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# High-Resolution Computer Graphics: A Research and Educational Tool in Production Economics

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"High-Resolution Computer Graphics: A Research and Educational Tool in Production Economics"

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This paper investigates some applications of high-resolution, three-dimensional PC-based computer graphics to research and education in agricultural production economics. Examples illustrate issues in the identification of stages of production for the multifactor case; special cases in which second order conditions for extrema are violated; neoclassical cost functions in two-product space; and CES-type functions in two-product space.

## High-Resolution Computer Graphics: A Research and Educational Tool in Production Economics

Over the past five years, high-resolution graphics imaging has become one of the most rapidly advancing components of computer science. Recent developments include (a) color monitors capable of displaying high-resolution images, (b) the availability of low-cost computer memory for storing the high-resolution images and quickly displaying images on-screen, and (c) the development of several new software packages designed to take advantage of the newly available hardware. These developments have opened new possibilities for the use of graphics images in research and education by theoretical and applied economists.

This paper illustrates some of the new applications of high resolution computer graphics to contemporary production economics problems. The use of computer graphics imaging within agricultural economics until recently has been very limited. Over a decade ago, Debertin, Pagoulatos and Bradford outlined some possibilities for three-dimensional graphics and provided some illustrations of applications to production economics. More recently, three dimensional illustrations of production surfaces are in texts by Debertin (1986) and by Beattie and Taylor. Bay and Schoney previously had proposed that computer graphics could also be used to analyze empirical data for production economics problems. Debertin (1985) suggested that two-dimensional graphics could be used as an aid for teaching about production and cost functions to undergraduates.

Most agricultural economics researchers are now skilled in mathematics and its economic applications, and making use of graphics (however refined) may at first appear to be a leap backward into the 1950s-- a time when few agricultural economists made extensive use of mathematics as an analytical tool. This view perhaps stems from a still widely-held belief that the graphical economics of the 1950s and earlier was inferior to mathematics as a tool for expressing economic logic. In that era, however, technical drafting was employed with only limited understanding of the specific underlying mathematical functions. Modern-day computer graphics has mathematics as its core and supplements, not replaces, conventional presentations of economic theory employing the calculus or other mathematics.

Even today, mathematics as applied to production economics without graphics has its limitations. For example, the use of techniques such as partial differentiation as a tool for examining characteristics such as the curvature of a production function surface or isoquant allows the researcher to evaluate the precise characteristics of the surface or isoquant only at the point being evaluated. As a result, the researcher may miss important characteristics of the function at points which are not evaluated. Hence, computer graphics allows the researcher to "see" characteristics of the function that might be overlooked if mathematics were the only research tool.

An analogy might be drawn between an airplane pilot and a researcher in agricultural economics. It might be possible to fly the airplane (analyze a function) relying entirely on the electronic instruments (mathematics). However, even though it is quite possible to fly a modern airplane without looking out the window, all airplanes still have windows. Window observation provides the pilot with important visual cues that may be far easier to interpret than relying solely on radar and other electronic instruments. Modern computer graphics similarly provides visual cues for the analysis of functions in economics, allowing the research to "see" relationships not readily apparent from the mathematics alone.

However, the modern pilot would not give up avionics in favor of relying only on window observations. Similarly, contemporary mathematical economics provides the basis for determining which problems should be analyzed employing graphics as well as suggesting specific functional forms to be used as the basis for the graphics imaging. Mathematics and computer graphics are complementary, not competing tools. The ability to see relationships contained within mathematical functions has opened new research possibilities for us and helped us to better understand production theory. The remainer of this paper cosists of examples chosen to illustrate capabilities of the software when applied to problems in production economics.

The illustrations contained in this paper, as well as the matrix algebra and other calculations needed to obtain needed parameters and calculations to generate the data for plotting, were done using a software program called *ASYSTANT* (Macmillan Software). The program requres, at mimimum, a PC based on the 286 or 386 chip with a math co-processor, and a graphics card capable of EGA level resolution or better. This software was written primarily for engineering applications as an upgrade for work formerly done on a scientific calculator. The economic applications presented here are not intuitively obvious from the documentation, and experimentation is required on the part of the user. Suggestions for generating data sets and setting up the software for economic analysis can be obtained by contacting the authors.

## The Three Stages of Production

Beattie argued that the area enclosed by the ridge lines in the neoclassical factor-factor model may contain portions of isoquants that do not belong within stage II. Is it possible to actually observe this for a specific function with the aid of computer graphics? A third degree polynomial with parameters capable of generating a three stage production function consistent with the geometry of the neoclassical theoryis a starting point. An approach for obtaining the appropriate parameters is outlined in Debertin, 1985. The function is

$$(1)\qquad y = x_1 + 1.5x_1^2 - 0.05x_1^3 + x_2 + 1.5x_2^2 - 0.05x_2^3 + 0.4x_1x_2$$

where $y$ is an output; $x_1$ and $x_2$ are inputs.

A plot of the surface indicates a region which increases at an increasing rate, as expected (Figure 1, left panel). The isoquants in this region might be either concave or convex (Figure 1, right panel). A plot of the isoquants clearly reveals that the second order conditions for cost minimization along an expansion path for any positive pair of input prices are violated for sufficiently small but positive levels of input use. Since the pair of ridge lines connect all points of zero or infinite slope on an isoquant map, and the area enclosed by the set of ridge lines includes some isoquants that are concave to the origin.

This region of concave isoquants near the origin can be uncovered using the calculus, although not very easily. Isoquants convex to the origin would, of course, require that the signed Hessian be positive, that is, $f_{11}f_{22} - f_{12}f_{21} > 0$. However, evaluating the sign on this determinant for possible values of $x_1$ and $x_2$ over the domain of the function would require a grid of all appropriate combinations of $x_1$ and $x_2$, and calculation of the sign on the determinant for each possible combination. These calculations could be made on a personal computer, using software such as *MathCAD* (Mathsoft Inc.).

Another advantage of the computer graphics approach is that it allows the researcher to quickly vary parameters of the production function and observe the resultant impacts both on the surface of the production function and on the shape of the isoquants. For example, increasing the parameter on the squared terms in the polynomial causes the concavity of the isoquants to become more extreme for small levels of input use.

## Conditions for the Existence of Extrema (Maxima or Minima)

First and second order conditions for the existence of a minimum or a maximum can be very difficult to visualize for a function of the form $y = f(x_1, x_2)$. To understand why, first consider a case where there is only one variable input, $x_1$. A maximum exists when $f_1 = 0$ and $f_{11} < 0$, a minimum when $f_1 = 0$ and $f_{11} > 0$, and a certain kind of an inflection point when $f_1 = 0$ and $f_{11} = 0$. The saddle point is sometimes envisioned as the two-factor analog to the inflection point of the single factor case; but, as will be shown, this is incorrect. There are cases where $f_1$ and $f_2$ are both zero which generate neither a maximum, minimum nor have the appearance of a saddle (point).

Consider an instance where $f_1 = 0$, $D_1 = f_{11}$ and $D_2 = f_{11} f_{22} - f_{12} f_{21}$. Then a maximum is obtained if $D_1 < 0$ and $D_2 > 0$, and a minimum occurs if $D_1 > 0$ and $D_2 > 0$. Note, however, that a non-positive $D_2$ might occur for a number of reasons. A "standard" saddle, parallel to the $x_1$, axis occurs if $f_{11} > 0$ and $f_{22} < 0$ and the product of $f_{12} f_{21}$ is smaller in absolute value than the product of $f_{11} f_{22}$. If, however, the signs on $f_{11}$ and $f_{22}$ are reversed, then the the saddle is parallel to the $x_2$ axis. Cases in which the second order conditions for a maximum or a minimum are violated by virtue of the fact that $f_{12} f_{21}$ is larger in absolute value than $f_{11} f_{22}$ are more unique. An example of a production function where this might occur is a polynomial which has a parameter on an interaction term sufficiently large such that the product $f_{12} f_{21}$ is greater than $f_{11} f_{22}$. The saddle no longer runs parallel to one of the axes, but rather lies on a ray along a plane extending from the origin (Debertin, 1986, p. 104).

An even more complicated illustration arises when the second cross partials $f_{12}$ are themselves a function of $x_1$ or $x_2$. Consider the function $y = -x_1^2 + x_2^2 - 15\, x_1^2 x_2^2$. Figure 2 illustrates the surface of this function and the corresponding contour map. The resultant surface is unlike any we have seen illustrated in calculus or economics textbooks. While it is possible to envision a simple saddle point without the aid of computer graphics, this sort of result from a complicated functional form would be virtually impossible to envision based solely on the mathematics. Furthermore, large parameters on interaction terms of economic functions or complicated interaction terms, certainly plausible if not commonplace in empirical research involving cost or production functions, can sometimes lead to quite surprising and unexpected results.

Contrary to conventional wisdom, not all inflection points have nonzero slopes. What happens if two inflection points meet, each with a zero slope? Clearly, based on first order conditions, a critical value exists, but not a maximum nor a minimum. Figure 3 illustrates the surface and the contour lines that arise from the polynomial

(2) $y = 8.824\ x_1 - 0.8824\ x_1^2 + 0.0294\ x_1^3 + 8.824\ x_2 - 0.8824\ x_2^2 + 0.0294\ x_2^3$.

This function could not appropriately be called a saddle, for there is no resemblance to the traditional saddle point. The contour lines (Figure 3, right) are particularly curious.

## Cost Functions in Two-Product Space

The neoclassical cost function, if extended to a two-product case, takes on a form nearly the same as that depicted in Figure 3. However, the inflection point generally does not have a zero slope; because if it did, marginal cost would be zero at the point of inflection. A slight modification of the parameters of the polynomial leads to a function with a positive marginal cost at the inflection point (Figure 4). An example of this function is:

(3) $C = vx = v[13.33\ y_1 - 0.667\ y_1^2 + 0.022\ y_1^3 + 13.33\ y_2 - 0.667\ y_2^2 + 0.022\ y_2^3]$

where $C$ is total cost, $y_1$ and $y_2$ are two products, x is an input (or bundle of optimally allocated inputs from the expansion path) and $v$ is the price of the input (unit of the optimally allocated input bundle). The right panel of Figure 4 clearly illustrates the surface and the contour lines for such a function. Economic interpretation of the contour lines is possible. Each contour line is actually an isocost line in 2-product space, representing the possible combinations of $y_1$ and $y_2$ that can be produced for a specific cost outlay $C^*$. Notice that on the isocost map (right panel, Figure 4), the portion representing increasing marginal cost corresponds with the NE corner, not the SW corner, as would be the case in factor space. Notice also that in this quadrant, isocost lines are concave, not convex to the origin. This concavity is consistent with neoclassical theory. The firm would wish to produce the greatest possible quantity of $y_1$ and $y_2$ for the cost outlay represented by the isocost line. Under most product price ratios, optimization would entail producing a combination of $y_1$ and $y_2$, rather than all of one of the two products.

## CES-like Counterparts in Product Space

Another product-space application application involves the use of a CES function (Arrow, *et al.*) to represent a two-output, one-input world. It is well known that the CES function of the form

(4) $y = A(ax_1^{-\rho} + bx_2^{-\rho})^{-1/\rho}$

is capable of generating isoquants ranging from a map of right angles ($\rho \rightarrow +\infty$) to a map of lines of constant slope downward and to the right ($\rho \rightarrow -1$) (See Henderson and Quandt, pp. 87-8; Debertin, 1986, pp. 203-6). Now consider the product-space analog of (4), namely

(5) $x = A(ay_1^{-\rho} + by_2^{-\rho})^{-1/\rho}$

If $\rho \rightarrow -1$, the lines of constant slope could represent production possibility curves. As $\rho$ becomes more negative than -1, realistic production possibility curves occur concave to the origin, with the limiting case of right angle isoquants bowed outward at $\rho = -\infty$. Figure 5 illustrates the product-space analog to the production surface in factor space, and the corresponding set of production possibilities curves for $\rho = -2$. Notice also that when this function is multiplied by the price of the input (or input bundle), $v$, then $x$ is transformed into $C$. The production possibility curves become isocost lines, concave to the origin as was the case in the NE corner of the right panel of Figure 4.

Figure 6 illustrates an extreme case of Figure 5 as $\rho$ becomes very negative (in this case, $\rho = -50$) and the production possibilities curves approach right angles, convex from below. Compare Figure 6 with that depicted in Case 1, p. 205 of Debertin, 1986, where the isoquants are approaching right angles concave from below; or turn Figure 6 upside down to see the factor space counterpart.

## Concluding Comments

This paper has provided some examples of the potential usefulness of high-resolution computer graphics as a research and educational tool for the analysis of functions in production economics. We have emphasized cases we believe are out of the ordinary, yet still of interest to the applied economist. We have not seen any of these examples in either the economics or mathematics texts. High resolution graphics of the more nearly commonplace cases, such as plots of surfaces and isoquants for Cobb-Douglas-type production functions, are also easily done, and could make excellent visual aids for advanced courses in production and other microeconomic theory courses.

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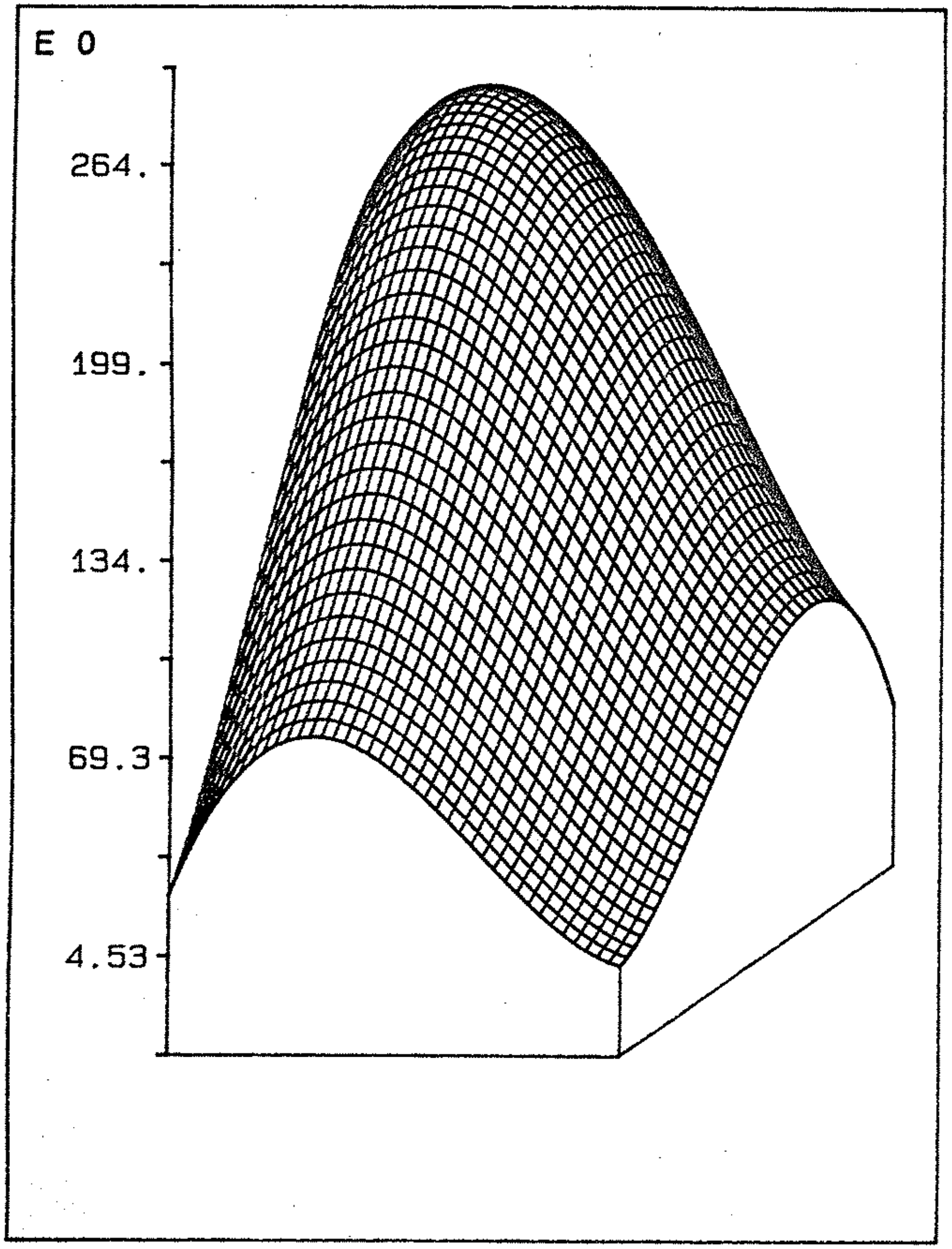


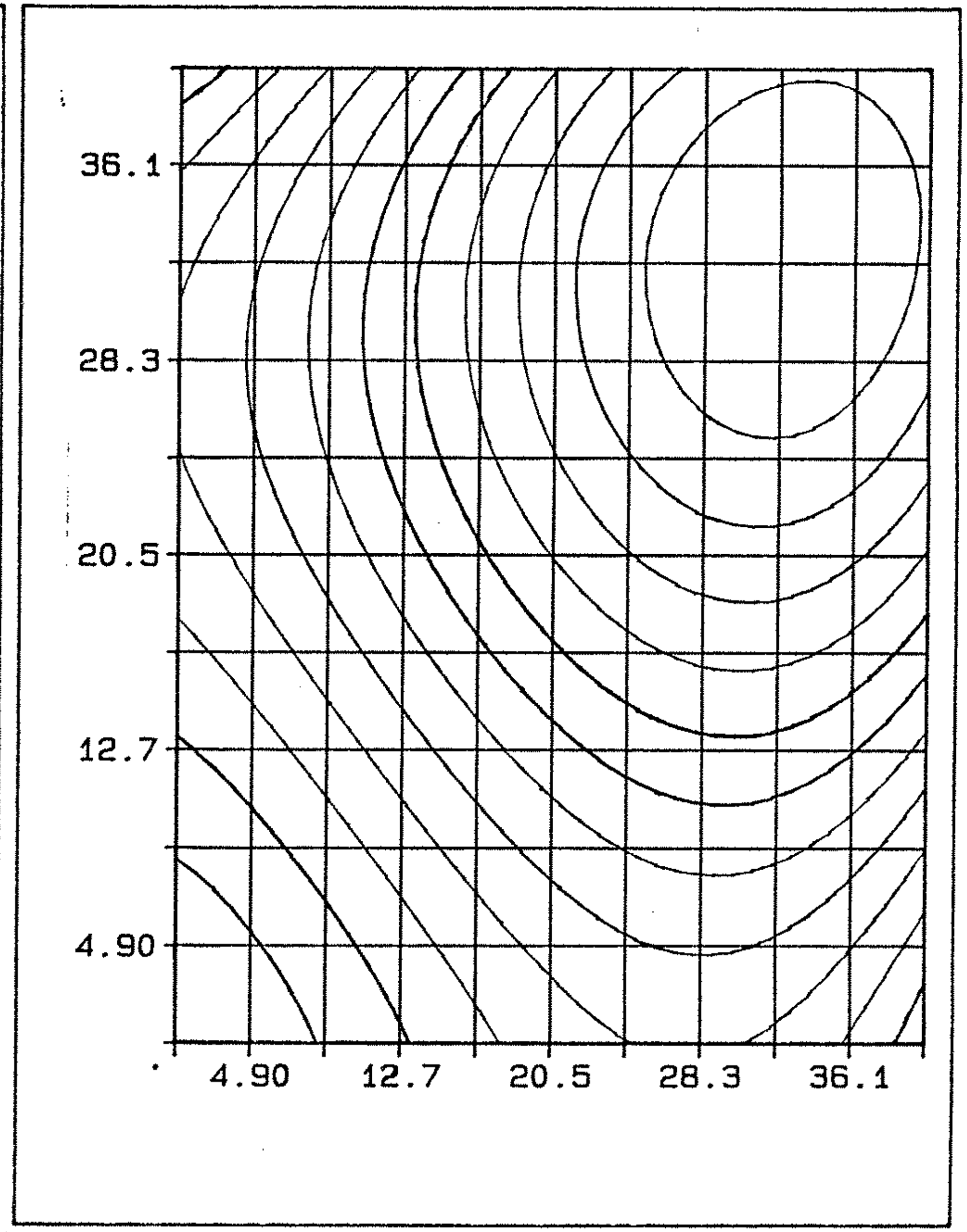




Figure 1. Surface and Isoquants for the Production Function

$$y = x_1 + 1.5x_1^2 - 0.05x_1^3 + x_2 + 1.5x_2^2 - 0.05x_2^3 + 0.4x_1x_2$$

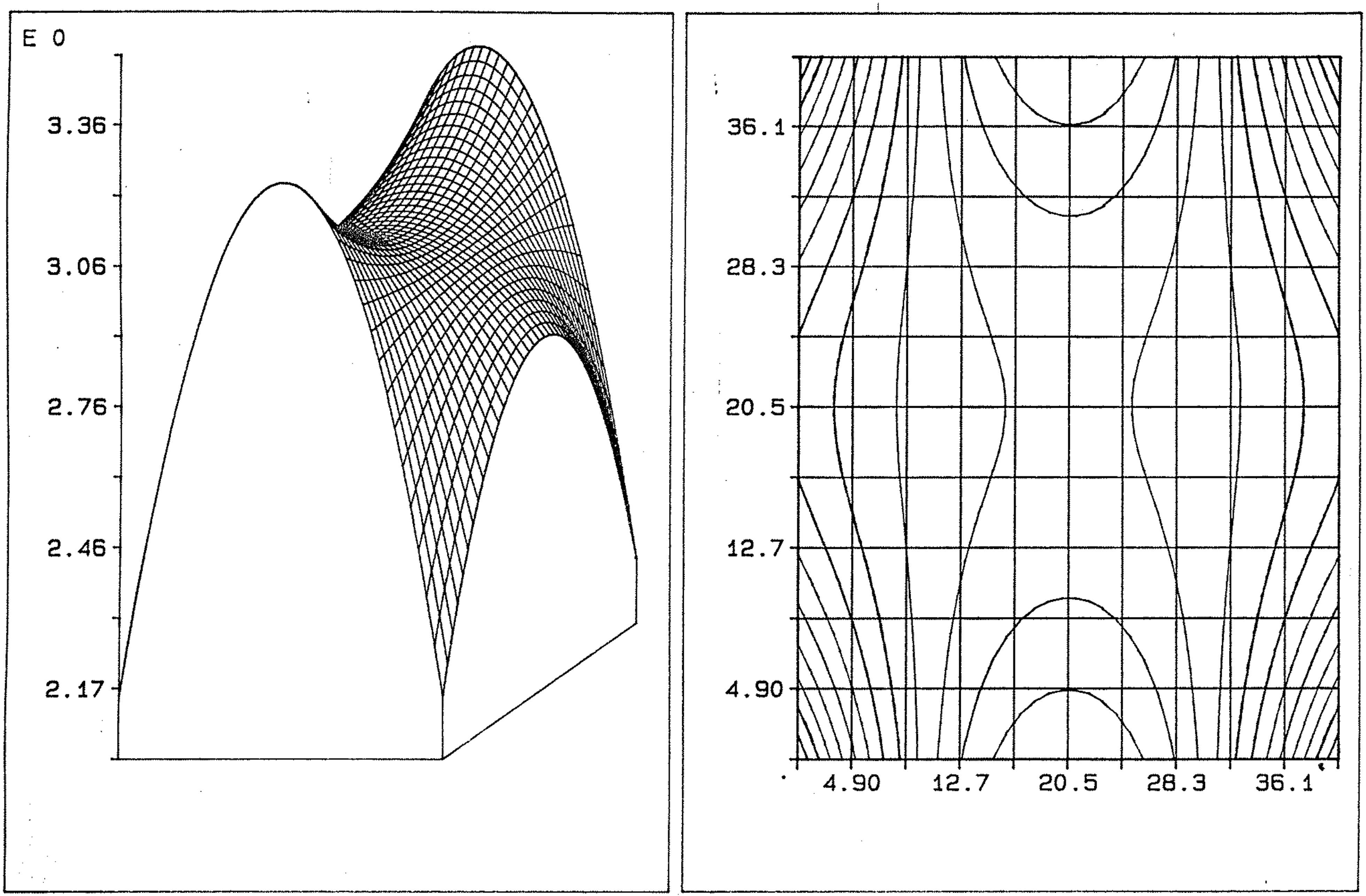
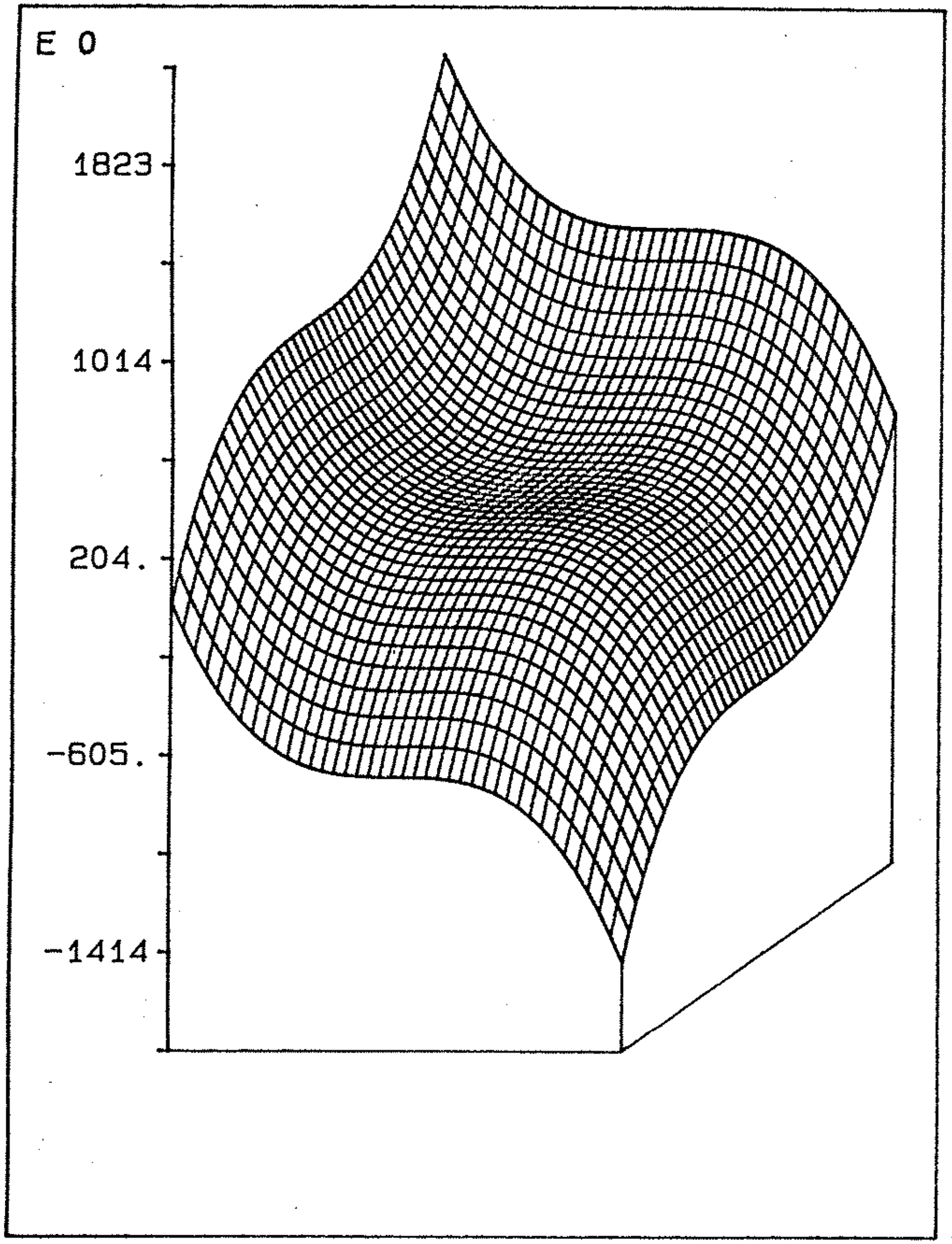



Figure 2. Surface and Contour Lines for the Function
$y = -x_1^2 + x_2^2 - 15x_1^2x_2^2$



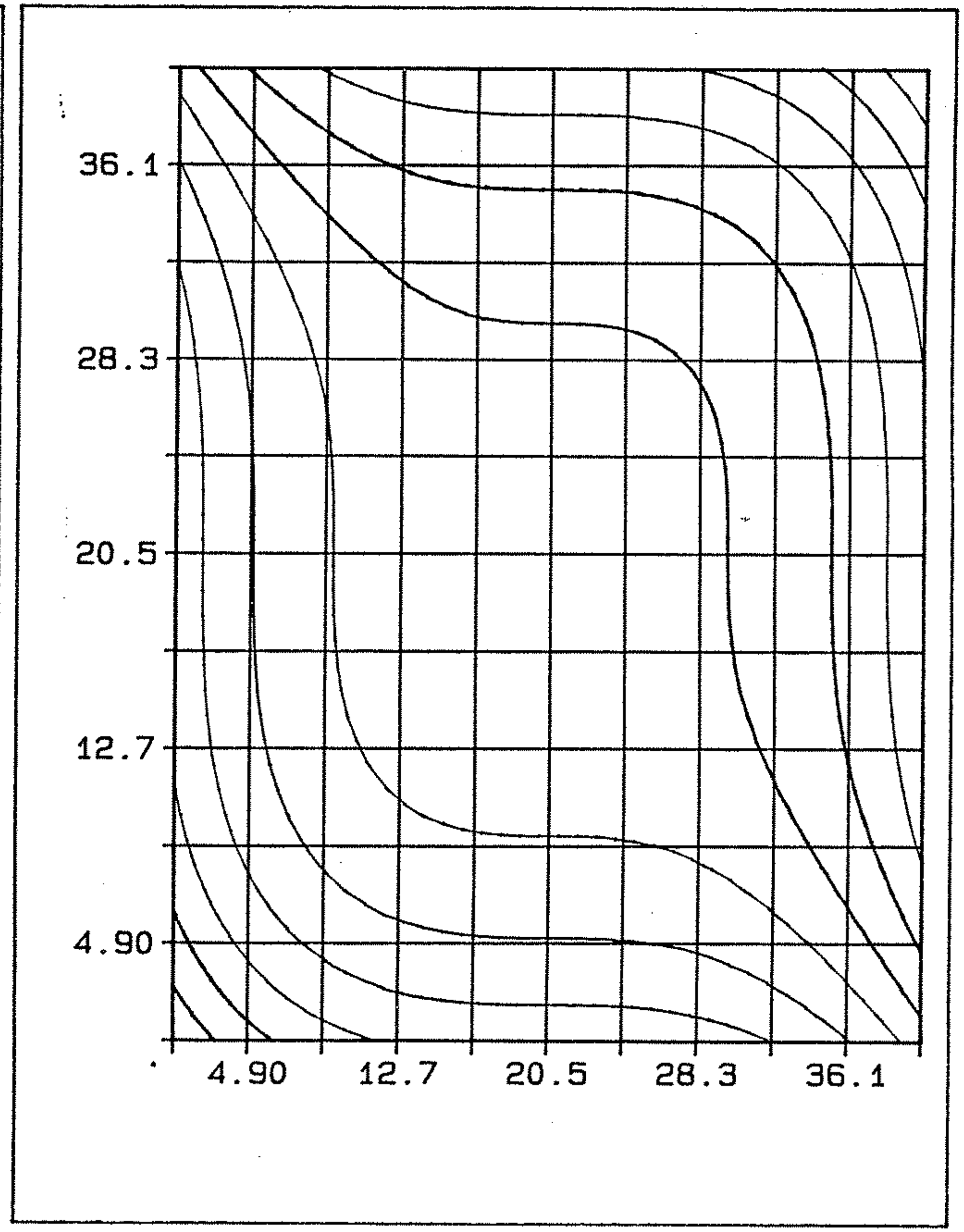




Figure 3. Surface and Contour Lines for the Polynomial

$$y = 8.824x_1 - 0.8824x_1^2 + 0.0294x_1^3 + 8.824x_2 - 0.8824x_2^2 + 0.0294x_2^3$$

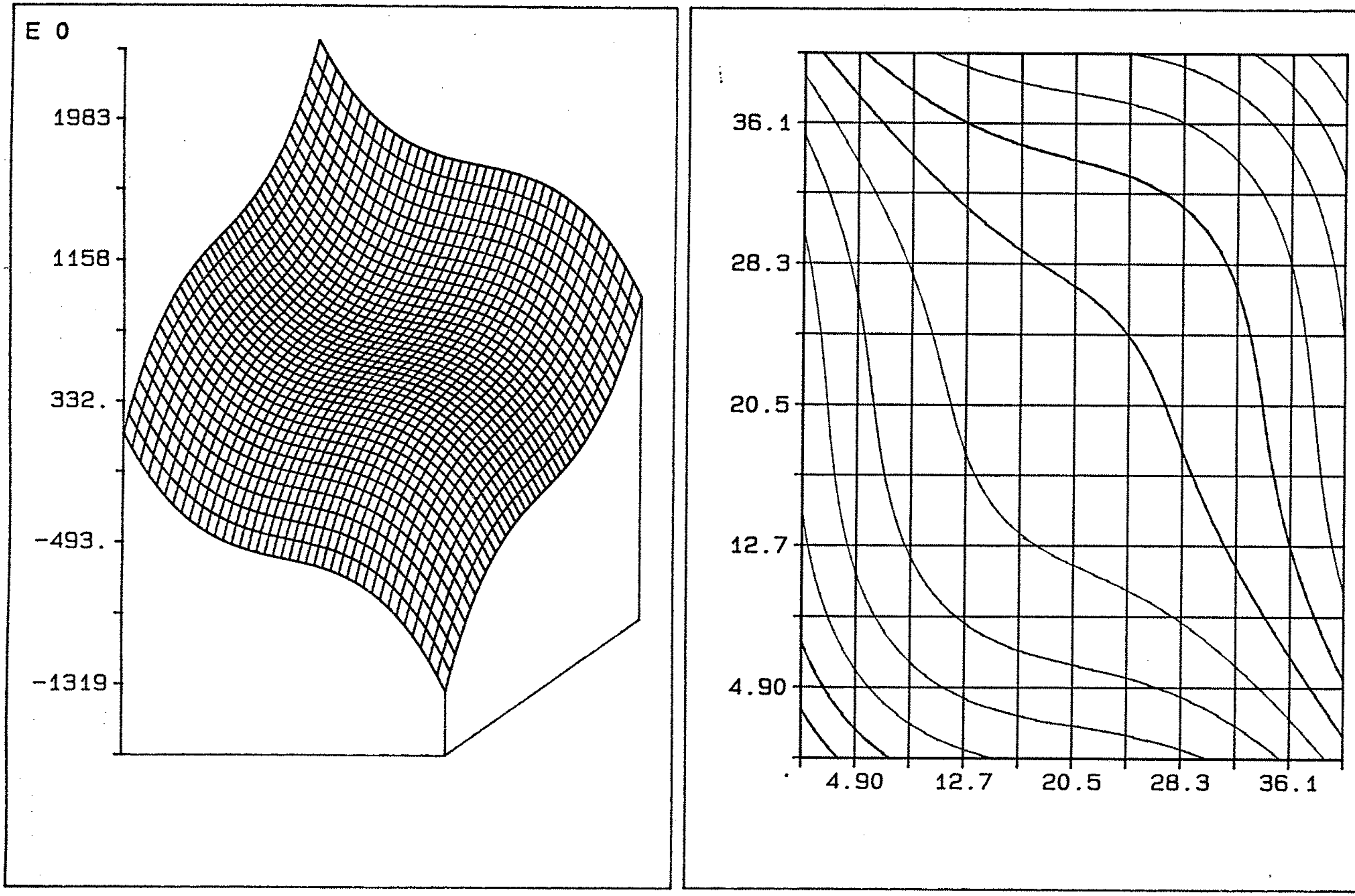
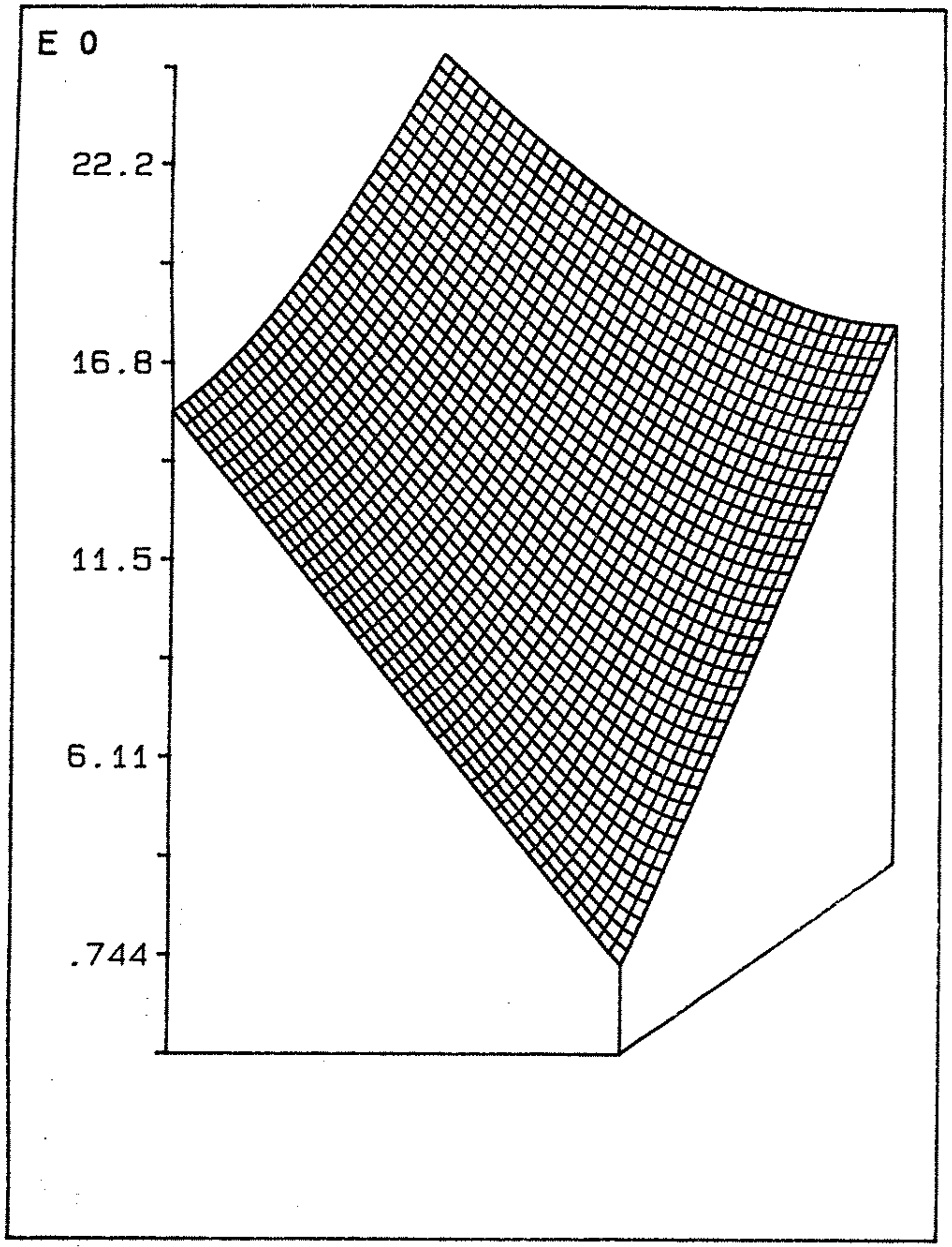



Figure 4. Surface and Contour (Isocost) Lines for the Cost function
$C = v[13.33y_1 - 0.667y_1^2 + 0.022y_1^3 + 13.33y_2^2 - 0.667y_2^2 + 0.22y_2^3]$



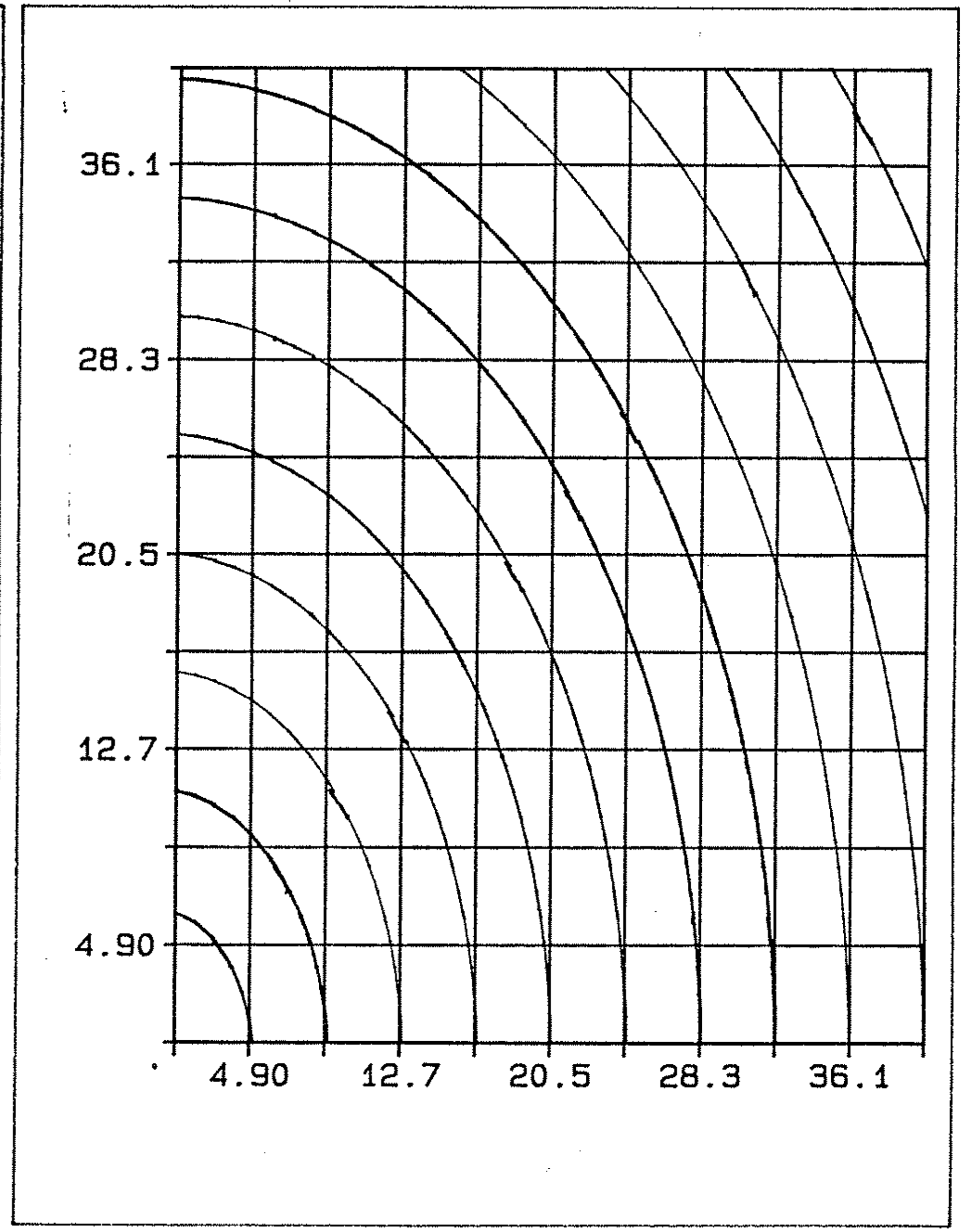




Figure 5. Surface and Isoresource Lines for Product-Space CES with Rho = -2.

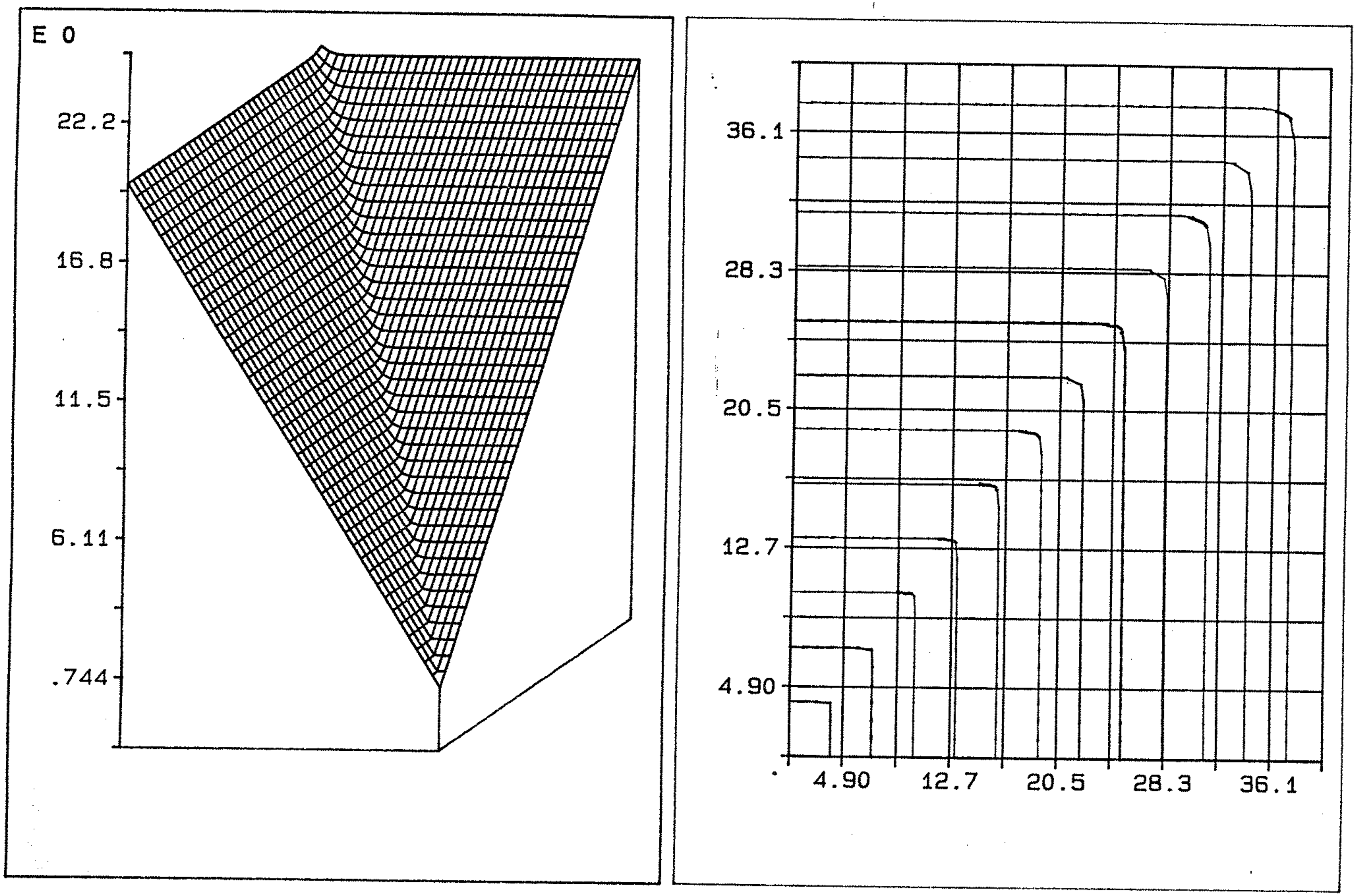



Figure 6. Surface and Isoresource Lines for Product Space CES with Rho - -50.