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The Theory of the Firm and the Concept of an Elasticity of Substitution of Product Space

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Much of the theory of the firm in product space is not nearly as well developed as the theory of the firm in factor space. For example, both general and agricultural economists have devoted considerable effort to developing functional forms representing production processes in factor space, but the companion effort in product space has been very limited. The purpose of this paper is to present some product-space concepts analogous to those commonly used in factor space. Concepts not specifically cited in past research, to the author's knowledge, are new. While several of these concepts have appeared previously in cited economics literature, the usefulness to agricultural economics research has not previously been made clear. Moreover, the author is not aware of any other single reference to these ideas.

In factor space, an equation for a production process involving n inputs and a single output is $y = f(x_1,\ldots,x_n)$ with an isoquant representing a fixed constant output as $y^{\circ} = f(x_1,\ldots,x_n)$. In product space, the analogous equation linking the production of m outputs to the use of a single input (or bundle of inputs), is $x = h(y_1,\ldots, y_m)$, and the product transformation function representing possible combinations of the y_i that can be produced from a fixed quantity of a single input (or input bundle, with the quantities of each input being held in fixed proportion to each other) is $x^{\circ} = h(y_1,\ldots, y_m)$. Although a considerable effort has been devoted to the development of explict specifications for production functions (Fuss, McFadden and Mundlak, Diewert, 1971), attempts at developing explicit product space counterparts have usually been simple modifications of production functions replacing the x_i with y_i, and substituting for the quantity of x in the product space model, a single input (or input vector $\mathbf{x} = \{x_1^{\circ},\ldots,x_n^{\circ}\}$ for y° in the factor space model. Efforts to derive product space functions by making assumptions about the specific form of the underlying production functions break down if the underlying production functions are not monotonic. Even if the underlying production functions are monotonic, the competition between products for inputs are normally not adequately represented with such an approach (Beattie and Taylor; Debertin) The standard presentation of the neoclassical theory of the firm usually specifies isoquants in factor space with a diminishing (or possibly constant) marginal rates of substitution. The standard presentation in product space specifies product transformation functions with an increasing (or possibly constant) rate of product transformation. A simple interchange of outputs and inputs may be inadequate, and the parameters of and in some cases the explicit form of the product space function (h) needed to generate product transformation functions consistent with neoclassical theory will be quite different from the parameters and form of the factor space production function (f).

Duality in Product Space

In product space, the total revenue function can play a role analogous to the cost function in factor space. Suppose that products (a) are either supplemental or competitive but not complementary with each other for the available resource bundle x°, and (b) rates of product transformation between output pairs are non-decreasing. These assumptions in product space are analogous to the free disposal and non increasing marginal rate of substitution assumptions (McFadden, 1978, pp. 8-9) in factor space.

Given the product space function $x = g(y_1, y_2, \ldots, y_m)$, the corresponding total revenue function that maximizes total revenue for a given input bundle x° is $r = \max[\mathbf{p'y}; g(\mathbf{y}) \leq x^{\circ}]$. If conditions (a) and (b) are met, then the revenue function exists, is continuous, is non-decreasing in each price in the product price vector $\mathbf{p}$, is linearly homogeneous in all product prices $\{p_1, \ldots, p_m\}$ (and in all outputs $\{y_1, \ldots, y_m\}$) and is convex in each output price for a given level of input x° (Hanoch, p. 292).

The product transformation functions needed for the existence of a corresponding dual revenue function are not necessarily more plausible in an economic setting than other product transformation functions, but are rather a mathematical convenience. A Cobb-Douglas like funtion in product space will not generate product transfomation functions consistent with (a) and (b), while under certain parameter assumptions, a CES like or translog like function in product space will generate product transformation functions consistent with these assumptions.

Cobb-Douglas Like Product Space

Consider first a Cobb-Douglas like analogy in product space. A Cobb-Douglas like two product one input model suggested by Just, Zilberman and Hochman (p. 771) from Klein is $y_1 y_2^{\delta} = Ax_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$. Now suppose there is but one input to the production process, and $Ax^{\alpha_1} = y_1 y_2^{\delta}$. Solving for input $x = (1/A)^{1/\alpha_1} y_1^{1/\alpha_1} y_2^{\delta/\alpha_1}$. The parameters α_1 and δ would normally be non-negative, since additional units of y_1 or y_2 can only be produced with additional units of the input bundle, and additional units on the input bundle produce additional units of outputs y_1 and y_2.

Rewriting in a slightly more general form, the product space function is $x = By_1^{\phi_1} y_2^{\phi_2}$. However, with positive parameters, in no case will this equation generate product transformation curves convex to the origin, for the Cobb-Douglas like function is quasi-concave for any set of positive parameter values. A Cobb-Douglas like function in product space cannot generate product transformation functions consistent with neoclassical theory and the usual constrained optimization revenue maximization conditions.

CES Like Functions in Product Space

Just, Zilberman and Hochman also suggest a possible CES like function in product space. A version of this function with one input and two outputs is $x = C[\lambda_1 y_1^{-\eta} + \lambda_2 y_2^{-\eta}]^{-1/\eta}$. The product transformation functions generated from the CES like function in product space are downsloping so long as λ_1 and λ_2 are positive, irrespective of the value of the parameter η.

The curvature on the product transformation function is given by the sign on $d^2y_2/dy_1^2 = -(1+\eta)(-\lambda_1/\lambda_2)y_2^{1+\eta}y_1^{-(2+\eta)}$. Since y_1, y_2, λ_1, $\lambda_2 > 0$, the sign is dependent on the sign on $-(1+\eta)$. In factor space, the values of ρ that are of interest are those that lie between -1 and $+\infty$, for these are the values that generate isoquants with a diminishing marginal rate of substitution on the input side. If the value of η is exactly -1, then the product transformation functions will be diagonal lines of constant slope λ_1/λ_2 and products are perfect substitutes.

However, the CES like function can generate product transformation functions with an increasing rate of product transformation. The five CES cases outlined by Henderson and Quandt in factor space include only values of ρ that lie between -1 and $+\infty$. In product space, the values of η that lie between -1 and $-\infty$ generate product transformation functions with an increasing rate of product transformation, since the second derivative is negative when $\eta < -1$. As $\eta \to -\infty$, the product transformation functions approach right angles, concave to the origin. Small negative values for η generate product transformation functions with a slight bow away from the origin. As the value of η becomes more negative, the outward bow becomes more extreme. In the limiting case, when $\eta \to -\infty$, y_2 is totally supplemental to y_1 when y_1 exceeds y_2; conversely y_1 is totally supplemental to y_2 when y_2 exceeds y_1. This is equivalent to the joint product (beef and hides) case. If η is a fairly large negative number (perhaps < -5) there exist many combinations of y_1 and y_2 where one of the products is "nearly" supplemental to the other. As $\eta \to -1$, the products become more nearly competitive throughout the possible combinations, with the diagonal product transformation functions when $\eta = -1$ the limiting case. Regions of product complementarity are not possible with a CES like product transformation function. Product transformation functions exhibiting a constant or an increasing rate of product transformation must necessarily intersect the y axes. Thus, there is no product space counterpart to the asymptotic isoquants generated by a Cobb-Douglas type function in factor space.

Alternative Elasticity of Substitution Measures in Product Space

Diewert (1973) extended the concept of an elasticity of substitution (which he termed elasticity of transformation) to multiple product-multiple input space. Hanoch also suggests that the elasticity of substitution in product space can be defined analogously to the elasticity of substitution in factor space. In the case of product space, revenue is maximized for the fixed input quantity x°, is substituted for minimization of costs at a fixed level of output y°(p. 292) in factor space. The elasticity of substitution in two product one input space (Debertin) is defined as ε_{sp} = % change in the ratio y_2/y_1 ÷ % change in the RPT or as $\phi_{sp} = [d(y_2/y_1)/dRPT][RPT/(y_2/y_1)]$.

Another way of looking at the elasticity of substitution in product space is in terms of its linkage to the rate of product transformation for CES like two-product space. Suppose that $Y = y_2/y_1$, or the output ratio. The rate of product transformation for CES like product space is defined as $RPT = Y^{(1+\eta)}$. The elasticity of substitution in product space is then (dlog Y)/(dlog RPT). Taking the natural log of both sides yields log RPT = $(1+\eta)$ log Y. Solving for log Y and logarithmically differentiating gives (dlog y)/(dlog RPT) = $1/(1+\eta)$. Assuming that $\eta < -1$, the elasticity of substitution in product space for a CES like function is clearly negative, but $\rightarrow 0$ as $\eta \rightarrow -\infty$.

The concept of an elasticity of substitution in product space is of considerable importance to agricultural economists, for it is a pure number that indicates the extent to which the agricultural products which can be produced with the same input bundle can be substituted for each other. Assuming competitive equilibrium, the inverse product price ratio p_2/p_1 can be substituted for the RPT, and the elasticity of substitution in product space can be rewritten as $\phi_{sp} = [d(y_2/y_1)/d(p_1/p_2)][(p_1/p_2)/(y_1/y_2)]$. As McFadden (1963) has indicated, there is no natural generalization of the two input elasticity of substitution when whe number of factors is greater than 2. The elasticity of substitution will vary depending on what is assumed to be held constant. However, the Allen, Morishima (Koizumi), and Shadow (McFadden) elasticities of substitution in factor space all collapse to the same number when n equals 2. Similarly, there is no natural generalization of product space elasticity of substitution when the number of products exceeds two.

In the case of farming, the elasticity of substitution in product space is a pure number that indicates the extent to which the revenue-maximizing farmer is able to respond to changes in relative product prices by altering the product mix. An elasticity of substitution in product space near zero would indicate that the farmer is almost totally unable to respond to changes in product prices by altering the mix of products that are produced and is the joint-product case. An elasticity of substitution in product space of $-\infty$ indicates that the farmer nearly always would be specializing in the production of the commodity with the favorable relative price. As relative prices change toward the other commodity, a complete shift would be made to the other commodity.

For most agricultural commodities, the elasticity of substitution in product space would be expected to lie between 0 and $-\infty$, indicating that to a certain degree, farmers will respond to changes in relative product prices by altering the product mix. Commodities which require very similar inputs would be expected to have very large negative elasticities of product substitution. Examples include Durum wheat versus Hard Red Spring wheat in North Dakota, or corn versus soybeans in the corn belt. Conversely, two dissimilar commodities requiring very different inputs would be expected to have elasticities of substitution approaching zero, and a change in relative prices would not significantly alter the output combination. In m product space, when m>2, the elasticity of substitution is $\varepsilon_{sp} = [d\log y_k - d\log y_i]/[d\log p_i - d\log p_k]$. This measure is representative of a two output two price (or TOTP) elasticity of product substitution analogous to the two input two price (TTES) elasticity of substitution in factor space, with the quantities of outputs other than i and k held constant.

The concept of an elasticity of substitution in product space is one mechanism for resolving the problems with the joint and multiple product terminology. The output elasticity of substitution is zero when outputs must be produced in fixed proportions (joint) with each other. The output elasticity of substitution is $-\infty$ when products are perfect substitutes for each other.

Other elasticity of product substitution concepts can be defined, each of which is analogous to a similar concept in factor space. For example, the one output one price (or OOOP) concept is Allen like and symmetric, or $\varepsilon_{spa} = \beta(d\log y_i)/(d\log p_k)$. The one intput one price (or OOES) concept in factor space is proportional to the cross price input demand elasticity evaluated at constant output. Similarly, the OOOP concept is proportional to the cross output price product supply elasticity evaluated at a constant level of input use. An own price OOOP can also be defined, that is proportional to the own price elasticity of product supply.

In factor space, the Allen elasticity of substitution is proportional to the cross price input demand elasticity evaluated at constant output. Normally, as the price of the jth input increases, more of the ith input, and less of the jth input would be used in the production process, as input x_i is substituted for input x_j, evaluated at constant output. Thus, the sign on the Allen elasticity of substitution in factor space is normally positive if inputs substitute for each other. However, in product space, the Allen like elasticity of substitution is proportional to the cross output price product elasticity of supply evaluated at a constant level of input use. Normally, as the price of the jth output increases, the amount of the jth output produced would increase, and the amount of the ith output produced would decrease, the opposite relationship from the normal case in factor space. Thus, while the Allen elasticity of substitution in factor space would normally have a positive sign, the Allen like elasticity of substitution would normally have a negative sign in product space. The negative sign is also consistent with the sign on the product elasticity of substitution for the CES like function derived earlier.

In the n input setting, Hanoch (p. 290) defines the optimal (cost minimizing) share for input x_j as a share of total variable costs as $S_j = w_j x_j^*/C$, where $C = \sum w_i x_i$, w_i = the price of the ith input, and output is constant. On the product side, define the revenue maximizing revenue share (R_k^*) for output y_k treating the input x^o (or input vector bundle $\mathbf{x}^o$) constant as $R_k^* = p_k y_k^*/R$, where p_k = the price of the kth output, $R = \sum p_i y_i$, $i=1,,,m$; and y_k^* = the revenue maximizing quantity of output y_k from the fixed input bundle $\mathbf{x}^o$. Invoking the revenue counterpart to Shephard's lemma (Beattie and Taylor, p. 235) gives $\partial R/\partial p_k = y_k^*$. The share of total revenue for optimal quantity of the kth output can then be rewritten as $R_k = d\log R/d\log p_k$.

In the m output case, the Allen like elasticity of substitution (or transformation) (A_{ik}^p) in product space between input x_i and x_j evaluated at a constant input price w_j is defined as $A_{ik}^p = (1/R_k)(E_{ij}^p)$, where E_{ij}^p = $d\log y_i/d\log p_k$, the cross price elasticity of supply for output y_i with respect to the kth product price. The Allen-like elasticity of substitution may be rewritten as $A_{ij}^p = d\log y_i/d\log R = A_{ki} = d\log y_k/d\log R$, since the inverse of the Hessian matrix for the underlying function h in product space is symmetric. In this context the Allen like elasticity of substitution in product space is the elasticity of y_i with respect to total revenue R, for a change in another price p_k, holding the quantity of the input (or input bundle) constant.

Yet another way of looking at the Allen like elasticity of substitution in product space is by analogy to the Allen elasticity of substitution defined in factor space defined in terms of the cost function and its partial derivatives. The Allen elasticity of substitution between the ith and jth input (A_{ij}^{f}) in factor space can be defined as in terms of the cost function and its partial derivatives, $A_{ij}^{f} = (CC_{ij})/(C_iC_j)$, where $C=h(w_1,\ldots,w_n, y^*)$; $C_i = \partial C/\partial w_i$; $C_j = \partial C/\partial w_j$; and $C_{ij} = \partial^2C/\partial w_i\partial w_j$. The corresponding revenue function definition in product space is $A_{ij}^{p} = (RR_{ij})/(R_iR_j)$, where $R=h(p_1,\ldots,p_n, x^*)$; $R_i = \partial R/\partial w_i$; $R_j = \partial R/\partial w_j$; and $R_{ij} = \partial^2R/\partial w_i\partial w_j$.

The two output one price (or TOOP) elasticity of product substitution is analogous to the two output one price Morishima or TOES elasticity of substitution in factor space. The Morishima like elasticity of substitution in product space (Koizumi) is $\varepsilon_{spm} = (\text{dlog } y_i - \text{dlog } y_k)/\text{dlog } p_k$. Like its factor-space counterpart, the Morishima like elasticity of substitution in product space is nonsymmetric. Fuss and McFadden (p. 241) note that in factor space, each elasticity of substitution can be evaluated based on constant cost, output or marginal cost. In product space, the total revenue equation is analogous to the cost equation in factor space. Hence, each elasticity of substitution in factor space may be evaluated based on constant total revenue, marginal revenue, or level of input bundle use. Generalization of the various product elasticity of substitution measures to m outputs involves making assumptions with regard to the prices and/or quantities of outputs other than the ith and jth output. A shadow like elasticity of substitution in product space is, like its factor space counterpart (McFadden), a long run concept, but in this case, all quantities of outputs other than i and j are allowed to vary.

Translog Like Functions in Product Space

The second order Taylor's series expansion of log y in log x_i, or translog production function (Christensen Jorgenson and Lau, 1971,1973), has received widespread use as a basis for cost-share equations used in the empirical estimation of elasticities of substitution in factor space. The slope and shape of the isoquants for the translog production function are dependent on both the estimated parameters of the function and the units in which the inputs are measured. Given the two input translog production function

$y = Ax_1^{\alpha_1} x_2^{\alpha_2} e^{\gamma_{12} \log x_1 \log x_2 + \gamma_{11}(\log x_1)^2 + \gamma_{22}(\log x_2)^2}$. The important parameter in determining the convexity of the isoquants is γ_{12}. The parameter γ_{12} is closely linked to the elasticity of substitution in factor space. A two output translog function in product space can be written as

$x = Ay_1^{\beta_1} y_2^{\beta_2} e^{\theta_{12} \log y_1 \log y_2 + \theta_{11}(\log y_1)^2 + \theta_{22}(\log y_2)^2}$. In two product space, the parameter θ_{12} would normally be expected to be negative, just as in factor space, γ_{12} would be expected to be normally positive. The indirect two output translog revenue function that represents the maximum amount of revenue obtainable for a specific quantity of input x^o, allowing the size of the input bundle to vary is

$\log R^* = \log D + \delta_1 \log p_1 + \delta_2 \log p_2 + \delta_{11} (\log p_1)^2 + \delta_{22} (\log p_{22})^2$
$+ \delta_{12} \log p_1 \log p_2 + \eta_{1x} \log p_1 \log x + \eta_{2x} \log p_2 \log x + \eta_x \log x + \eta_{xx}(\log x)^2$

Beattie and Taylor (p. 235-6) derive the revenue counterpart to Shephards lemma. They show that $\partial R^*/\partial p_j = y_j^*$. Thus, if the firm's revenue function is known, systems of product supply equations can be derived by differentiating the revenue function and performing the indicated substitution. Factor prices are treated as fixed constants in such an approach. Differentiating with respect to the jth product price, say p_1, yields $d\log R^*/d\log p_1 = \delta_1 + 2\delta_{11} \log p_1 + \delta_{12} \log p_2 + \eta_{1x} \log x$. Economic theory imposes a number of restrictions on the values that the parameters of such a function in the m output case. These restrictions are similar to those imposed on the parameters of cost share equations in factor space. First, total revenue from the sale of m different products is $R = \sum R_i$ $i = 1,\ldots,m$. Thus, if the revenue from m-1 of the revenue share equations is known, the remaining revenue share is known with certainty, and one of the revenue share equations is redundant.

Young's theorem holds in product just as it does in factor space. Thus, $\delta_{ij} = \delta_{ji}$, which is the same as the symmetry restriction in factor space. Any revenue function should be homogeneous of degree one in all product prices. This implies that $\sum\delta_i = 1$, and $\sum\delta_{ij} = 0$. In product space, the assumption corresponding to constant returns to scale in factor space is that there is a constant increase in revenue associated with an increase in the size of the input bundle. This implies $dR^*/dx = \delta_x = 1$, $\sum\delta_{ix} = 0$ for $i = 1,\ldots,n$; and $\delta_{xx} = 0$. These assumptions are as plausible in product space as the analogous assumptions are with regard to indirect cost functions in factor space.

Brown and Christensen derive the constant output Allen elasticities of substitution in factor space from $\sigma_{ij} = (\theta_{ij} + S_iS_j)/S_iS_j$, where S_i, S_j = the cost shares attributed to factors i and j, respectively. θ_{ij} = the restricted regression coefficient from the $\log r_i \log r_j$ term in the cost share equation. The analogous formula for deriving the Allen like elasticities of substitution in product space is $\sigma_{ijp} = (\delta_{ij} + R_iR_j)/R_iR_j$. As indicated earlier, the parameter δ_{ij} will usually be negative, and the Allen like elasticity of substitution in product space (σ_{ijp}) for most commodities is negative.

Concluding Comments

Many possibilities exist for empirical analysis linked to agriculture based on the models developed in this paper. One of the simplest approaches would be to estimate estimate revenue share equations for major commodities in U.S. agriculture for selected time periods (following the approach used by Aoun for estimating cost share equations for agricultural inputs in factor space) and derive various elasticity of substitution measures in product space. These revenue share parameter estimates would be used to estimate product elasticity of substitution measures for the various major agricultural commodities in the United States. Such an empirical analysis could stress the implications for current ag policy in terms of determining how farmers alter their product mix over time in the face of changing government price support programs such as those contained in the 1985 farm bill.

Another possibility is to estimate changes in the product space elasticity of substitution measures over time. Some thirty years ago Heady and others discussed the impacts of specialized versus flexible facilities using a product space model. One way of looking at a facility specialized for the production of a specific commodity is that it represents product space in which the elasticity of substitution is near zero. A flexible facility is represented by a product space elasticity of substitution that is strongly negative.

It is also possible to think in terms of an analogy to a Hicks' like technological change in product space. In product space, technological change occurring over time may favor the production of one commodity at the expense of another commodity. If, as the state of technology improves over time, and no shift is observed in the proportions of the y_i to y_j over time, then the technology is regarded as Hicks like neutral in product space. Technology that over time shifts the output expansion path toward the production of the j th commodity, then the technology is regarded as Hicks like favoring for product y_j. If technological change causes the output expansion path to shift away from the production of commodity y_i, then the technological change could be referred to as y_i inhibiting technological change. As technological change occurs for a specific agricultural commodity, presumably that commodity is favored relative to others in a product space model. For example, has technological change over the past thirty years tended to favor the production of soybeans relative to other grains? Such an approach might be useful in assessing the economic impacts of genetic improvements in specific crops or classes of livestock.

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