_L(w)$ and $r_U(w)$. This $r_a^*(w)$ satisfies $r_L(w) \leq r_a(\mathrm{w}) \leq r_U(w)$ for all values for $w$, but is not of the same functional form as those bounds. The earliest FORTRAN computer programs (The Fortran Company, Tucson, AZ, USA) used to implement SDRF made the assumption that the lower and upper bound risk aversion functions are each a constant; that is, $r_L(w) = a$ and $r_U(w) = b$. This assumption does not imply, however, that $r_a^*(w)$ is a constant. Instead, $r_a^*(w)$ takes on the value '*a*' for some values of $w$, and the value '*b*' for the remaining values of $w$. Under SDRF, no assumption of constant absolute risk aversion is made even when the absolute risk aversion measure is required to lie between two constants.

The SERF procedure presented in this journal by Hardaker *et al*. (2004a), and further illustrated in a book by Hardaker *et al*. (2004), also identifies an efficient set for decision makers whose risk aversion measures lie between a lower and upper bound. For the case where these lower and upper bounds on absolute risk aversion are constants, SERF determines an efficient set by first computing the certainty equivalent (CE) of the various random alternatives for very closely spaced values for the constant absolute risk aversion measures that fall between these bounds, that is, the CE is computed for all CARA decision makers whose constant risk aversion measure falls in the interval $[a, b]$. SERF then defines as efficient only those alternatives whose

CE is the largest for at least one constant magnitude of absolute risk aversion in the interval [*a*, *b*].

The SERF procedure finds efficient alternatives rather than eliminating those alternatives that are inefficient. Thus, SERF has the advantage of not requiring a pair-wise comparison between all pairs of elements in the set of alternatives. While this is a desirable feature, it comes at the cost of requiring that each of the risk aversion measures being considered is constant, that is, assuming all decision makers are CARA when the bounds are a constant. Hardaker *et al.* (2004a) also indicate that the SERF procedure can be implemented when the lower and upper bounds on relative rather than absolute risk aversion are assumed to be constant. With this assumption SERF requires that all relative risk aversion measures lies between the specified constant lower and upper bounds, and that the relative risk aversion measures of all decision makers are also constant (constant relative risk aversion, CRRA).

In summary, SERF assumes a lower and upper bound on risk aversion of some functional form, CARA and CRRA are explicitly discussed, and then also assumes that all risk aversion measures that lie between these bounds are of the same form as the lower and upper bounds themselves. This latter assumption is not part of the SDRF procedure or any of its implementations, and is a strong requirement and leads to smaller efficient sets.

To illustrate how these two procedures differ, and to verify that SERF and SDRF can lead to different efficient sets for even the simplest of cases, the following example with just two elements in the choice set is provided. In this example, SERF, which makes the stronger assumption concerning risk preferences, leads to an efficient set with just one element, while the SDRF efficient set contains two elements. Because there is only one pair of alternatives, the pair-wise nature of the SDRF procedure does not impact the efficient set determination.

Consider a choice set with two alternatives. The first alterative, denoted X, gives an outcome of 1 with probability 1/2, and outcomes of 2 or 4, each with probability 1/4. Alternative Y gives an outcome of 0 or 2, each with probability 1/4, and an outcome of 3 with probability 1/2. These two alternatives have the same mean value and variance. The set of decision makers to be considered are those whose absolute risk aversion measures $r_{\mathrm{a}}(w)$ satisfy $0.1 \leq r_{\mathrm{a}}(w) \leq 4$.

By direct computation, one can verify that alternative X is preferred to alternative Y by all decision makers whose utility function $u(w)$ takes the form $u(w) = -\mathrm{e}^{-cw}$ with values for $c$ in the interval [0.1, 4]. This set of agents display constant absolute risk aversion greater than or equal to 0.1 and less than or equal to 4. In fact, X is preferred to Y for all CARA utility functions no matter what the risk aversion level. As preference for X over Y is unanimous for this group, the CE for X lies above that for Y for all these CARA utility functions and risk aversion levels. Thus, the SERF procedure yields X as the only efficient element in this two-element choice set.

Now, consider a utility function $u(w)$ whose risk aversion measure lies between 0.1 and 4, but whose risk aversion measure is not required to be a constant. Specifically consider $u(w)$ such that $u(w) = -e^{-0.1w}$ for $w < 2.83$ and $u(w) = -e^{-0.283} + e^{-11.32} - e^{-4w}$ for $w \geq 2.83$. This utility function is constructed to be continuous, and to have risk aversion measure $r_a(w) = 0.1$ for $w < 2.83$ and $r_a(w) = 4$ for $w > 2.83$. This risk aversion measure is the $r_a^*(w)$ which minimizes Eu(X) − Eu(Y), and was identified using the algorithm suggested in Meyer (1977). As indicated in the earlier discussion, this minimizing risk aversion measure lies between the specified lower and upper bounds of 0.1 and 4, respectively, and is equal to the lower bound, 0.1, for some values 0f $w$, and equal to the upper bound, 4, for the remaining values of $w$. Direct computation verifies that for this decision maker, alternative Y is strictly preferred to alternative X. Because many utility functions, including all CARA utility functions satisfying these upper and lower bounds on risk aversion, prefer X over Y, this verifies that both X and Y are efficient under the SDRF assumptions on risk aversion.

The example shows that for all utility functions whose absolute risk aversion measure lies between 0.1 and 4, the efficient set of these two alternatives contains both X and Y. That is, some decision maker would choose X over Y, and others would choose Y over X. If all decision maker are also assumed to have risk aversion measures which take the same form as the upper and lower bound function, that is, they are CARA, then the efficient set is reduced to just alternative X. SERF leads to a smaller efficient set because, in addition to lower and upper bounds on risk aversion, SERF assumes that all decision makers are CARA.

To understand the example further, notice that alternatives X and Y are constructed to have the same mean value, and to also have the same variance. Alternative X, however, has more of its risk/variance to the right of the mean, while alternative Y has more of its risk/variance to the left of the mean. This difference between the two alternatives is what leads to the finding that X is preferred to Y by all decision makers who are CARA. At the same time, the risk aversion function, $r_a^*(w)$ that minimizes [EU(X) - EU(Y)] has the least possible aversion to risk to the left of 2.83, and the largest possible aversion to risk to the right of 2.83. As a consequence, for this risk aversion function, Y is preferred to X because most of the risk associated with Y is in the region where the decision maker is least sensitive to risk. Thus, a reversal of the ranking under CARA is obtained. Using the CE for CARA preferences when determining efficient sets, as SERF does, fails to include in the efficient set those alternatives that would be selected by decision makers whose risk aversion functions are not constant, but do lie between the specified bounds.

It is also the case that to construct this example, the cumulative distribution functions (CDF) for X and Y must cross one another at least two times, and not just once. CDFs that do not cross, or cross only one time are referred to as simply related by Hammond (1974), who shows how the assumption of constant absolute risk aversion can be profitably used to determine efficient sets. This point was recognized and exploited by McCarl, who explicitly assumes that all decision makers under consideration have absolute risk aversion measures that are constant (McCarl 1988, p;.25). Hardaker *et al*. (2004a) noted several times that McCarl's Riskroot procedure gives the same efficient set as does SERF. Indeed, this is the case and follows from the fact that constant absolute risk aversion is assumed in both procedures.

To summarize, SERF and SDRF each assume a lower and upper bound on risk aversion. SERF also assumes that all risk aversion measures are of the same form as these lower and upper bound functions. When either procedure is implemented, the assumption made most often is that these lower and upper bounds are constants. For this case, SERF assumes all decision makers are CARA, while SDRF makes no such assumption. As a consequence of this additional rather strong requirement on risk preferences, the efficient set identified under SERF is typically smaller than that identified using SDRF.

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