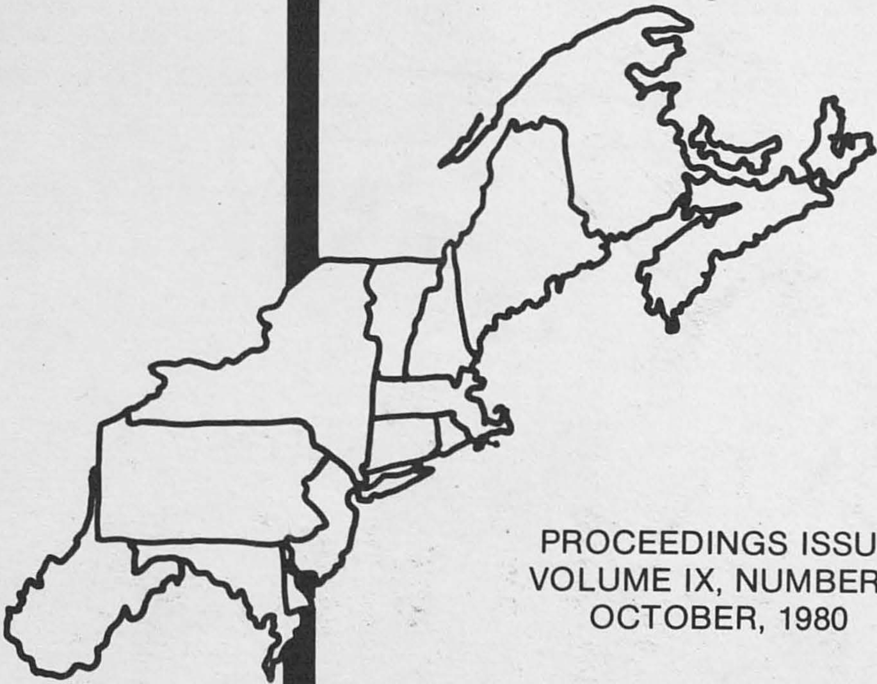
JOURNAL OF THE

Northeastern Agricultural Economics Council



PROCEEDINGS ISSUE
VOLUME IX, NUMBER 2
OCTOBER, 1980

THE FRONTIER PRODUCTION FUNCTION: A TOOL FOR IMPROVED DECISION MAKING

Richard A. King

INTRODUCTION

A frontier production function may be thought of as a "best practice" production function (Førsund and Jansen) or a function that expresses the maximum product obtainable from various combinations of factors given the existing state of technical knowledge. It is the theoretical counterpart to farm enterprise budgets or processing plant budgets derived by economic engineering methods to describe the best possible production processes.

Professor Leontief has suggested that a production function be visualized as the shape of a sheet thrown over an array of hat pins whose locations identify particular input combinations and whose heights represent the levels of output which result. Given data for a sample of firms, it is understandable that not all pins will touch the sheet. If we remove these "short" pins, the result will be a frontier production function. Most econometric procedures use both short and long pins to produce an *average* production function rather than a *frontier* production function which would represent the best practice input-output relationship.

An *efficient unit isoquant* can be used to represent a frontier production function. A unit isoquant is simply the relationship between one or more inputs which are measured in terms of a ratio to the level of output. Variables are of the form X_i divided by Y rather than expressing Y as a function of the X_i's. It is my intent to convince you that the unit isoquant representation of a frontier production function can be a very useful tool for teachers, researchers and extension folk.

THE UNIT ISOQUANT

The notion of an efficient unit isoquant (EUI) developed by Farrell in 1957 was introduced to the profession at the Western Farm Economics Association meeting in Los Angeles in 1966 (Boles, Bressler, and Seitz) and at the Southern Agricultural Economics Association in 1967. Although it has been used sporadically since then, there is much to be gained by more general use. In my opinion it stands alongside supply and demand curves as an analytical device.

By transforming absolute levels of input to the form of input per unit of output, it is possible to examine the substitutability between any pair of inputs and to demonstrate the impact of changes in relative input prices on cost-minimizing factor combinations. The unit isoquant offers a way of comparing the performance of individual farms or marketing firms by the use of a simple scatter diagram. Used in this way, it can be a valuable addition to the kit of tools carried by our extension folks.

The unit isoquant is also convenient for comparing the shape of alternative functional forms that might be estimated, including the effects of size of unit as well as factor-factor tradeoffs. A unit isoquant completely captures all the information provided by a Cobb-Douglas function, which assumes no scale effects.

There is a sense of panic that comes over some students when asked to think about a production function that has more than a single input. Computer programs are available for plotting three dimensional diagrams for the student, but there is something to be said for providing a comprehensible graphic technique that the student can readily master to complement the efficient, but often mysterious computer package.

As an analytical tool there is much to be said for the EUI. Technical change can be seen as a drift of the frontier toward the origin. Using statistical methods to estimate an average function from a scatter of observations, it is not possible to distinguish a shift that has been brought about through the exit of less efficient firms from the industry from a shift that comes from a general movement toward the origin of many firms in an industry. It is possible to contrast the factor combinations that characterize recent entrants into an industry with the combinations of older members of the industry. In short, the efficient unit isoquant has a variety of uses that have yet to be fully exploited.

FARRELL'S DEFINITION OF FIRM EFFICIENCY

How does one measure the relative efficiency of different firms in an industry? One answer is to construct simple input-output ratios such as labor used per unit of output or capital investment per unit of output. The difficulty with simple ratios of this type is that while a firm may rank high in efficiency when measured in terms of output per unit of labor, it may do this only at the cost of a large amount of capital per unit of output. That is to say, the firm with a low labor/output ratio may have a high capital/output ratio and *vice versa.* It is clear that some method is needed by which all of the important inputs can be considered simultaneously.

Figure 1 illustrates the two-input, single output case and is a reproduction of Figure 1 in the Farrell paper. The two axes represent the rate of use of each input per unit of output. The curve SS′ is to be regarded as the efficient unit isoquant. This curve represents the smallest quantity of factor 1 which can be used to produce one unit of output as the amount of factor 2 used is varied. All points on this line and those more distant from the origin are attainable while all points between the line SS′ and the origin are not attainable.

Now consider a firm represented by point P. We draw line OP from the origin to that observation. This line intersects the efficient unit isoquant at point Q. The length QP then is a measure of the excess use of the two factors relative to what is technically feasible, represented by the length OQ. We measure *technical efficiency* as the ratio of the length OQ to the length OP. Thus, all points on the efficient unit isoquant are 100 percent technically efficient and all points lying above the isoquant are less than 100 percent efficient.

Let the relative prices of factor 1 and factor 2 be represented by the slope of line AA′ which is tangent to the efficient unit isoquant at point Q′. It is clear that while point Q lies on the efficient unit isoquant, the resources required at this point are more costly than the resources which would be required at point Q′. (This is true because any factor line parallel to the line AA′ but farther from the origin represents a larger outlay for the factors F_1 and F_2.) The length RQ is a measure of the price inefficiency associated with the selection of the technically efficient, but more costly, point Q as compared with the minimum outlay point Q′. We construct the index of *price efficiency* by forming the ratio OR/OQ.

Richard A. King is M. G. Mann Professor, Department of Economics and Business, North Carolina State University, and President, American Agricultural Economics Association.

We may now combine these two indexes to obtain a measure of *economic efficiency*. This is the ratio OR/OP. This ratio turns out to be equivalent to the product of technical efficiency and price efficiency,

$$\frac{OQ}{OP} \times \frac{OR}{OQ}$$

In summary then, given the efficient unit isoquant, the relative prices of the factors, and any observed position of a firm either on that isoquant or above and to the right of the isoquant, it is possible to form an index of technical efficiency, an index of price efficiency and the product of the two, an index of economic efficiency for that firm.

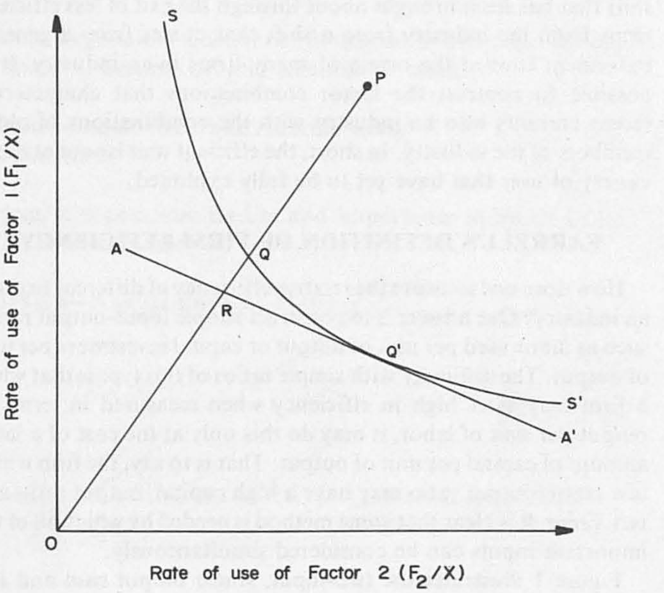


FIGURE 1.
The efficient unit isoquant (Farrell [1957], p. 254)

ESTIMATING THE EFFICIENT UNIT ISOQUANT

We turn now to the problem of empirically estimating an efficient unit isoquant. Suppose that for our two-input, single-output world we have data on the quantity of each input used by each firm and the quantity of output derived from those inputs in a specified time period. Think of each firm (P) as being represented by a hatpin located in the X_1, X_2 plane, the height of the pin representing quantity of output. We can take the suggestion of Farrell and divide each input by output, thereby collapsing the three dimensional Y, X_1, X_2 space to a two dimensional plane in X_1/Y and X_2/Y as in Figure 2.

One procedure for describing these data points would be to estimate a regression line for the scatter of observations using least-squares or some comparable procedure. A serious weakness of doing this is simply that at best it describes the *average* of all firms rather than providing information about the *most efficient* firms.[1] It leaves in all the "short" pins.

The proposal made by Farrell is that we describe the relationship by constructing an envelope-type curve that passes through the points nearest the origin, i.e., through the observations for the most efficient plants. In Figure 2 the efficient unit isoquant is drawn through the observations for firms *a*, *b*, *c*, and *d*. Firm *a* lies furthest to the left of all observations, indicating that it uses least of factor 2. The isoquant is therefore drawn vertically from point *a*. Similarly, firm *d* uses least of factor 1 per unit of output so the isoquant is drawn horizontally to the right of this observation. Between these two points the curve consists of a series of line segments connecting the four firms *a* through *d*. Alternative schemes are proposed by Aigner *et al.* and Timmer.

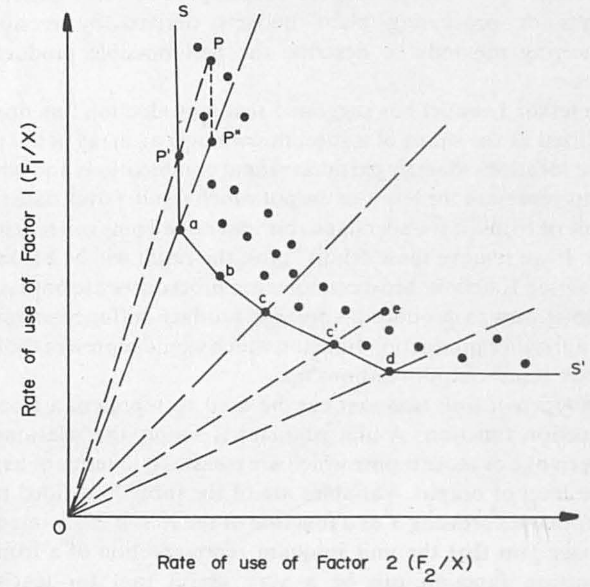


FIGURE 2.
Hypothetical example of the "pessimistic" estimate of the efficient unit isoquant—two inputs (Bressler [1967], p. 5)

You will recall that the technical efficiency of firm *e* which does not lie on the efficient unit isoquant is found by drawing a line to this point from the origin. The line *oe* crosses the efficient unit isoquant at point e′ which lies on the line segment *cd*. Technical efficiency is measured by the ratio oe′/oe as suggested earlier.

Two choices are open for estimating price efficiency. One procedure is to use the tangent representing market prices of the factors. However, if there is reason to believe that substantial differences exist among firms in the relative prices paid for the factors, it is possible to substitute a factor price line representing "own" prices for the line representing "market" prices of factors.

Bressler (see Bressler and King, p. 406) shows that economic efficiency is equivalent to the inverse ratio of average cost. The envelope curve to economic efficiency observations is strictly equivalent to the envelope curve to average cost observations. Note that both economic efficiency and average cost indexes are independent of *proportional* changes in factor prices but in general are not independent of changes in *relative* factor prices.

SCALE AND EFFICIENCY

Of particular interest to economists is the relationship between efficiency and scale of operation. Individual firm data can be sorted by size group and efficient unit isoquants constructed for each group as outlined above for the industry as a whole. It is often possible to separate the efficiency index of each firm into components associated with its performance relative to other firms

---

[1]For a contrary view see Marc Nerlove (1965, Chapter 5). Estimates of average production functions are reported in Binswanger, Nerlove (1965, 1967), Ringstad, and Zellner and Revankar.

in its size group as well as with the performance of one size group relative to other groups.

Figure 3 may be helpful in visualizing the insights provided by the Farrell approach. In Figure 3a, we have a two-input, single-output production function with levels of output Y_I, Y_{II} and Y_{III}. The slope of these isoquants represents the marginal rate of substitution. If the production function is linear homogeneous, such as the Cobb-Douglas form, a log transformation of the inputs results in linear isoquants as shown in Figure 3b. In the constant returns to scale case, illustrated in Figure 3c, there is little difficulty in empirically verifying whether or not the observations are satisfactorily represented by a single linear unit isoquant.

In the event that constant returns to scale is not a satisfactory representation, we find two alternatives in Figure 3d. For one case we label these unit isoquants from low output to high output moving away from the origin thus reflecting decreasing returns to scale ($b_1 + b_2 < 1$). In the event that increasing returns to scale exist, these unit isoquants are labeled from high output to low as we move away from the origin ($b_1 + b_2 < 1$). With cross section data and shift variables for size group we should be able to handle the matter of scale reasonably well in our two factor world, a necessary step if we are to separate scale effects from inefficiencies.

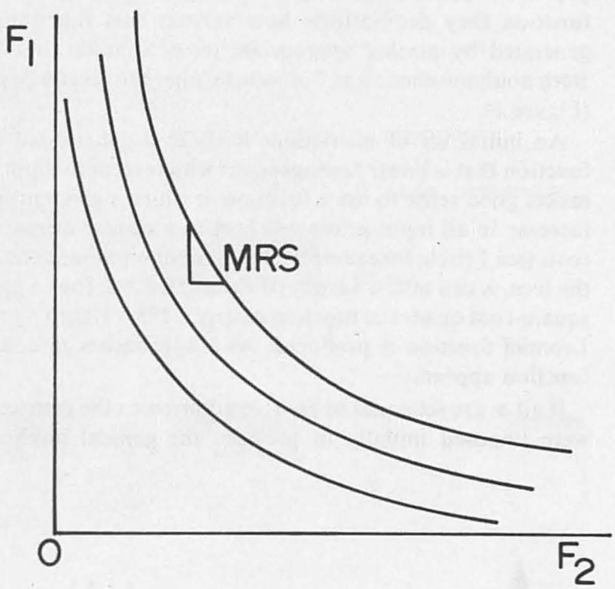


FIGURE 3a.
Production function

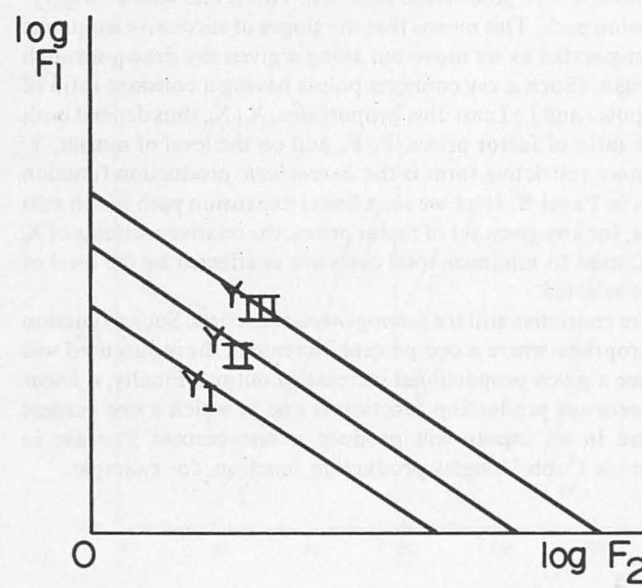


FIGURE 3b.
C-D production function, inputs in logarithms

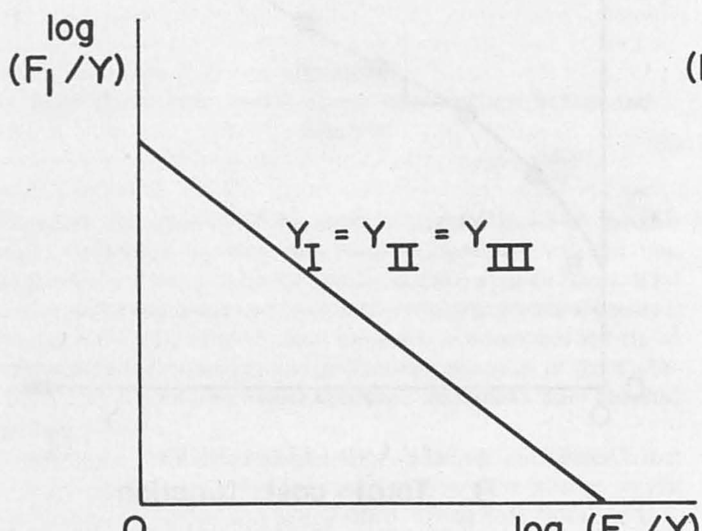


FIGURE 3c.
Unit isoquant, constant returns

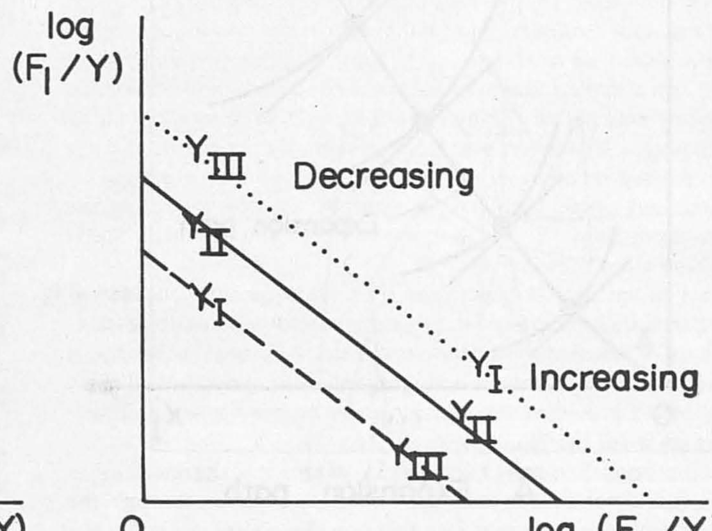


FIGURE 3d.
Unit isoquant, varying returns to scale

PRODUCTION FUNCTION—COST FUNCTION RELATIONS

Selection of an appropriate functional form can be a complicated piece of business. Following the example set in "How the Grinch Stole Christmas" we will start out slow. Panel A in Figure 4 depicts the usual textbook expansion path derived from the points of tangency between isoquants and lines representing the ratio of the prices of factors, X_i and X_j. Panel B depicts the total cost function that uses these tangency points to produce a minimum cost function BB′ which corresponds with the expansion path AA′ in Panel A.

There are several classes of production function that we need to identify. These are illustrated in Figure 5. The most general form is a *nonhomothetic* production function. This is one with a "wiggly" expansion path. This means that the slopes of successive isoquants are not parallel as we move out along a given ray drawn through the origin. (Such a ray connects points having a constant ratio of the inputs i and j.) Least cost proportions, X_i/X_j, thus depend both on the ratio of factor prices, P_i/P_j, and on the level of output, Y.

A more restrictive form is the *homothetic* production function shown in Panel B. Here we see a linear expansion path which tells us that, for any given set of factor prices, the relative amounts of X_i and X_j used to minimize total costs are unaffected by the level of output selected.

More restrictive still is a *homogeneous* function. Such a function is appropriate where a one percent increase in the inputs used will produce a given proportional increase in output. Finally, a *linear homogeneous* production function is one in which a one percent increase in all inputs will produce a one percent increase in output—a Cobb-Douglas production function, for example.

Returning to a point made earlier, a unit isoquant will completely describe a linear homogeneous production function. Thus, if we plot our observed data points in terms of inputs per unit of output, we can immediately visualize whether or not such a function is an appropriate representation of the underlying production function. As we will show later, a scatter diagram of this type can be helpful in identifying the presence of economies of scale or technical change over time.

A FAMILY OF COST FUNCTIONS

At the risk of seeming to go off the deep end for a few moments, let me try to clarify some of the terminology that one encounters in the production and cost function literature by referring to a recent paper by Berndt and Khaled. Using a generalized Box-Cox function they demonstrate how various cost functions can be generated by placing appropriate restrictions on that function. Both nonhomothetic and homothetic alternatives can be produced (Figure 6).

An initial set of restrictions' leads to a generalized Box-Cox function that is linear homogeneous with respect to input prices. It makes good sense to use a function in which a given proportional increase in all input prices will lead to a similar increase in total costs (see Frisch, for example). In this nonhomothetic cost function the term λ can take a variety of values. If $\lambda = 2$, then a generalized square-root quadratic function emerges. If $\lambda = 1$, then a generalized Leontief function is produced. As λ approaches zero, a translog function appears.

If all ϕ_i are set equal to zero, in addition to the restrictions that were imposed initially to produce the general nonhomothetic

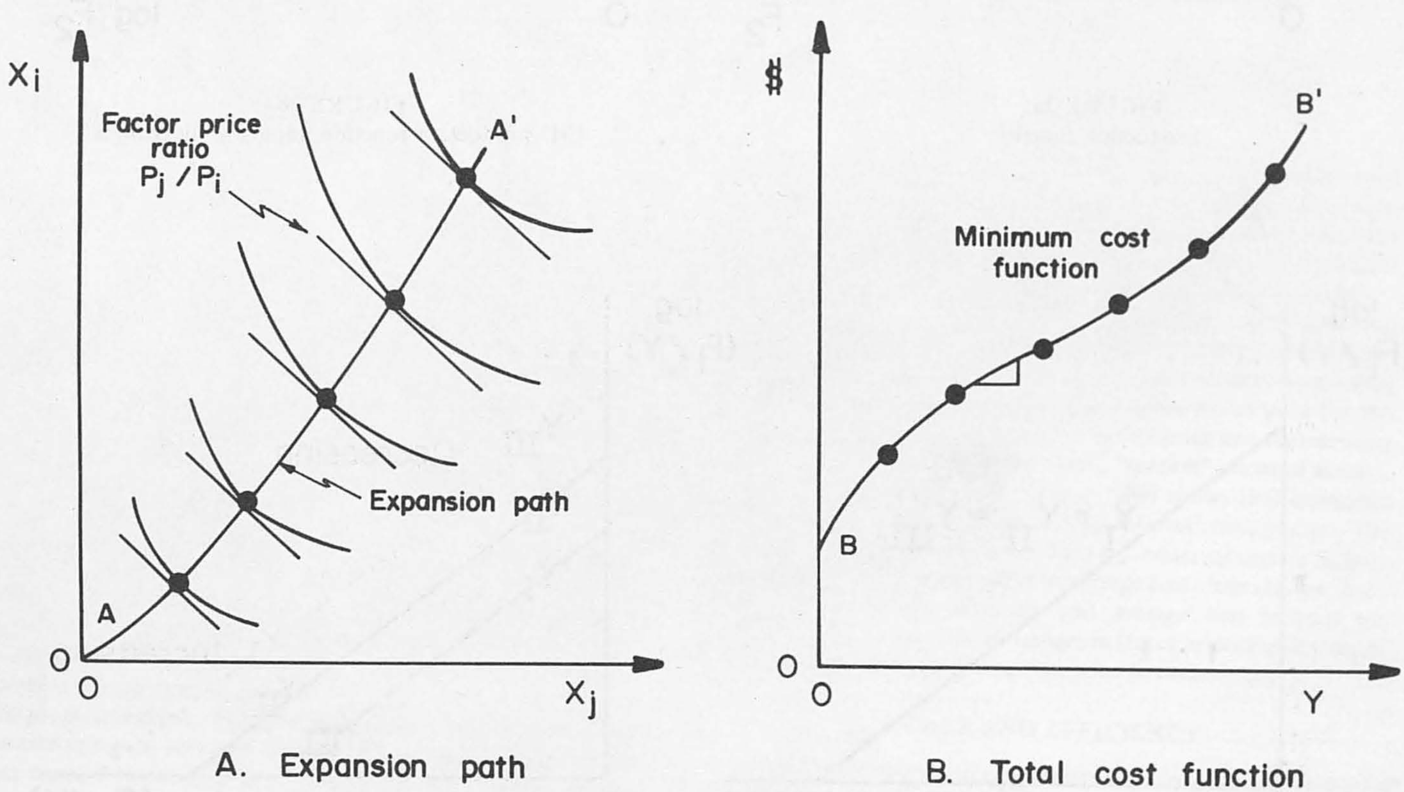


FIGURE 4.
Relation between factor price ratio, expansion path and cost function

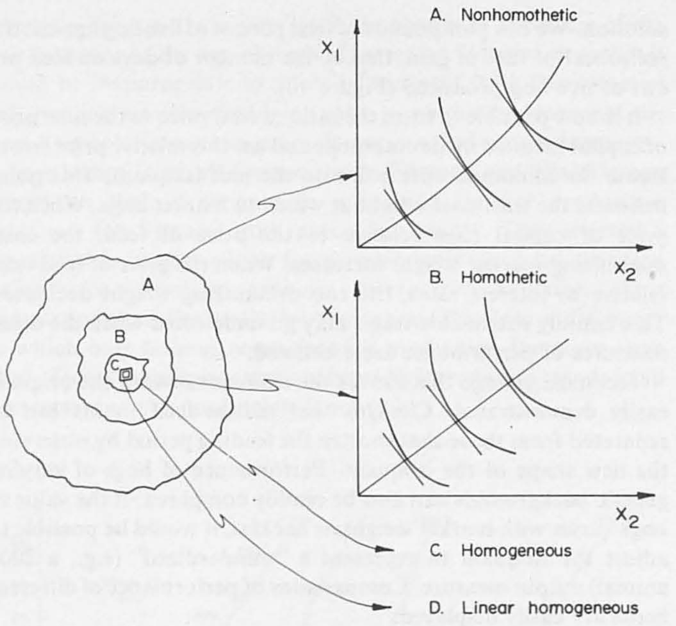


FIGURE 5.
Alternative functional forms

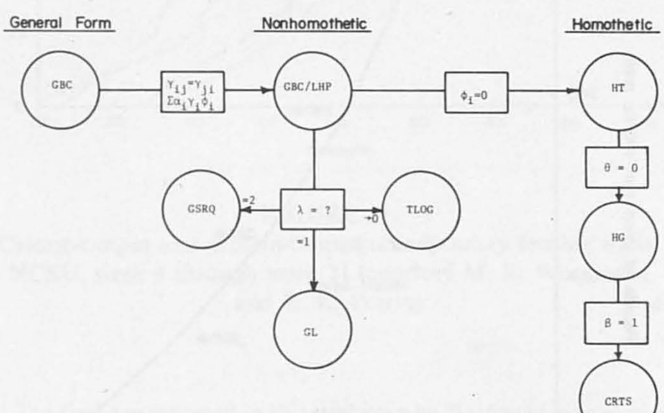


FIGURE 6.
Generalized Box-Cox family of cost functions (Berndt and Khaled)

function, the result is an homothetic function in which variable scale elasticities are allowed (nonhomogeneous). Recall that homotheticity means that the expansion path must be linear. If $\theta = 0$ as well, a homogeneous function results in which scale elasticity is constant. Finally, if $\beta = 1$ then we generate a constant returns to scale or linear homogeneous production function of the Cobb-Douglas type. (See mathematical appendix for detailed explanation.)

The beauty of the formulation used by Berndt and Khaled is that it is possible to apply statistical tests to determine whether or not the indicated restrictions are justified. Using data from the U.S. manufacturing sector for the period 1947-71 they find that the homotheticity restrictions which would lead to one of the forms in the second column of Figure 6 are decisively rejected. That is to say, the expansion path for U.S. manufacturing is not linear. However, should one elect a model in the second column they found little basis for choosing one over another among that set of homothetic functional forms.

Using unit isoquants we can see the changes measured by Berndt and Khaled more easily than by interpreting the parameters of the equations. In the following figures we find inputs per unit of output displayed for each of three pairs of inputs. In Figure 7 we can observe a drift toward the origin in the use of capital and labor per unit of output. This is a mixture of the effects of technical change and economies of scale. Similarly in Figure 8 we observe a drift to the left in energy and intermediate material inputs. In contrast to these two, however, Figure 9 shows the complementarity between capital and energy use. These inputs behave like cages and layers in egg production!

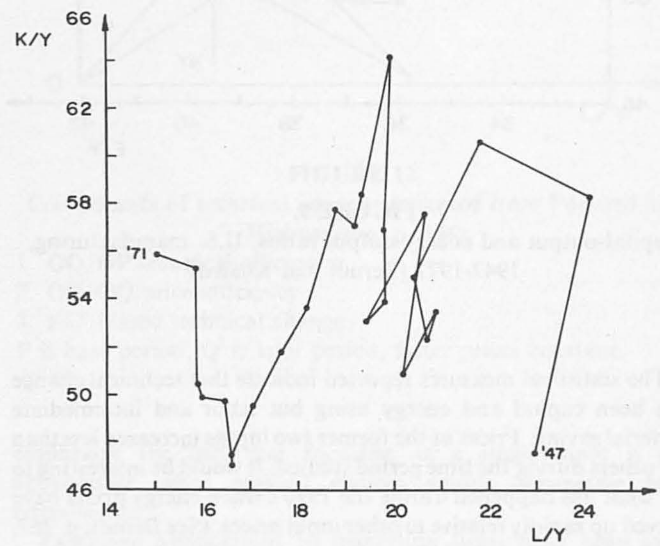


FIGURE 7.
Capital-output and labor-output ratios, U.S. manufacturing, 1947-1971 (Berndt and Khaled)

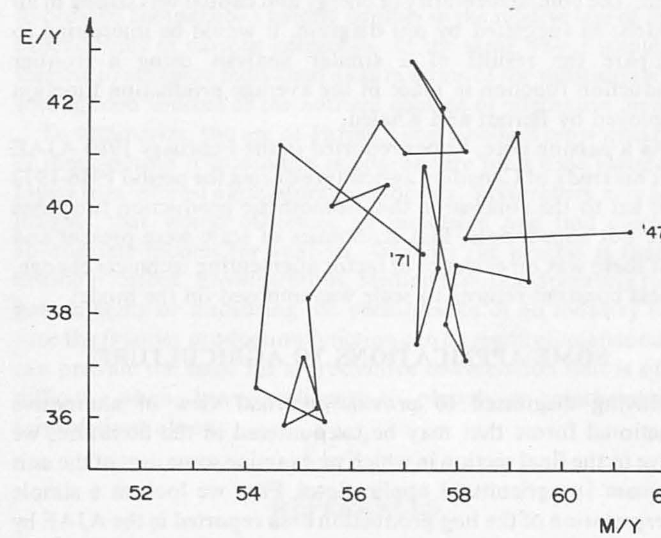


FIGURE 8.
Energy-output and intermediate materials-output ratios, U.S. manufacturing, 1947-1971 (Berndt and Khaled)

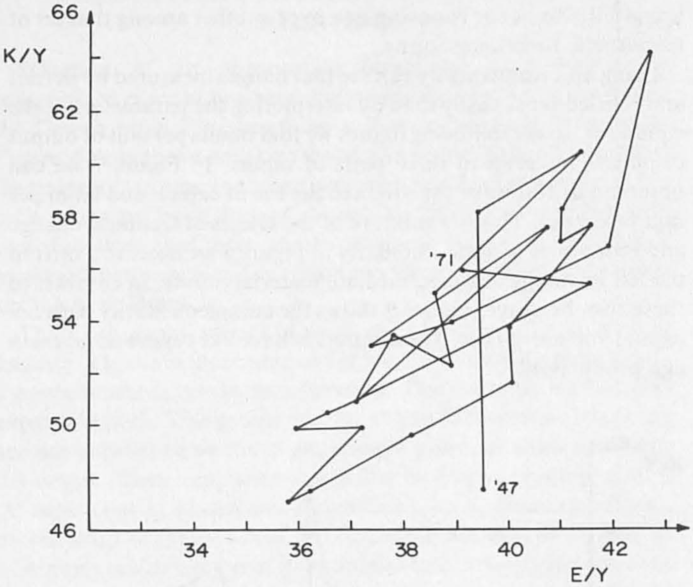


FIGURE 9.
Capital-output and energy-output ratios, U.S. manufacturing, 1947-1977 (Berndt and Khaled)

The statistical measures reported indicate that technical change has been capital and energy using but labor and intermediate material saving. Prices of the former two inputs increased less than the others during the time period studied. It would be interesting to see what has happened during the 1970's when energy prices have moved up rapidly relative to other input prices. (See Berndt, p. 267, for price and quantity indexes.)

Productivity gains were largely explained by the exploitation of economies of scale while little disembodied technical change was measured. Homotheticity, homogeneity, and constant returns to scale are all rejected. Neutrality of technical change is rejected less decisively. The generalized square-root and translog forms of the Box-Cox model were rejected, but not the generalized Leontief form. The complementarity of energy and capital was strong in all models, as suggested by our diagram. It would be interesting to compare the results of a similar analysis using a frontier production function in place of the average production function employed by Berndt and Khaled.

As a passing note, Lopez reported in the February 1980 AJAE that his study of Canadian agriculture during the period 1946-1977 also led to the conclusion that homothetic production functions were not appropriate, that economies of scale were present and that there was no evidence of factor augmenting technical change, unless constant returns to scale was imposed on the model.

SOME APPLICATIONS TO AGRICULTURE

Having disgressed to provide a broad view of alternative functional forms that may be encountered in the literature, we move to the final section in which we examine some uses of the unit isoquant in agricultural applications. First we look at a simple interpretation of the hog production data reported in the AJAE by George Ladd and Craig Gibson.

The usual measures of performance of hog and cattle feeding trials are (1) pounds of feed per pound of gain and (2) rate of gain in pounds per day. Have you ever attempted to plot these two measures together? The unit isoquant format offers an easy solution. We can plot pounds of feed per cwt of live hog against the reciprocal of rate of gain, that is, the number of days on feed per cwt of live hog produced (Figure 10).

It is now possible to form the ratio of feed price to the time price of capital invested in the enterprise and use this relative price line to locate the minimum cost point on the unit isoquant. This point indicates the least-cost weight at which to market hogs. When the price of capital rises relative to the price of feed, the cost-minimizing market weight increases. When the price of feed rises relative to interest rates, the cost-minimizing weight decreases. This entirely reasonable result may go undetected when the usual measures of performance are examined.

Technical change that moves the isoquant toward the origin is easily demonstrated. Changes that reduce feed inputs can be separated from those that shorten the feeding period by observing the new shape of the isoquant. Performance of hogs of varying genetic backgrounds can also be readily compared. If the value of hogs varies with market weight or backfat, it would be possible to adjust the isoquant to represent a "standardized" (e.g., a 200# animal) output measure. Comparisons of performance of different herds are easily displayed.

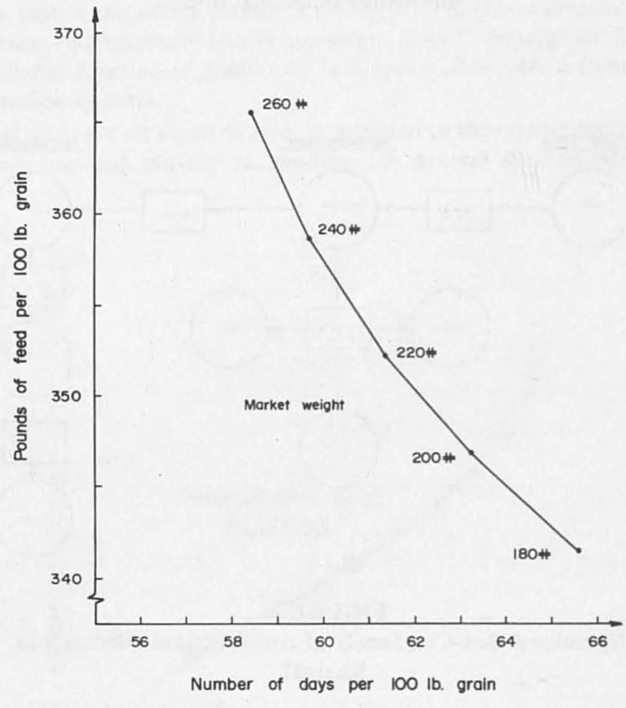


FIGURE 10.
Feed-output and time-output ratios for hogs of varying market weights (Ladd and Gibson)

A more complicated example is encountered in the interpretation of a turkey feeding experiment conducted at North Carolina State University. Again using the unit isoquant format we find four feeding systems compared in Figure 11. Each observation refers to aggregate feed intake and aggregate weight of the birds as they grow older. First of all, it is immediately apparent that larger quantities of all inputs are needed per pound of live turkey as they grow to heavier weights. Unless heavier birds bring a higher price per pound, there would be no incentive to feed turkeys past the earliest possible slaughter weight. A second observation, not as dramatic but equally of interest, is the change in curvature of the unit isoquant as the age of the bird increases. It seems clear that it would be inappropriate to select a functional form (homothetic) that requires a linear expansion path, given this evidence that the least-cost calorie/protein ratio changes as bird weight increases.

In a recent issue of *Econometrica*, Førsund and Hjalmarsson report on their experience in estimating frontier production functions for Swedish milk processing plants. They selected a functional form that allows for variable scale elasticity, that is, a homothetic, nonhomogeneous production function. This contrasts with earlier studies by Aigner and Chu, Seitz and Timmer in which only homogeneous frontier production functions were used. Technical progress was analyzed by introducing trends in all parameters of the production function.

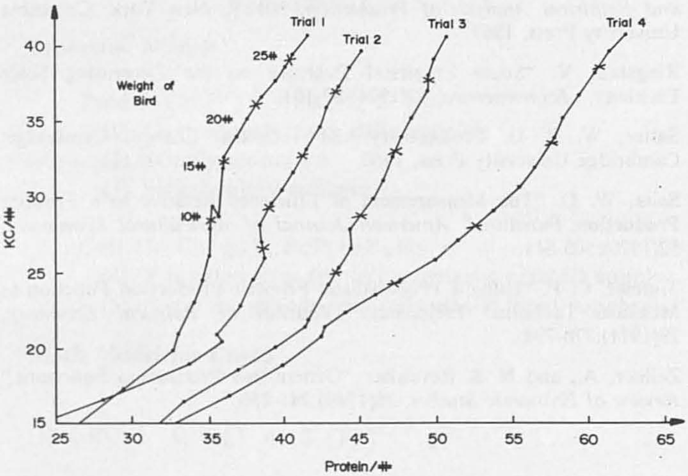


FIGURE 11.

Calorie-output and protein-output ratios, turkey feeding trials, NCSU, week 4 through week 21 (courtesy M. K. Wolgenant and R. K. Perrin)

The findings reported in this study can be illustrated as in Figure 12 using the approach suggested by Salter. Factor use in the base period is represented by the point P and in the later period by the point Q′. Improvement in technical efficiency (neutral) is measured by the ratio OQ/OP and improvement in price efficiency by the ratio OR/OQ. Biased technical change favoring a larger labor/capital ratio is suggested by the length RQ′. Estimated trends in the scale elasticity function suggest a doubling of the optimal size of plant during the study period from 1964 to 1973. The authors provide dramatic diagrams showing upward shifts in the production function and in the scale elasticity function and a rapid drift toward the origin of the efficiency frontier. Input coefficients in 1973 were roughly 40 percent of those in 1964, reflecting rapid technical advance and great structural change in the milk processing industry of Sweden.

What about Extension uses of the frontier production function? It is possible that this would be an effective form in which to summarize the experiences of dairy or poultry farms as gathered annually in farm record projects. Feed/labor input combinations or alternative roughage/concentrate systems could be easily displayed in unit isoquant form, identifying individual farms in a fashion that would make comparisons with other farms more understandable than the usual tabular form. It would also serve to

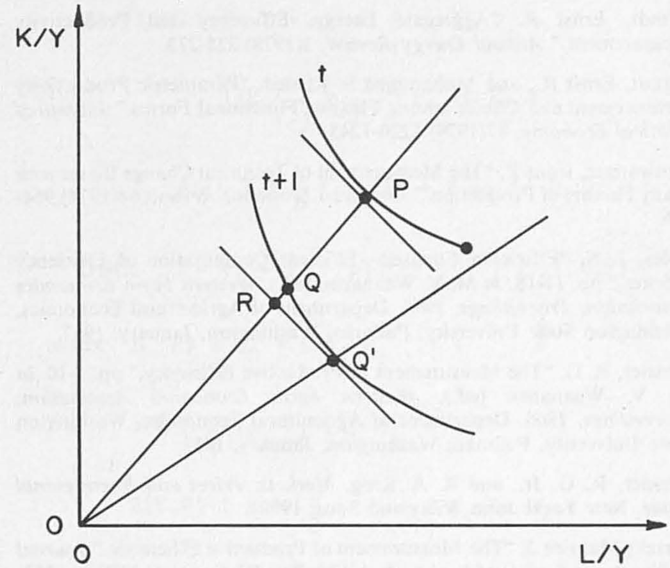


FIGURE 12.

Components of technical advance (adapted from Førsund and Hjalmarsson, p. 894)

1. OQ/OP technical efficiency
2. OR/OQ price efficiency
3. RQ′ biased technical change

P is base period, Q′ is later period, faster prices constant.

emphasize the fact that focusing on a single input is not appropriate when making choices among alternative farm adjustments.

Extension applications to marketing firms have been quite successful in some work done at N.C. State (Mathia and Hammond). While the estimation of the underlying production or cost function may be useful as a research activity, it is likely that businessmen are able to relate more easily to graphic demonstrations of the variability among similar firms in a given year or to changes in their own operations over time.

Household accounts can be analyzed using this general format to provide comparisons among families in the relative size of their expenditures in various categories. In the same way it would be feasible to compare individual diets in terms of the substitutability among food sources or the nutrient content of alternative menus.

To summarize, the use of Farrell's unit isoquant representation of input-output relationships is appropriate for a wide variety of topics that interest agricultural economists. Researchers may gain insights that would otherwise go unnoticed and find clues to appropriate models for analysis. Whether the process is that of raising livestock, growing crops, feeding families, operating town governments or measuring the performance of an industry over time the frontier production function can be readily understood. It can provide the basis for a productive conversation that is often difficult when heavy reliance is placed on mathematical formulations alone.

REFERENCES

Aigner, D. J., and S. F. Chu. "On Estimating the Industry Production Function." *American Economic Review*, 58(1968):226-239.

Aigner, D. J., C. A. K. Lovell, and P. Schmidt. "Formulation and Estimation of Stochastic Frontier Production Function Models." *Journal of Econometrica*, 6(1977):21-37.

Berndt, Ernst R. "Aggregate Energy, Efficiency and Productivity Measurement." *Annual Energy Review*, 3(1978):225-273.

Berndt, Ernst R., and Mohammed S. Khaled. "Parametric Productivity Measurement and Choice among Flexible Functional Forms." *Journal of Political Economy*, 87(1979):1220-1245.

Binswanger, Hans P. "The Measurement of Technical Change Biases with Many Factors of Production." *American Economic Review*, 64(1974):964-976.

Boles, J. N. "Efficiency Squared—Efficient Computation of Efficiency Indexes," pp. 11-18. *In* M. V. Waananen (ed.), *Western Farm Economics Association, Proceedings, 1966*, Department of Agricultural Economics, Washington State University, Pullman, Washington, January, 1967.

Bressler, R. G. "The Measurement of Productive Efficiency," pp. 1-10. *In* M. V. Waananen (ed.), *Western Farm Economics Association, Proceedings, 1966*, Department of Agricultural Economics, Washington State University, Pullman, Washington, January, 1967.

Bressler, R. G. Jr., and R. A. King. *Markets, Prices and Interregional Trade*. New York: John Wiley and Sons, 1970.

Farrell, Maurice J. "The Measurement of Productive Efficiency." *Journal of The Royal Statistical Society*, Vol. 120, Part III, Series A, 1957, pp. 253-281.

Førsund, Finn R., and Lennart Hjalmarsson. "Frontier Production Functions and Technical Progress: A Study of General Milk Processing in Swedish Diary Plants." *Econometrica*, 47(1979):883-900.

Førsund, F. R., and E. S. Jansen. "On Estimating Average and Best Practice Homothetic Production Functions via Cost Functions." *International Economic Review*, 18(1977):463-476.

Frisch, R. *Theory of Production*. Dordrecht: D. Reidel Publishing Co., 1965.

Ladd, G. W., and C. Gibson. "Microeconomics of Technical Change: What's a Better Animal Worth?" *American Journal of Agricultural Economics*, 60(1978):236-240.

Lopez, R. E. "The Structure of Production and the Derived Demand for Inputs in Canadian Agriculture." *American Journal of Agricultural Economics* 62(1980):38-45.

Mathia, Gene A., and Leigh H. Hammond. "Measuring Economic Efficiency: An Application to Apple Marketing Facilities," pp. 195-212. *In* Proceedings, Marketing Section, Association of Southern Agricultural Workers, January 30—February 1, 1967. Department of Agricultural Economics, Texas Technological College, Lubbock, Texas.

Nerlove, Marc. "Returns to Scale in Electricity Supply. *In* C. Christ, *et al.* (eds.), *Measurement in Economics: Studies in Mathematical Economics and Econometrics in Memory of Yehuda Grünfeld*. Stanford: Stanford University Press, 1963.

__________. *Estimation and Identification of Cobb-Douglas Production Functions*. Chicago: Rand McNally and Company, 1965, Chapter 5.

__________. "Recent Empirical Studies of the CES and Related Production Functions," pp. 55-122. *In* Murray Brown (ed.) *The Theory and Empirical Analysis of Production*, NBER, New York: Columbia University Press, 1967.

Ringstad, V. "Some Empirical Evidence on the Decreasing Scale Elasticity." *Econometrica*, 42(1974):87-101.

Salter, W. E. G. Productivity and Technical Change. Cambridge: Cambridge University Press, 1960.

Seitz, W. D. "The Measurement of Efficiency Relative to a Frontier Production Function." *American Journal of Agricultural Economics*, 52(1970):505-511.

Timmer, C. P. "Using a Probabalistic Frontier Production Function to Measure Technical Efficiency." *Journal of Political Economy*, 79(1971):776-794.

Zellner, A., and N. S. Revankar. "Generalized Production Functions." *Review of Economic Studies*, 36(1969):241-250.

MATHEMATICAL APPENDIX

DERIVATION OF GENERALIZED BOX-COX FAMILY OF COST FUNCTIONS[1]

1. $$GBC = [1 + \lambda [\alpha_0 + \sum_i \alpha_i P_i(\lambda) + 1/2 \sum\sum_{ij} \gamma_{ij} P_i(\lambda) P_j(\lambda)]]^{1/2} Y^{\beta(Y,P)}$$

where $\beta(Y,P) \equiv \beta + \frac{\Theta}{2} \ln Y + \Sigma_i \phi_i \ln P_i$

and $P_i(\lambda) \equiv (P_i^{\lambda/2} - 1)/\lambda/2$

If $\gamma_{ij} = \gamma_{ji}$

$\sum_i \alpha_i = 1 + \lambda\alpha_0$

$\sum_i \gamma_{ij} = \lambda/2$

and $\sum_i \phi_i = 0$

Then (1) becomes (2)

2. GBC = Linear Homogeneous in Factor Prices

$$NHT = [2/\lambda \sum_i \sum_j \gamma_{ij} P_i^{\lambda/2} P_j^{\lambda/2}]^{1/2} Y^{\beta(Y,P)}$$

If $\lambda = 2$

Then (2) becomes (3)

3. Generalized Square Root Quadratic

$$GSRQ = [\sum_i \sum_j \gamma_{ij} P_i P_j]^{1/2} Y^{\beta(Y,P)}$$

If $\lambda = 1$

Then (2) becomes (4)

4. Generalized Leontief

$$GL = 2 \sum_i \sum_j \gamma_{ij} P_i^{1/2} P_j^{1/2} Y^{\beta(Y,P)}$$

If $\lambda \to 0$

Then (2) becomes (5)

5. Translog

$$T\,LOG = \alpha_0 + \Sigma \alpha_i \ln P_i + 1/2 \Sigma \Sigma \gamma_{ij} \ln P_i \ln P_j + \beta \ln Y + \frac{\Theta}{2} (\ln Y)^2 + \Sigma \phi_i \ln P_i \ln Y$$

If $\phi_i = 0, i = 1, \ldots n$

Then (2) becomes (6)

6. Homothetic Nonhomogeneous

$$HT \quad [2/\lambda \sum_i \sum_j \gamma_{ij} P_i^{\lambda/2} P^{\lambda/2}]^{1/\lambda} Y^{\beta + \Theta/2 \ln Y}$$

If $\Theta = 0$

[1]Berndt and Khaled.

Then (6) becomes (7)

7. Homogeneous of Degree $1/\beta$

HG $[2/\lambda \sum_i \sum_j \gamma_{ij} P_i^{\lambda/2} P_j^{\lambda/2}]^{1/\lambda} Y^\beta$

If $\beta = 1$

Then (7) becomes (8)

8. Constant Returns to Scale

CRTS = $[2/\lambda \sum \sum \gamma_{ij} P_i^{\lambda/2} P_j^{\lambda/2}]^{1/\lambda} Y$

BERNDT-KHALED MODEL, U.S. MANUFACTURING, 1947-71

Economic Model:

Production f^n $Y = f(K, L, E, M)$

(1) Continuously twice differentiable
(2) Strictly monotone
(3) Strictly quasi-concave

Cost f^n $C = g(Y, P_K P_L P_E P_M)$

(4) Y is exogenous (perfectly inelastic product supply)
(5) P_i are exogenous (perfectly elastic input supplies)

Math Model for Cost f^n

(1) GBC: $C = [1 + \lambda G(P)]^{1/2} Y^{\beta(Y,P)}$

where $G(P) \equiv \alpha_0 + \sum_i \alpha_i P_i(\lambda) + 1/2 \sum_i \sum_j \gamma_{ij} P_i(\lambda) P_j(\lambda)$

$$\beta(Y,P) \equiv \beta + \frac{\Theta}{2} \ln Y + \varepsilon_i \phi_i \ln P_i$$

$$P_i(\lambda) \equiv \frac{P_i^{\lambda/2} - 1}{\lambda/2}$$

(2) GBC : LHP $C = [2/\lambda \sum_i \sum_j \gamma_{ij} P_i^{\lambda/2} P_j^{\lambda/2}]^{1/2} Y^{\beta(Y,P)}$

(3) GSRQ: $C = [\sum_i \sum_j \gamma_{ij} P_i P_j]^{1/2} Y^{\beta(Y,P)}$

(4) GL: $C = 2 \sum_i \sum_j \gamma_{ij} P_i^{1/2} P_j^{1/2} Y^{\beta(Y,P)}$

(5) TLOG: $\ln C = \alpha_0 + \sum_i \alpha_i \ln P_i + 1/2 \sum_i \sum_j \gamma_{ij} \ln P_i \ln P_j + \beta \ln Y + \frac{\Theta}{2} (\ln Y)^2 + \sum_i \phi_i \ln P_i \ln Y$

GBC: $C = [1 + \lambda G(P)]^{1/2} Y^{\beta(Y,P)} e^{T(t,P)}$

where $G(P) \equiv \alpha_0 + \sum_i \alpha_i P_i(\lambda) + 1/2 \sum_i \sum_j \gamma_{ij} P_i(\lambda) P_j(\lambda)$

$$\beta(Y,P) \equiv \beta + \frac{\Theta}{2} \ln Y + \sum_i \phi_i \ln P_i$$

$$T(t,P) \equiv t(\tau + \sum_i \tau_i \ln P_i)$$

$$P_i(\lambda) \equiv \frac{P_i^{\lambda/2} - 1}{\lambda/2}$$

GBC|LHP:

General form:

$$C = A^{1/\lambda} Y^B e^D$$

where $A = \frac{2}{\lambda} \sum_i \sum_j \gamma_{ij} P_i^{\lambda/2} P_j^{\lambda/2}$

$$B = \beta + \frac{\Theta}{2} \ln Y + \sum \phi_i \ln P_i$$

$$D = t(\tau + \sum_i \tau_i \ln P_i)$$

Estimated GBC|LHP function for U.S. manufacturing, 1947-71

$$A^{1/\lambda} = [3.2852 \sum \sum \gamma_{ij} P_i^{.3044} P_j^{.3044}]^{1.6426}$$

$$B = .8693 - .0045 \ln Y + \sum \phi_i \ln P_i$$

$$D = t(.0005 + \sum_i \tau_i \ln P_i)$$

X_i	γ_{ij} $(\gamma_{ij} = \gamma_{ji})$				ϕ_i	τ_i	Calculated values[1]	
	K	L	E	M			$\sum_j \gamma_{ij}$	α_i
K	-.0588	-.1503	.0401	-.0609	-.0392	.0011	-.2299	-.7553
L		.2141	.1192	-.2739	-.0360	-.0004	-.0909	-.2986
E			-.0425	-.0231	-.0292	.0006	.0937	.3078
M				.4042	.1044	-.0013	.0463	.1521
Σ	-	-	-	-	0	0	-	-.5940

$\lambda = .6088$ $\phi = -.0090$

$\beta = .8693$ $\tau = .0005$

$\alpha_0 = 2.6183$ $P_i(\lambda) = 3.2852\,(P_i^{.3044} - 1)$

[1] $\sum_j \gamma_{ij} = \frac{\lambda}{2}\alpha_i \; ; \; \sum_i \alpha_i = 1 + \lambda\,\alpha_0$