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Valuing Agricultural Mortgage-Backed Securities

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ABSTRACT

A model to value Federal Agricultural Mortgage Corporation (Farmer Mac) agricultural mortgage-backed securities (AMBS) is developed and numerically solved. The results suggest prepayment penalties currently being used by Farmer Mac reduce yields on AMBS considerably. Even with prepayment penalties, it can be advantageous for profit maximizing mortgagors to optimally prepay or even default on agricultural mortgages. The model is used to quantify prepayment and default risk by valuing the embedded options in the mortgages. Monte Carlo simulation is also used to determine the probability of optimal prepayment given the term structure assumption used to develop the model.

Key Words: *agricultural mortgage-backed securities, default, dynamic programming, simulation, prepayment.*
JEL: G13, G21.

The mid-1980s were a difficult time for agricultural lending. As Barkema, Drabenstott, and Froerer (1988) note, an agricultural recession led to widespread loan defaults, causing the Farm Credit System (FCS) to lose over $2 billion in 1985. Mounting losses combined with a legislative desire to decrease budget expenditures resulted in a reorganization of the agricultural lending system that culminated in the Agricultural Credit Act of 1987. Some of the most significant changes brought about by this legislation are found in Title VII, which established the Federal Agricultural Mortgage Corporation. The Federal Agricultural Mortgage Corporation, also known as Farmer Mac, is a federally chartered corporation charged with providing a secondary market for agricultural real estate loans.

Most research relating to mortgages and secondary mortgage markets has been directed at residential mortgages. This is likely attributable to the size of the residential (non-farm) secondary mortgage market which in 1998 represented 81.9 percent of all mortgage debt and was a staggering $4.738 trillion (U.S. Census Bureau). By contrast, commercial mortgage debt made up 16.4 percent of all mortgage debt while farm mortgage debt made up the balance of 1.6 percent. While small relative to the other categories, farm mortgage debt continues to grow and was a record $95 billion in 1998 (U.S. Census Bureau).

The three classifications of mortgage debt above share similarities and differences that have implications for valuation models of agricultural mortgage-backed securities (AMBS). Commercial and agricultural mortgages are similar in many respects, not the least of which is that loan performance is more readily tied to the financial performance of the mortgaged asset. As a result, prepayment and default are likely influenced by the income earning ability

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of the mortgaged asset as well because volatile commodity prices and/or commodity yields affect the ability of the mortgagor to service the mortgage. By contrast, loan performance is more closely tied to demographic variables and the (typically non-volatile) personal income of the mortgagor in the case of residential mortgages.

As Kelly and Slawson (2000) note, residential mortgages are also highly standardized relative to commercial mortgages where terms are more complex and heterogeneous. While agricultural mortgages are also fairly standardized, the terms tend to be different than those for residential mortgages. For example, the typical residential mortgage requires a minimum down payment of 5 percent of the lesser of purchase price and appraised value. Agricultural real estate mortgages are more apt to impose a minimum down payment of 25 percent to 33 percent depending on the mortgagee. The higher down payment helps mitigate the additional risk of default attributable to the mortgaged asset's financial performance volatility. Higher down payments also insulate the mortgagee from some of the relative illiquidity associated with agricultural real estate (another key difference between agricultural and residential real estate). Other differences include repayment frequency, which is typically semiannual for agricultural mortgages, and maturities that rarely go beyond 20 years.

Another important difference is that residential mortgages can be prepaid and rarely impose prepayment penalties in such an event. This is in stark contrast to commercial mortgages that in some cases cannot be prepaid and agricultural mortgages where prepayment triggers a penalty.[1] All Farmer Mac I loans made after 1996 include prepayment penalties (FAMC 1999). Interestingly, a recent GAO survey of lenders indicated that they would likely use Farmer Mac more if prepayment penalties were eliminated. The report recognizes that the elimination of prepayment penalties would increase the prepayment risk faced by Farmer Mac, which might necessitate charging borrowers higher interest rates. Higher interest rates might also precipitate higher default risk in the event of a economic downturn in the agricultural sector.

Given the key differences between agricultural mortgages and other mortgages and the fact that no pricing models of the current incarnation of the Farmer Mac program currently exist in the literature, the central purpose of this paper is to present an AMBS pricing model that is more consistent with some of the features of agricultural real estate and mortgages. To this end, we apply a variant of existing analytic models of mortgage-backed security pricing to the case of AMBS.

The model significantly extends Chhikara and Hanson in numerous ways, most notably with respect to prepayment penalties. Because sub-optimal prepayment and default are closely tied to the financial performance of the mortgaged asset in the agricultural case, we model agricultural land values as a diffusion process and allow the probabilities of sub-optimal default and prepayment to be functionally related to the service flow of the asset. Prepayment penalties used by Farmer Mac are analyzed to determine the implications for the cost of capital facing potential mortgagors and the risk protection they provide investors. The model is also used to value the embedded options to (optimally) default and prepay and to determine equilibrium interest rates that might induce a potential borrower to take a loan with a prepayment penalty. We also empirically analyze the extent to which Farmer Mac prepayment penalties actually preclude optimal prepayment given the term structure we assume.

An Analytical Model to Value AMBS

Derivative securities take their name from the fact that they "derive" their value from the

[1] For example, Farm Credit System banks offer three classes of loans in this category, which are referred to as the *Prepayment Premium Loan Options* (PPLO). Under the Multiflex option, loans can be prepaid or converted to another type of loan with little or no penalty. Another PPLO, called the *Flex option*, corresponds to Farmer Mac's partial open prepayment structure and offers a lower rate than the Multiflex option. The Exceptional Rate option is a PPLO that offers the lowest interest rate, but does not allow prepayment during the fixed rate period without assessment of a severe penalty, much like Farmer Mac's yield maintenance provisions.

value of some other asset. Valuing derivative securities is typically done by determining the set of assets that influence the value of the derivative and assuming the evolution of the value of these assets can be modeled with stochastic differential equations. Next, Ito's lemma is applied to determine the dynamics of the derivative and arbitrage or equilibrium arguments are made so the resulting model can be solved.

As an example, suppose the time t value of a (derivative) security depends on the value of another asset whose level is given by $X(t)$. Let $F[X(t), t]$ denote the value of the derivative and assume $X(t)$ follows geometric Brownian motion where $dX = \alpha X dt + \beta X dZ$. The dynamics of $F(\cdot)$ can be found by applying Ito's lemma to get $dF = F_X dX + \frac{1}{2} F_{XX} dX^2 + F_t dt$ where subscripts denote partial differentiation. Provided the security pays no other cash flows, dF then represents the capital gain from holding the security which should equal some expected return in equilibrium. That is, $\mathrm{E}(dF) = \rho F dt$. Substituting for dX and dX^2 above and taking the relevant expectation implies $\frac{1}{2}(\beta X)^2 F_{XX} + (\alpha X) F_X + F_t - \rho F = 0$ is an equation whose solution characterizes the value of the derivative. Additional arbitrage or equilibrium arguments can sometimes be made to eliminate the generally unobservable parameter ρ.

The AMBS model we develop is based on existing pricing models for interest rate contingent claims [see for example, Brennan and Schwartz (1977), Buser and Hendershott (1984), Cunningham and Hendershott (1984), Foster and Van Order (1984), Cox, Ingersoll, and Ross (1985a and 1985b), Green and Shoven (1986), Stanton (1995), and Deng, Quigley, and Van Order (2000)].[2]

Pooled loans are assumed to be fully amortizing mortgages for productive agricultural real estate with outstanding principal $F(t)$ at time t. The loans are homogeneous with respect to terms and have a fixed continuously compounded coupon rate, r^*, for a term to maturity of T years. The amortizing feature of the loans implies a payment of $C = r^*F(0)/(1 - e^{-r^*T})dt$ is required to retire $F(t)$ by the maturity time T. In the absence of prepayment or default, the dynamics of the loan principal balance is described by the ordinary differential equation $dF(t) = [r^*F(t) - C]dt$ implying principal outstanding at any time t is given by the solution to this ordinary differential equation, namely, $F(t) = \{F(0)[1 - e^{-r^*(T-t)}]\}/(1 - e^{-r^*T})$. We assume the mortgagor can prepay the loan at any time, but faces a prepayment penalty for doing so. Prepayment penalties are denoted by $\theta_i[r(t), L(t), t]$, where $i = ym$ denotes yield maintenance and $i = pp$ denotes partial-open prepayment.[3]

Uncertainty in the economy is characterized by the probability space $(\mathbf{\Omega}, \mathcal{F}, \mathbf{Q})$ in which $\mathbf{\Omega}$ is the state space, $\mathcal{F}$ is the σ-algebra representing measurable events, and $\mathbf{Q}$ is the risk-neutral probability measure. The spot rate of interest evolves according to the stochastic differential equation $dr(t) = \kappa[\mu - r(t)] + \sigma r(t)^{1/2} dZ(t)$ with the usual interpretation of the parameters.[4] $Z(t)$ is a $\mathbf{Q}$-Brownian motion with $\mathrm{E}^{\mathrm{Q}}[dZ(t)] = 0$ where E^{Q} represents the expectation operator under the risk-neutral probability measure $\mathbf{Q}$.

Land values, $L(t)$, are assumed to follow a diffusion given by $dL(t) = (\alpha - \nu)L(t)dt + \beta L(t) dW(t)$ where α is the instantaneous total expected return, β is the instantaneous proportionate variance, and ν represents the rate at which income flows to the owner of the land from employing it in an agricultural capacity.[5]

[2] Existing pricing models are set in continuous time and we maintain this convention in what follows principally because the stochastic calculi are particularly well suited for this type of analysis.

[3] As the analytic model to be developed does not depend on the functional form of any specific prepayment penalty, a discussion of the functional form of θ_i and its relevant argument(s) is deferred to a later section of the paper.

[4] In their intertemporal general equilibrium model, Cox, Ingersoll, and Ross (1985b) derive the dynamics of the specified spot interest rate under very specific assumptions relating to the agents and the economy. As we are relying on this specific diffusion, we are also relying on all the assumptions Cox, Ingersoll, and Ross (1985b) made to derive it.

[5] This "income flow" is analogous to the "service flow" found in the residential real estate literature, neither of which the mortgage-backed security holder has

$W(t)$ is a **P**-Brownian motion with $E^P[dW(t)] = 0$, and E^P is an expectation operator under probability measure **P**.

"Sub-optimal default" is modeled as a Poisson random variable, $x(t)$, which equals zero as long as the mortgagor does not default on the loan. This type of default arises stochastically for any number of (unspecified) reasons and differs from "optimal default" because the latter is the mortgagor's response to a decline in the underlying asset's value. The incidence of sub-optimal default is represented by $x(t)$ instantaneously jumping to one and causes the loan to exit the pool. Therefore, the dynamics of sub-optimal default are given by:

$$(1) \quad dx(t) = \begin{cases} 1 & \text{with probability } \phi^d[r(t), L(t), t]dt \\ 0 & \text{with probability } 1 - \phi^d[r(t), L(t), t]dt, \end{cases}$$

where $\phi^d[r(t), L(t), t]\, dt$ is the instantaneous probability of default occurring at time t which, as indicated, can depend on the spot rate, land values, and time.

"Sub-optimal prepayment" is also modeled as a Poisson random variable, $y(t)$, which equals zero as long as the mortgagor does not prepay the loan. This type of prepayment arises stochastically for any number of (unspecified) reasons and differs from "optimal prepayment" which is the mortgagor's response to a decline in interest rates. The incidence of sub-optimal prepayment is represented by $y(t)$ instantaneously jumping to one. As in the case of sub-optimal default, the loan also ceases to exist when prepayment occurs. Therefore, the dynamics of sub-optimal prepayment are given by:

$$(2) \quad dy(t) = \begin{cases} 1 & \text{with probability } \phi^p[r(t), L(t), t]dt \\ 0 & \text{with probability } 1 - \phi^p[r(t), L(t), t]dt. \end{cases}$$

The instantaneous probability of prepayment occurring at time t is $\phi^p[r(t), L(t), t]dt$ and, as shown, also can depend on the spot rate, land values, and time.[6]

Finally, the price of a contingent claim on the loans in the pool is given by the value functional $V \equiv V[L(t), r(t), x(t), y(t), t]$ where the arguments are as defined by the preceding assumptions. From this point forward we also suppress explicit time and functional dependence where no confusion can arise.

The Fundamental PDE for AMBS

Given the preceding assumptions, the fundamental PDE characterizing the value of AMBS can be shown to be

$$(3) \quad \left(\frac{\sigma^2 r}{2}\right)\frac{\partial^2 V}{(\partial r)^2} + (\sigma\beta\rho L\sqrt{r})\frac{\partial^2 V}{\partial L \partial r} + \left(\frac{\sigma^2 L^2}{2}\right)\frac{\partial^2 V}{(\partial L)^2} + [\kappa(\mu - r) - \lambda r]\frac{\partial V}{\partial r} + [(r - \nu)L]\frac{\partial V}{\partial L} + \frac{\partial V}{\partial t} - rV + \phi^d(F - V) + \phi^p(F - V + \theta_t) + C = 0.$$

Equation (3) is similar to equations presented by Titman and Torous (1989), and Kau *et al.* (1992), with a couple of exceptions. The equation is also recognized as the fundamental equation characterizing a number of interest-rate contingent claims including the risky mortgage, mortgage insurance, as well as mortgage-backed securities. One difference between equation (3) and the PDE characterizing residential mortgage-backed securities is the existence of the prepayment penalty, θ_t. Another difference, which we return to in a

any claim to. Also, by "agricultural capacity" we mean that the mortgaged asset is being farmed, either by the mortgagor directly or indirectly through a leasing arrangement with a farmer. In the case of a farmer mortgagor, $\nu L(t)$ represents the residual return to land. In the case of an absentee owner, the precise form of the leasing arrangement determines the interpretation of $\nu L(t)$. For example, in a cash rental agreement $\nu L(t)$ is the cash rent the farmer pays to the landowner for the right to farm the land.

[6] The specific functional forms of the probabilities, $\phi^d[r(t), L(t), t]dt$ and $\phi^p[r(t), L(t), t]dt$ are addressed in a later section of paper.

later section of the paper, is the nature of the probabilities of sub-optimal default and prepayment.

In equation (3), λ is the market price of risk and ρ is the instantaneous correlation coefficient between interest rates and land values. All the parameters in equation (3) can be observed (and therefore estimated) except λ. However, according to Kau *et al.* (1992, 1995), the parameter can be set equal to zero under either of two (different) assumptions. The market price of risk can be assumed to be included in the term structure parameters κ and μ or it can be assumed the local expectations hypothesis (LEH) holds. Under this latter assumption, $\lambda = 0$ because the LEH implies that the spot interest rate $r(t)$ contains all information available at time t regarding future interest rates. More detailed information about the LEH and a technical mathematical definition can be found in Musiela and Rutkowski (1998). Consistent with much fixed-income research, it is assumed $\lambda = 0$ because the LEH holds.

To fully specify the AMBS model, boundary conditions and an initial condition for the PDE (3) are required. The initial condition is simply $V(L, r, x, y, T) = 0$ given the amortizing feature of the mortgage. As noted above, the mortgagor possesses the option to call the loan at any time, but is subject to a prepayment penalty for doing so. While sub-optimal prepayment is governed by a Poisson process, optimal prepayment of the mortgage is driven by the interest rate diffusion process and the profit-seeking motive of the mortgagor. When the spot interest rate falls below some trigger level or value, the loan will be optimally called by the mortgagor. This optimal call policy results in the principal outstanding serving as a boundary for the value of the mortgage, $V(L, r, x, y, t) \leq F(t)$.

Similarly, it is optimal for the mortgagor to default at any time t if the value of the mortgaged asset falls below the market value of the mortgage. Therefore, $V(L, r, x, y, t) \leq L(t)$ prior to maturity [Schwartz and Torous (1992)]. We also assume the solution to (3) has bounded derivatives and that the following conditions hold

$$\text{(4)} \qquad \lim_{r \to \infty} V(L, r, x, y, t) = 0 \quad \text{and}$$

$$\lim_{L \to 0} V(L, r, x, y, t) = 0.$$

These boundary conditions are relatively standard given equation (3). For more detail, see for example, Titman and Torous, (1992).

Farmer Mac Prepayment Penalties

Unlike mortgage-backed securities issued by other GSEs such as Ginnie Mae or Freddie Mac, AMBS issued under the Farmer Mac I program have a guaranteed yield. The guaranteed yield is supposed to make AMBS more attractive to investors than standard mortgage-backed securities. To be able to promise investors a guaranteed yield on its securities without over exposing itself to risk, Farmer Mac includes a prepayment penalty in the terms of the loans it pools.

Yield maintenance is the most common prepayment penalty used by Farmer Mac and assesses the mortgagor a penalty such that the security holder is made "whole" in terms of the expected cash flows over the life of the loan. The yield maintenance prepayment penalty used by Farmer Mac is given by

$$\text{(5)} \qquad \theta_{ym}[r(t), t, \tau] = \max\left\{\eta F(t),\ F(t)(r^* - R) \times \frac{[1 - e^{-R(\tau - t)}]}{R}\right\}, \quad \forall\, \tau < T$$

where η is equal to 1 percent, and $R \equiv R(r, t, \tau)$ is the yield on the interpolated Treasury Constant Maturity maturing on the "yield maintenance date" which is denoted by τ.[7] Notice $\tau < T$ because in practice, the "yield maintenance date" occurs (six months) before loan maturity.

The economy that supports the assumed spot-rate dynamics also allows for a complete characterization of the term structure. That is,

[7] Equation (5) is actually the continuous time analogue of the discrete time yield maintenance penalty equation Farmer Mac uses.

bonds of any maturity can be priced under the assumptions laid out by Cox, Ingersoll, and Ross (1985b) and these prices can then be used to infer the corresponding yield needed in equation (5). The time t price, $P(r, t, \tau)$, of a bond maturing at τ is $P(r, t, \tau) = A(t, \tau) \exp[-B(t, \tau)r(t)]$ where $A(t, \tau)$ and $B(t, \tau)$ are coefficient functionals given by equation (23) of Cox, Ingersoll, and Ross (1985b). The yield-to-maturity, $R(r, t, \tau)$, for bonds priced in this manner is given by equation (25) of Cox, Ingersoll, and Ross (1985b), namely,

$$\text{(6)} \qquad R(r, t, \tau) = \frac{[rB(t, \tau) - \log A(t, \tau)]}{(\tau - t)}.$$

Intuitively, yield maintenance is designed to capture the present value of the interest that the investor forgoes as a result of the prepayment. It does appear this penalty overstates the actual interest lost over the loan's life because of the fully amortizing feature of the loan. Also note that this type of prepayment penalty is a function of the r and t state variables, but not L.

A more recent development is partial open prepayment loans, which Farmer Mac introduced in 1998. Under this plan, the mortgagor pays a prepayment penalty for an initial period of the loan's life, after which no prepayment penalty is assessed. The structure currently in use assesses a declining penalty for the first two and a half years,

$$\text{(7)} \qquad \theta_{pp}(t) = \begin{cases} \delta_1 F(t) & 0 < t \leq t_1 < T \\ \delta_2 F(t) & t_1 < t \leq t_2 < T \\ \delta_3 F(t) & t_2 < t \leq t_3 < T \\ 0 & t_3 < t \leq T \end{cases}$$

where t_1 represents the time of the first scheduled payment, t_2 is one year after t_1, and t_3 is two years after t_1. Additionally, δ_j, $j = 1, 2, 3$ represents the percentage of $F(t)$ that is paid in the form of a penalty. Currently, $\delta_1 = 9$ percent, $\delta_2 = 8$ percent, and $\delta_3 = 7$ percent for Farmer Mac partial open prepayment loans.

The Empirical Model to Value AMBS[8]

Several discretization techniques may be used (in lieu of an analytic solution) to solve PDEs like equation (3) such as finite differencing or simulation. Numerical integration or differencing is the most common method [see e.g. Dunn and McConnell (1981); Brennan and Schwartz (1985); Kau *et al.* (1992, 1995)]. However, the presence of multiple state variables coupled with frequent embedded early exercise opportunities greatly complicates the implementation of a differencing methodology [Schwartz and Torous (1989)]. Therefore, a combination Monte Carlo simulation/dynamic programming approach was developed to solve the PDE (3) and value the AMBS.

Simulation Methods and Dynamic Programming

Monte Carlo simulation is often used to price options and other derivative securities [see e.g. Boyle (1977); Schwartz and Torous (1989); Boyle, Broadie, and Glasserman (1997)]. Broadie and Glasserman (1997) present the state of the art in numerical option pricing and also appear to have pioneered the most contemporary pricing technique. In their approach, they utilize simulation combined with dynamic programming to develop two estimates of the price of an American stock option. This methodology simulates a non-recombining lattice of stock prices and then proceeds backward through a portion of the lattice to determine an optimal exercise policy and two current values of the option. The two option price estimates, one of which is biased high while the other is biased low, are proven to be asymptotically consistent estimators of the "true" option price.

One problem with this methodology is the excessive storage requirements necessary to

[8] In this section, time is denoted with subscripts rather than the previous convention to highlight the difference between the continuous time analytic model and the discrete time empirical model used to solve the analytic model.

implement the technique.[9] The approach is more appropriate for Bermudan options than for the problem at hand because of its reliance on the generation of all the paths for the state variables before the application of the backward recursion required of the dynamic programming algorithm. One feasible way to circumvent the storage problem is to utilize a path-wise simulation method, which trades storage for computation time. In this approach, state variable paths are simulated stochastically one at a time and the method of dynamic programming is applied to each simulated path to generate one current value of the AMBS. This process is repeated a large number of times and the value of the AMBS is determined by calculating the average of the current values. This average will converge to the true value given that the distributional and other assumptions of the model hold. This is the approach implemented to solve equation (3).

The diffusions that are simulated when pricing AMBS are the discretized versions of the term-structure-diffusion equation and risk-neutralized land-value-diffusion equation. The risk-neutralized land-value-diffusion equation is determined in the usual way by finding an appropriate change of measure for the land value diffusion. Such a change in measure is easily obtained given the Cox, Ingersoll, and Ross economy and the assumption that land values follow geometric Brownian motion.

Time-steps in the empirical model are set at 1/12 which is consistent with monthly realization of the stochastic elements of the model. Functions developed by Press *et al.* (1993) are used to draw two correlated standard normal variates (one for the spot rate and one for land values) and two Poisson random variates (one each for sub-optimal prepayment and default) at each time step. After one complete set of time paths has been simulated, one possible time zero value of the value of the AMBS is calculated by backing up along this set of paths and applying the dynamic programming algorithm.

The mortgagor makes two decisions at each time point—whether or not to optimally prepay or optimally default. Because exercise of each option is triggered by different conditions, it is necessary to implement a hierarchy to check for optimal exercise. At each point the decision to optimally prepay is examined first. Optimal prepayment is governed by boundary conditions, though in the context of the empirical model optimal prepayment will occur if $r^* > r_t + \xi + \zeta + \gamma_t$, where ξ is the percentage loan markup, ζ is the percentage cost of refinancing, and γ_t is the percentage cost of the prepayment penalty. Recall that prepayment penalties are determined via the yield maintenance or partial-open prepayment equations and are measured in dollar terms. Therefore, prepayment penalties must be converted into their basis point equivalent. The conversion is accomplished by amortizing the prepayment penalty and remaining loan balance over the remaining number of periods and determining an equivalent basis point cost of the penalty.

If it is optimal for the mortgagor to prepay at time t, the value of the AMBS is

$$(8)\qquad V_t = \begin{cases} F_t + C + \theta_{i,t} & \text{if a payment is scheduled at } t \\ F_t + \psi_t + \theta_{i,t} & \text{otherwise} \end{cases}$$

where the variable ψ_t measures accrued interest from the time of the previous payment and t. Accrued interest is necessary because it is assumed there are monthly exercise opportunities, which differs from the frequency of payments (i.e. payments are semi-annual for Farmer Mac mortgages).

If optimal prepayment is unwarranted, the next decision to consider is whether to opti-

[9] As Broadie and Glasserman (1997) point out, their technique is exponential in the number of exercise opportunities. If four state variable paths are simulated with monthly exercise opportunities for 30 years (as might be the case when pricing home loans and assuming that optimal prepayment and default are monthly occurrences), the number of terminal nodes will be on the order of 1.679×10^{10}. In addition, the total number of values that must be stored is even greater because the entire lattice must be saved for the dynamic programming application. If each value is stored as a (single precision) floating point variable with a storage requirement of 8 bytes, it is apparent that the memory and storage requirements for this methodology quickly make it impracticable (approximately 128 gigabytes to store just the terminal nodes).

mally default. If optimal default occurs, the value of the AMBS is simply $V_t = F_t$. Technically, Farmer Mac does try to collect a prepayment penalty in the event of default. However, the actual incidence of penalty collection is low enough that this can be ignored. Additionally, optimal default should only occur in a month in which a payment is due because the mortgagor will try to maintain control of the asset as long as possible before defaulting.

If neither optimal prepayment or optimal default occur, the final conditions to check for are sub-optimal prepayment and default. Sub-optimal prepayment occurs if the Poisson random variable is equal to 1 at time t. If sub-optimal prepayment occurs, the value function is the same as equation (8). Likewise, sub-optimal default will occur if the Poisson random variable is equal to 1. The value function under sub-optimal default is also the same as that of optimal default, namely, $V_t = F_t$.

If neither of the mortgagor's options are exercised and sub-optimal prepayment or default does not occur, the scheduled payment is passed through to the AMBS investor and the loan is continued. In this case, the value of the AMBS is given by the dynamic programming recursive relation $V_{t-\Delta t} = C + V_t/(1 + r_{t-\Delta t})$. Intuitively, this relation represents the continuation value of the mortgage. Notice also that the notation $V_{t-\Delta t}$ explicitly shows the backward recursive nature of the dynamic programming algorithm and allows for a non-stochastic implementation of the algorithm because the path of each state variable is stochastically simulated before the algorithm is applied. Successful implementation of the path-wise simulation/dynamic programming approach allows for a numerical approximation to V by generating a distribution of AMBS values at all points in time.

Recall that the simulation/dynamic programming approach detailed here was designed to circumvent some of the problems associated with storage intensity by trading storage for computation time. It should be noted that numerically approximating V in the manner suggested is still no small task. High initial interest rate scenarios can take over 180 minutes to determine a mean value of V at time zero on a Pentium II with a 450-mHz processor.[10]

Sub-optimal Probabilities

The functional forms of sub-optimal prepayment and default can take many forms. Dunn and McConnell use Federal Housing Authority (FHA) experience to characterize the frequency of sub-optimal prepayment. Later work, such as that by Kau *et al.* (1992) and Hanson and Chhikara (1993), uses Public Securities Association (PSA) experience to represent nonfinancial termination. PSA experience seeks to capture the reduced level of prepayment by mortgagors early in the life of a loan while allowing for higher probability of prepayment as time passes. Use of PSA experience to represent sub-optimal prepayment in an agricultural setting probably misrepresents the incidence of sub-optimal prepayment because PSA experience is derived from (primarily monthly) residential mortgage prepayment data. Also, as noted by Brennan and Schwartz (1985), PSA does not distinguish between optimal and sub-optimal prepayment which necessarily implies PSA overstates the frequency of sub-optimal prepayment.

In agriculture, the ability of a land owner to service a mortgage for agricultural real estate is heavily tied to the financial performance of the mortgaged asset. This idea is also consistent with commercial and Farm Credit Association lenders' preferences for self-liquidating loans. Sub-optimal default is inevitable if conditions in the agricultural economy (i.e. low commodity prices and/or low commodity yields) are poor. Similarly, favorable conditions in the agricultural economy can bring about significant income in a given year such

[10] High initial interest rates are computationally intensive because the spot rate diffusion implies interest rates will gravitate toward their long-term mean value. As such, the spot rate falls over time, implying more potential for prepayment. To determine whether prepayment under yield maintenance should occur, forward rates must be determined and the prepayment penalty must be converted to a basis point equivalent, both of which add significantly to the computation time.

that the probability of (sub-optimal) prepayment is increased.[11]

In the present model we link the probability of sub-optimal prepayment and default to conditions in the agricultural economy by specifying ϕ^p and ϕ^d to be functionally related to the financial performance of the mortgaged asset through νL, the income the mortgagor receives from the use of the asset in an agricultural capacity. While such a linkage is plausible, it offers the advantage of marginal complexity. That is, no additional state variables need to be specified and the model does not become more complex than it presently is.

While any functional form could be used, for simplicity the probability of prepayment is assumed to be a linear function of the difference between actual and expected income flow. Therefore, the probability of sub-optimal prepayment is given by

$$\text{(9)} \qquad \phi_t^p = \phi_0^p + \phi_1^p[\nu L_t - E_t^Q(\nu L_t)]$$
$$\phi_0^p, \phi_1^p > 0,$$

where ϕ_0^p and ϕ_1^p are constants. Given the assumed parameter signs in equation (9), the probability of sub-optimal prepayment increases as the actual flow of income exceeds expectations.

It remains to define the nature of the expected income flow, $E^Q(\nu L_t)$, appearing in equation (9). One way to specify the term is to take the expectation of the risk-neutralized diffusion equation for the residual return to land which yields $E^Q(dL) = (r - \nu)Ldt$. This result can be viewed as a first-order, linear, ordinary differential equation with variable coefficients (given the expectation E^Q). An integral representation of a solution to this equation is

$$\text{(10)} \qquad E^Q[L(t)] = L(0)\exp\left\{\int_0^t [r(s) - \nu]\, ds\right\}$$

assuming that the initial land value equals $L(0)$. Given the expectation then, the income flow at t depends only on the initial land value and the spot rate path up to t.

Because ν is a constant, we have $E^Q[\nu L(t)] = \nu E^Q[L(t)]$. Using this result and discretizing equation (10) results in

$$\text{(11)} \qquad \nu E_t^Q(L_t) = \nu L_0 \prod_{s=0}^{t} (1 + r_s - \nu).$$

Thus, in risk-neutral terms the expected income flow at t is simply the initial (time zero) income flow compounded at the difference between the spot rate and the rate of income flow for t periods (months). Because the dynamic programming algorithm requires a recursive relationship at each point, equation (11) is implemented in the empirical model as $E_t^Q(\nu L_{t+\Delta t}) = \nu L_t(1 + r_{t+\Delta t} - \nu)$ at each point.

Sub-optimal default is also assumed to be dependent on the difference between actual and expected income flow. The functional form of the sub-optimal default function is similar to that specified in (9), namely

$$\text{(12)} \qquad \phi_t^d = \phi_0^d - \phi_1^d[\nu L_t - E_t^Q(\nu L_t)]$$
$$\phi_0^d, \phi_1^d > 0,$$

where ϕ_0^d and ϕ_1^d are constants. Thus, the probability of sub-optimal default increases as expected income flow exceeds actual income flow. The numerical implementation of equation (12) is carried out in an analogous manner to that of equation (9).

Data

Term structure parameters used is the analysis are estimated using the procedure suggested by Nowman using monthly yield data made available by the Federal Reserve for U. S. Treasury Constant Maturity securities for the period April 1953 to July 2000 (566 observations). The estimation reveals κ equals 0.007773, σ^2 equals 0.000257 and μ equals 6.9183 percent. The presence of β, ν, and ρ in equation (3) also necessitates an estimate of the volatility of land values, the rate of income flow, and the correlation coefficient between land values and interest rates. The parameter

[11] It should be noted that, for simplicity, we ignore delinquency and curtailment even though these are more apt to precede outright default and prepayment in the manner suggested.

β was also estimated using Nowman's technique while techniques suggested by Gemmill were used to estimate ν and ρ from cash rent data published by the USDA ERS for 1967 to 1994 (28 annual observations). Cash rent is assumed to proxy the income flow the mortgagor could receive (or actually does receive in the case of an absentee owner) if the land were rented. The estimation reveals β equals 13.4566 percent annually, ν equals 4.0076 percent annually, while ρ equals -0.0542.

One of the big unknowns for investors of Farmer Mac securities is borrower prepayment behavior for agricultural mortgages and Farmer Mac continues to work to help resolve this issue. Data are not available for the estimation of the parameters of the sub-optimal functions given by equations (9) and (12) so these values are assumed. Empirically, the linear probability model describing sub-optimal prepayment and default are

$$(13) \quad \phi_t^p = \begin{cases} 0.00 & [\nu L_t - E(\nu L_t)] < -1.00 \\ 0.02 + 0.02 \times [\nu L_t - E(\nu L_t)] & -1.00 \leq [\nu L_t - E(\nu L_t)] \leq 2.00 \\ 0.06 & 2.00 < [\nu L_t - E(\nu L_t)] \end{cases}$$

and

$$(14) \quad \phi_t^d = \begin{cases} 0.00 & [\nu L_t - E(\nu L_t)] > 0.12 \\ 0.02 - 0.12 \times [\nu L_t - E(\nu L_t)] & -4.00 \leq [\nu L_t - E(\nu L_t)] \leq 0.12 \\ 0.50 & [\nu L_t - E(\nu L_t)] < -4.00 \end{cases}$$

As the empirical model prices AMBS per \$100 of outstanding loan balance, the actual numerical values used in equations (13) and (14) are less tangible than might be expected. For illustrative purposes, Figure 1 shows an example of the sub-optimal prepayment and default probability functions generated by (13) and (14) given the assumed parameter values.

The functional forms for the probabilities of sub-optimal prepayment and default contrast to prepayment and default probabilities presented by Schwartz and Torous (1992). There, the authors define the probability of prepayment to be zero when there is a positive probability of default, and vice versa. The specifications used herein permit the coexistence of positive probabilities of sub-optimal prepayment and default, but generally the incidence of each is indirectly related. For example, when the difference between actual and expected income flow is equal to zero, there is a 2-percent probability of sub-optimal prepayment and a 2-percent probability of sub-optimal default. As the difference increases (decreases), the probability of sub-optimal prepayment increases (decreases) while the probability of sub-optimal default decreases (increases). The coexistence of positive probabilities of sub-optimal prepayment and default is realistic in agricultural given the volatility of agriculture income.

In terms of the mortgage, the initial loan balance is assumed to be \$100 while the loan mark-up (ξ) is assumed to be 200 basis points. Such a spread is typical of most agricultural mortgages. Consistent with convention, refinancing costs (ζ) are assumed to be 50 basis points [see e.g. Bhattacharya and Koren (1998)] and the specific mortgage analyzed is a 20-year, fixed-rate mortgage with constant, semi-annual payments.

Results and Discussion

Presented in Tables 1 and 2 are AMBS prices per \$100 of outstanding loan balance at time zero under alternative spot prices, land values, prepayment penalties, and prepayment and default assumptions. The main difference between the two tables is that Table 1 represents values when equations (13) and (14) are used for sub-optimal mortgage termination while those of Table 2 are for PSA-based sub-optimal prepayment and a fixed 3-percent probability of sub-optimal default (included for comparison purposes). Not surprisingly, the results are nearly identical because the linear probability model, by construction, induces behavior similar to that of PSA prepayment.

As shown in Tables 1 and 2, the value of AMBS is an increasing function of the spot rate of interest when prepayment penalties are in place. This is because in the event of pre-

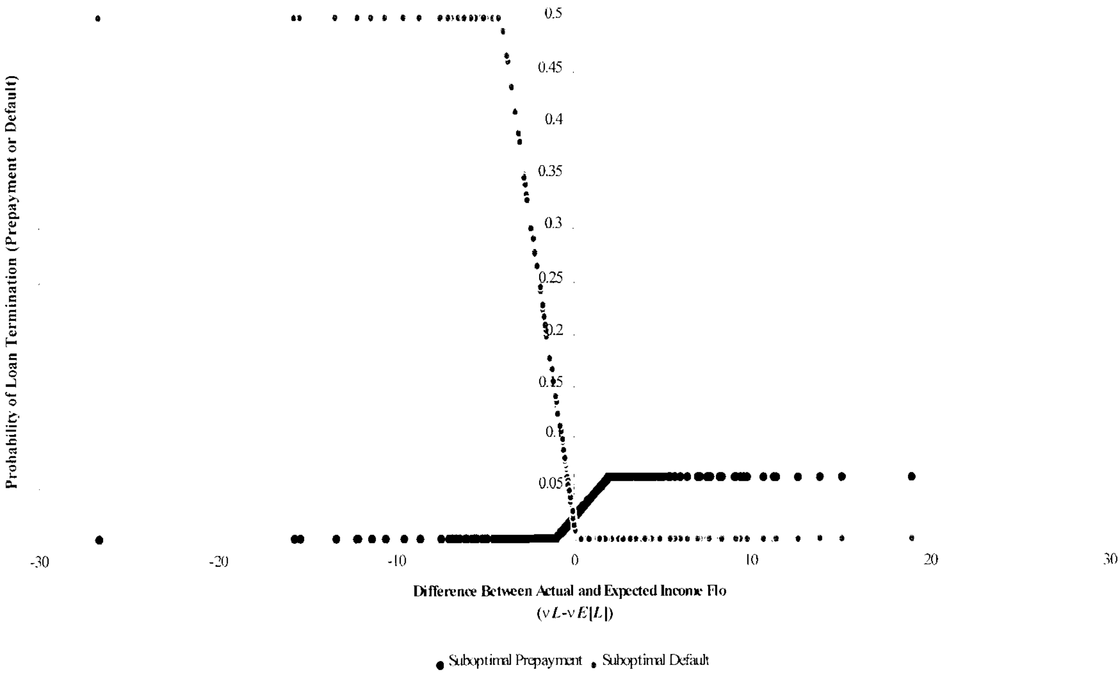

Figure 1. Suboptimal prepayment and default function behavior in relation to the difference between actual and expected income flow

payment (which is more readily induced by high initial spot rates), both yield maintenance and partial-open prepayment compensate the investor for lost interest income. With the exception of low initial spot rates, yield maintenance also ensures a higher AMBS value when compared to partial-open prepayment. This result was anticipated given that yield maintenance is in place until six months before maturity while partial-open prepayment imposes no prepayment penalty after the first two and one-half years. Given the assumed term structure, when the initial spot rate is below the long-term mean rate, the yield curve is upward sloping, implying the prepayment penalty for yield maintenance will always be less than that for partial-open prepayment loans [see equations (5) and (7)].

As the initial spot rate increases above the long-term mean rate, there is more and more downward pressure on rates which means there is potentially more and more incentive for optimal prepayment (which most often triggers a penalty—especially under yield maintenance). A situation when no prepayment penalty is in place is also presented in Tables 1 and 2 and graphed in Figure 2 for illustrative purposes. When no penalty is in place, the value of AMBS are generally a decreasing and convex function of the spot rate. This is also because there is continually more and more incentive to optimally prepay as the initial spot rate increases above the long-term mean rate, but there is no penalty in place to insure the investor against such an occurrence and thereby increase the security's value.

Also shown in Tables 1 and 2 is the sensitivity of the AMBS value to the initial land price. Higher initial land prices imply lower loan-to-value ratios and higher income flow, both of which lower the probability of default (optimal and sub-optimal). However, as a practical matter, it appears that initial land values have limited impact on the value of the AMBS on a per $100 of initial loan balance basis. This is likely because unlike prepayent, default is rarely an inevitable conclusion given the down payment required and assumed probabilities.

Table 3 presents the yields associated with

Table 1. Means and Standard Errors of the Value of Farmer Mac AMBS Per $100 of Initial Loan Balance Under Alternative Initial Spot Rates, Land Values, and Prepayment Penalties with Income Department Probability of Suboptimal Prepayment and Default

	No Penalty			Yield Maintenance			Partial-Open Prepayment		
$L(0)$	$125	$150	$200	$125	$150	$200	$125	$150	$200
$r(0)$									
0.50%	100.2342	100.3271	100.8717	99.7005	99.7488	99.9674	100.5892	100.6626	101.0251
	0.0963	0.0783	0.1325	0.0159	0.0190	0.0275	0.1207	0.1024	0.1770
1.50%	100.4993	101.8259	105.5052	99.7457	100.1633	100.5108	100.9060	101.2770	101.2354
	0.1318	0.1650	0.3509	0.0502	0.0554	0.0934	0.2476	0.3295	0.4486
2.50%	100.9625	103.7257	108.1987	100.0458	100.6227	100.5873	101.3029	101.0484	99.6823
	0.2701	0.2641	0.4688	0.0910	0.0646	0.1526	0.4789	0.5351	1.0323
3.50%	100.7213	103.8504	105.8982	100.3886	100.9878	100.9218	101.5561	101.3998	100.3608
	0.3417	0.3905	0.6662	0.1101	0.1116	0.1467	0.4260	0.7269	0.9789
4.50%	100.1091	102.2129	102.9433	101.3015	101.8641	101.9660	102.3628	102.0742	101.8846
	0.2807	0.4282	0.5210	0.1122	0.1748	0.1421	0.8258	0.6294	0.8326
5.50%	99.0103	100.2663	100.4994	104.5676	105.4548	105.5017	103.3006	103.5431	103.4072
	0.5005	0.3450	0.4268	0.2498	0.3372	0.2335	0.4721	0.3801	0.6809
6.50%	98.1301	98.8625	98.9272	107.8466	108.9743	109.0396	104.3134	104.3663	104.2242
	0.2515	0.3766	0.3697	0.3280	0.4067	0.4043	0.6404	0.4396	0.5803
7.50%	97.3022	97.6071	97.6640	111.1795	111.8882	112.0043	104.9743	105.0817	105.1260
	0.3493	0.3099	0.3136	0.4560	0.7183	0.6609	0.3415	0.4730	0.5319
8.50%	96.5071	96.6782	96.6777	113.7201	114.3060	114.2451	105.7692	105.8830	105.8678
	0.3014	0.3933	0.4347	0.5923	0.5299	0.5995	0.3649	0.4283	0.3712
9.50%	95.8947	96.0554	96.0960	116.1242	116.1606	116.3766	106.1688	106.2723	106.0626
	0.2021	0.1990	0.2642	0.8445	0.6631	0.8150	0.3836	0.2192	0.2924
10.50%	95.5628	95.5627	95.5126	117.9497	117.7632	118.1661	106.4244	106.5478	106.5009
	0.3135	0.2689	0.2747	1.0365	0.9674	0.8641	0.1684	0.2674	0.3333
11.50%	95.1997	95.2772	95.2580	119.5319	119.5410	119.1680	106.6330	106.6836	106.6422
	0.2416	0.2832	0.2660	1.2428	0.9264	1.3999	0.2378	0.2718	0.2212
12.50%	94.8068	94.9631	94.8948	120.6168	120.7209	120.9233	106.8085	106.8899	106.7900
	0.1967	0.2921	0.2426	1.3495	0.9317	1.1309	0.2387	0.2144	0.2977
13.50%	94.7832	94.7678	94.7734	122.1855	121.7935	122.0073	106.9340	106.8804	106.9767
	0.3481	0.2928	0.2470	0.9600	1.1037	1.0585	0.1788	0.2561	0.1441
14.50%	94.5741	94.6357	94.6876	122.5290	123.2139	122.7105	106.9517	106.9669	106.9856
	0.3001	0.2164	0.2218	1.5954	1.7118	1.1070	0.2308	0.1977	0.2259

Note: The top entry in each cell is the mean AMBS value while the number below is its standard error.

Table 2. Means and Standard Errors of the Value of Farmer Mac AMBS Per $100 of Initial Loan Balance Under Alternative Initial Spot Rates, Land Values, and Prepayment Penalties with PSA Suboptimal Prepayment and Fixed 3% Probability of Suboptimal Default

	No Penalty			Yield Maintenance			Partial-Open Prepayment		
$L(0)$	$125	$150	$200	$125	$150	$200	$125	$150	$200
$r(0)$									
0.50%	99.8559	99.9962	100.6199	99.6433	99.7119	99.9518	99.9349	100.0551	100.4745
	0.0817	0.0897	0.1491	0.0139	0.0191	0.0298	0.0881	0.0874	0.1499
1.50%	100.3459	102.2331	107.6066	99.7284	100.2676	101.0236	100.2359	100.8669	101.0887
	0.1506	0.1841	0.3553	0.0429	0.0449	0.0676	0.1730	0.2123	0.5172
2.50%	101.1811	104.9300	110.8675	100.1446	100.9585	101.5099	100.7324	101.1445	100.3157
	0.2601	0.3729	0.6046	0.0673	0.0848	0.0694	0.3100	0.6481	0.8900
3.50%	101.2313	105.1332	108.3563	100.5319	101.2970	101.6256	101.2648	101.2612	101.1574
	0.4469	0.4708	0.6280	0.1265	0.0816	0.0930	0.5890	0.8727	0.9068
4.50%	100.5963	103.5594	104.6163	101.1838	102.0196	101.9660	102.2285	102.1524	101.7493
	0.3348	0.4323	0.3905	0.1345	0.1613	0.1421	0.6770	0.8307	0.7437
5.50%	99.5044	101.3780	101.8085	105.1429	106.0293	106.0804	103.0246	103.1232	102.8263
	0.2807	0.4104	0.4433	0.3286	0.2697	0.2774	0.5495	0.8062	0.9386
6.50%	98.5838	99.5225	99.5861	100.1556	110.3334	110.0784	103.8397	104.0527	104.1625
	0.3914	0.4335	0.5195	0.4124	0.4108	0.4853	0.5407	0.6015	0.6919
7.50%	97.6019	98.0678	97.9775	112.8285	113.4003	113.6081	104.8432	105.0850	104.6490
	0.3154	0.4719	0.3329	0.4064	0.6042	0.4208	0.5612	0.6124	0.5824
8.50%	96.7398	96.8765	97.0301	115.6916	116.1087	116.3723	105.2491	105.5937	105.5441
	0.2874	0.2898	0.3527	0.8029	0.6492	0.8016	0.4341	0.3840	0.4648
9.50%	96.0998	96.1927	96.2686	118.2646	119.0097	118.8993	105.7192	105.7675	106.0083
	0.3458	0.2788	0.2239	1.1114	0.8981	0.9965	0.4371	0.4800	0.4463
10.50%	95.5021	95.6673	95.6521	120.4844	120.3728	120.3270	106.2608	106.2048	106.2532
	0.1844	0.2790	0.2274	1.0064	0.9465	0.8862	0.3180	0.3086	0.2799
11.50%	95.0896	95.0323	95.0935	122.4189	122.1544	122.7070	106.4290	106.6004	106.5116
	0.3379	0.2364	0.2797	0.9410	0.7869	1.1391	0.3310	0.2988	0.2597
12.50%	94.7100	94.6994	94.6490	123.8943	123.7844	123.9698	106.7653	106.7099	106.7198
	0.3133	0.2218	0.2569	0.9699	1.3618	0.9972	0.1722	0.2217	0.2591
13.50%	94.5019	94.4735	94.4263	125.3100	125.5823	125.1883	106.8557	106.7971	106.8001
	0.3148	0.2465	0.3426	1.2359	1.4287	1.4326	0.2194	0.3173	0.2760
14.50%	94.3177	94.3507	94.1701	126.3861	126.7400	126.4023	106.8411	106.8042	106.8732
	0.1998	0.1560	0.2271	1.4726	1.2711	1.2583	0.2934	0.1980	0.2797

Note: The top entry in each cell is the mean AMBS value while the number below is its standard error.

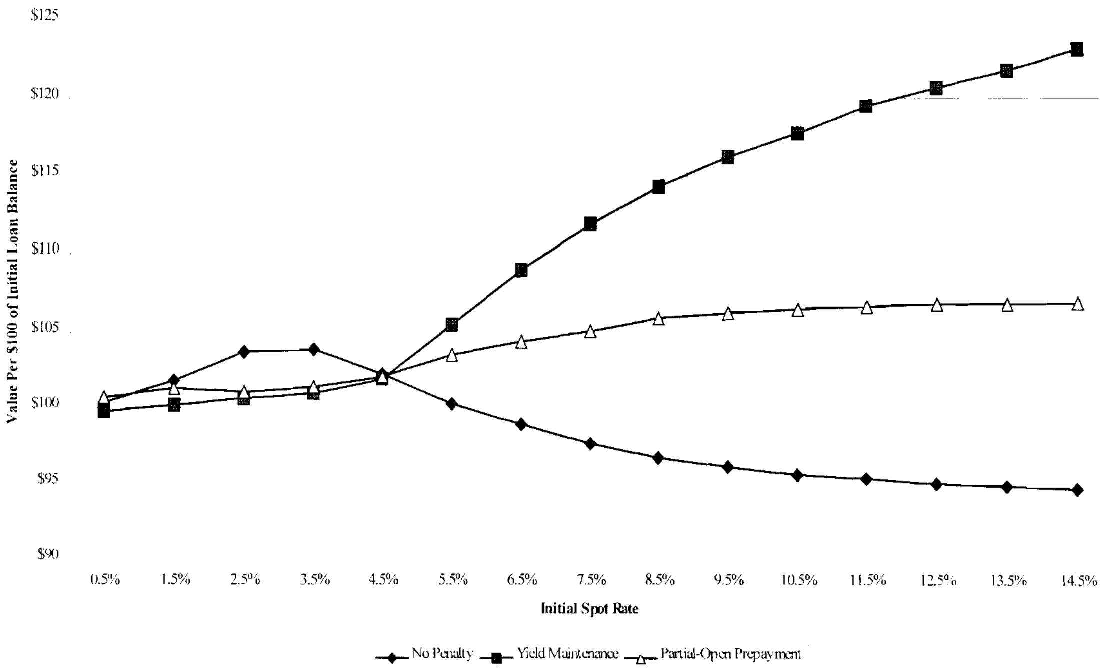

Figure 2. The value of Farmer Mac AMBS for alternative initial spot rates and prepayment penalties

Table 3. Mean Annual Yields on Farmer Mac AMBS Per $100 of Initial Loan Balance Under Alternative Initial Spot Rates, Land Values, and Prepayment Penalties

	No Penalty			Yield Maintenance			Partial-Open Prepayment		
$L(0)$	$125	$150	$200	$125	$150	$200	$125	$150	$200
$r(0)$									
0.50%	2.50%	2.49%	2.45%	2.54%	2.53%	2.52%	2.47%	2.47%	2.44%
1.50%	3.49%	3.38%	3.10%	3.55%	3.52%	3.49%	3.46%	3.43%	3.43%
2.50%	4.47%	4.24%	3.88%	4.55%	4.50%	4.50%	4.44%	4.46%	4.58%
3.50%	5.51%	5.23%	5.05%	5.54%	5.48%	5.49%	5.43%	5.45%	5.54%
4.50%	6.59%	6.39%	6.32%	6.48%	6.42%	6.41%	6.38%	6.40%	6.42%
5.50%	7.75%	7.61%	7.59%	7.17%	7.09%	7.08%	7.30%	7.28%	7.29%
6.50%	8.90%	8.81%	8.80%	7.85%	7.73%	7.73%	8.21%	8.20%	8.22%
7.50%	10.06%	10.02%	10.01%	8.49%	8.42%	8.41%	9.15%	9.14%	9.13%
8.50%	11.24%	11.22%	11.22%	9.19%	9.13%	9.14%	10.07%	10.05%	10.05%
9.50%	12.42%	12.39%	12.39%	9.89%	9.88%	9.86%	11.03%	11.01%	11.04%
10.50%	13.57%	13.57%	13.58%	10.63%	10.65%	10.60%	12.00%	11.98%	11.99%
11.50%	14.74%	14.73%	14.73%	11.37%	11.37%	11.42%	12.98%	12.97%	12.98%
12.50%	15.93%	15.90%	15.92%	12.16%	12.15%	12.13%	13.97%	13.95%	13.97%
13.50%	17.07%	17.07%	17.07%	12.88%	12.93%	12.90%	14.96%	14.97%	14.95%
14.50%	18.25%	18.23%	18.22%	13.74%	13.65%	13.72%	15.97%	15.97%	15.96%

Note: These yields are based on prices reported in Table 1.

the values of AMBS presented in Table 1. As shown, yields are an increasing function of the spot rate and a slightly decreasing (or at least fairly constant) function of initial land values. Consider an initial spot rate of 8.50 percent and an initial land value of $150 with a corresponding loan-to-value ratio of two-thirds (i.e. $100/$150). Given the mortgage assumptions above, the loan would be made at a 10.50 percent contractual rate (spot plus markup). Yet the equilibrium price of the loan (conditional on the profit maximizing behavior of the mortgagor) implies the yield on AMBS laying claim to the cash flows of the loan is 11.22 percent when no prepayment penalties are in place. From Table 1, AMBS would sell at discount with no prepayment penalty; hence the yield is above 10.50 percent. The implication is that not having a prepayment penalty in place is an imperfect means of funding such a loan. This is because the equilibrium price of the security (conditional on the optimal prepayment and default behavior of the mortgagor) implies a yield that is actually higher than the contractual rate on the loan.

Such is not the case with prepayment penalties however. With prepayment penalties the yields are 9.13 percent with yield maintenance and 10.05 percent with partial open prepayment under the same scenario. The value to Farmer Mac of having prepayment penalties in place in this setting, then, is 209 basis points for yield maintenance and 117 basis points for partial open prepayment. These amounts can also be interpreted as amounts that Farmer Mac could offer to banks to pass on to mortgagors to make their loans more competitive and compensate borrowers for agreeing to a loan with a prepayment penalty.

More concise information regarding the value of prepayment penalties is presented in Table 4 which shows the embedded call option values to the mortgagor and the value of prepayment penalties to Farmer Mac under alternative initial spot rates and spot rate volatilities. For example, at 5.0-percent annual spot rate volatility and a 7.50-percent initial spot rate, the gross value of the embedded call option (the mortgagor's right to prepay) is $21.93 per $100 of initial loan balance. This value is calculated as the difference between the value of two (default-free) AMBS, one that can be prepaid without penalty and one that cannot be prepaid at all.

However, in reality this gross value is split between mortgagor and mortgagee when prepayment penalties are in place. With yield maintenance, the gross value to the mortgagor drops to $5.19 per $100 of initial loan balance because the difference of $16.75 (i.e. $21.93 − $5.19) is passed onto Farmer Mac when the prepayment penalty is in place. Similarly, the same spot-rate scenario indicates that under partial open prepayment the mortgagor's right to prepay is valued at $12.14 per $100 of initial loan balance while the value of having partial open prepayment in place to Farmer Mac is $9.80 per $100 of initial loan balance. Commensurate with traditional option pricing theory, the value of the embedded call increases with increases in the initial spot rate and volatility of the spot rate. Also important to note is the fact that prepayment penalties mitigate prepayment, but do not preclude it on average. Although not presented, depending on the initial spot rate mean prepayment times range between 18 months and three years.[12]

A similar analysis is possible regarding the embedded put option in mortgages, namely, optimal default. Table 5 presents values of the option to default under alternative land values and land value volatilities. As shown, the value of the mortgagor's option to default is a decreasing function of land value and an increasing function of land value volatility. By construction, high initial land values are associated with low initial loan-to-value ratios (high down payments) which is why some very low option prices are noted in Table 5. When initial land values are high, income flow is also high. Both imply a low probability of default that when coupled with low land value volatility leads to the low option prices. Because the incidence of default in such cases is

[12] The fact that yield maintenance does not preclude prepayment is an interesting result examined in the next section. Either the penalty itself is mis-specified and too small to preclude prepayment and/or the term structure itself is mis-specified.

Table 4. Value of Embedded Call (Prepayment) Option Under Alternative Spot Rates, Spot Rate Volatilities, and Prepayment Penalties

σ	$r(0)$	Gross Call Option Values	Call Option Value with Yield Maintenance	Call Option Value with Partial-Open Prepayment	Value of Yield Maintenance	Value of Partial-Open Prepayment
2.50%	4.50%	0.2760	0.0259	0.0218	0.2501	0.2542
	5.50%	2.3704	0.2195	0.7335	2.1509	1.6369
	6.50%	8.9237	0.8244	4.5530	8.0993	4.3707
	7.50%	17.1899	1.9603	10.7354	15.2296	6.4545
	8.50%	24.1112	3.6592	16.2487	20.4520	7.8625
	9.50%	30.0738	6.0292	20.9239	24.0446	9.1499
5.00%	4.50%	4.9518	0.9020	0.9270	4.0498	4.0248
	5.50%	10.3170	2.4125	3.7142	7.9045	6.6028
	6.50%	16.1937	4.6014	7.7353	11.5923	8.4584
	7.50%	21.9342	5.1873	12.1389	16.7469	9.7953
	8.50%	27.3904	8.3947	16.5865	18.9957	10.8039
	9.50%	32.5442	9.7257	21.0341	22.8185	11.5101
10.00%	4.50%	11.7568	2.8053	1.8696	8.9515	9.8872
	5.50%	17.2576	5.5552	5.6781	11.7024	11.5795
	6.50%	22.7265	9.0032	10.0325	13.7233	12.6940
	7.50%	28.1101	12.6259	14.5898	15.4842	13.5203
	8.50%	33.3276	16.3169	19.2322	17.0107	14.0954
	9.50%	38.4118	20.0190	23.8841	18.3928	14.5277

Note: A Land value equal to $150 per $100 of initial loan balance is assumed.

Table 5. Value of Embedded Put (Default) Options Under Alternatives Land Values and Land Value Volatilities

	β		
$L(0)$	6.75%	13.50%	27.00%
$125	0.7386	0.7386	5.9089
$140	0.1388	0.1388	4.0599
$155	0.0183	0.0183	2.7941
$170	0.0008	0.0008	1.9983
$185	0.0002	0.0002	1.4305
$200	0.0000	0.0000	1.0424

Note: An initial spot rate equal to the long-term mean of 6.92% is assumed.

extremely rare, the option has low or no value to the mortgagor.

Precluding Prepayment

In this section the incidence of optimal prepayment in spite of prepayment penalties is investigated. To conduct the analysis, a simulation model of mortgage prepayment was developed. A 20-year mortgage with semi-annual payments was assumed with an initial loan-to-value ratio of two-thirds, loan markup of 2 percent, and refinancing cost of 0.50 percent. The spot rate diffusion path over the life of the loan was simulated 20,000 times and, at each scheduled payment, equation (5) or (7) was applied to determine the relevant prepayment penalty. The penalty was converted to a basis point cost (γ_t) and added to the loan markup (ξ) and refinancing cost (ζ) to determine a hurdle or trigger rate necessary to induce prepayment. Recall that optimal prepayment occurs whenever $r^* > r_t + \xi + \zeta + \gamma_t$.

Table 6 presents the probability of optimal prepayment during the first three years of the loan under yield maintenance and partial open prepayment penalties for alternative initial spot rates and spot rate volatilities. As shown, low initial spot rates rarely induce optimal prepayment over the first three years of the loan for any level of spot rate volatility. However, as the initial spot rate increases and/or the volatility of the spot rate increases, an increased incidence of optimal prepayment is ob-

Table 6. The Conditional Probability of Optimal Prepayment During the First Three Years of the Life of a Loan for Alternative Initial Spot Rates and Spot Rate Volatilities

		Yield Maintenance					*Partial Open Prepayment*		
		$r(0)$ = 4%	$r(0)$ = 7%	$r(0)$ = 10%			$r(0)$ = 4%	$r(0)$ = 7%	$r(0)$ = 10%
σ = 5%	*6 mos.*	0.0000	0.0000	0.0000	σ = 5%	*6 mos.*	0.0000	0.0000	0.0012
	12 mos.	0.0000	0.0000	0.0024		*12 mos.*	0.0000	0.0001	0.1015
	18 mos.	0.0000	0.0001	0.0267		*18 mos.*	0.0000	0.0001	0.3140
	24 mos.	0.0000	0.0002	0.1067		*24 mos.*	0.0000	0.0012	0.6060
	30 mos.	0.0000	0.0002	0.2390		*30 mos.*	0.0000	0.0016	0.7534
	36 mos.	0.0000	0.0005	0.3811		*36 mos.*	0.0000	0.1187	0.9896
σ = 10%	*6 mos.*	0.0001	0.0029	0.0089	σ = 10%	*6 mos.*	0.0000	0.0045	0.0677
	12 mos.	0.0000	0.0115	0.0693		*12 mos.*	0.0000	0.0266	0.2681
	18 mos.	0.0000	0.0237	0.1723		*18 mos.*	0.0000	0.0427	0.4104
	24 mos.	0.0000	0.0326	0.2764		*24 mos.*	0.0000	0.0665	0.5658
	30 mos.	0.0000	0.0397	0.3721		*30 mos.*	0.0000	0.0792	0.6515
	36 mos.	0.0000	0.0454	0.4555		*36 mos.*	0.0000	0.3001	0.8899
σ = 20 %	*6 mos.*	0.0013	0.0281	0.0493	σ = 20%	*6 mos.*	0.0002	0.0396	0.1577
	12 mos.	0.0007	0.0655	0.1645		*12 mos.*	0.0001	0.0995	0.3488
	18 mos.	0.0004	0.0903	0.2690		*18 mos.*	0.0001	0.1320	0.4570
	24 mos.	0.0002	0.1147	0.3572		*24 mos.*	0.0001	0.1708	0.5623
	30 mos.	0.0002	0.1284	0.4305		*30 mos.*	0.0001	0.1853	0.6186
	36 mos.	0.0001	0.1372	0.4924		*36 mos.*	0.0010	0.3859	0.8092

Note: An initial land value of $150 per $100 of initial loan balance is assumed.

served.[13] The fact that yield maintenance does not preclude optimal prepayment is consistent with research on commercial mortgages by Lefcoe (1999).

Similar results are noted for partial open prepayment loans. Notice the relatively high probability of prepayment in month 36 (i.e. the 6th payment on the loan). Recall that partial open prepayment loans only have prepayment penalties during the first two and one-half years of the loan which implies that the 6th payment is the first time when prepayment is not penalized. Interestingly, increased spot-rate volatility appears to affect the probability of optimal prepayment differently depending on the prepayment penalty involved. For example, for an initial spot rate of 10 percent, increasing the spot rate volatility increases the probability of prepayment under yield maintenance. However, under partial open prepayment this occurrence is only noted during the first 18 months of the loan. After that time, increases in the volatility of the spot rate actually decrease the probability of prepayment.

Summary and Conclusions

The purpose of this research has been to develop and analyze a model of Farmer Mac mortgage-backed securities and the prepayment penalties used by Farmer Mac. Agricultural properties are similar to commercial properties in that they are income producing and impose some form of prepayment restriction. The lenders' preferences for self-liquidating loans was captured by tying the service or income flow of the mortgaged property to the probabilities of sub-optimal default and sub-optimal prepayment. Like other mortgage-backed security models, the model developed here allows for a quantification of default and prepayment risk by uncovering the embedded call and put options in the mortgage. Another innovation of this research was the implementation of a path-wise Monte Carlo simulation/dynamic programming approach to numerically solve a complex partial differential equation characterizing the value of the security. Finally, we analyze prepayment penalties used by Farmer Mac to determine their ability to actually preclude prepayment.

The results indicate that yield maintenance generally offers investors more prepayment risk protection than partial open prepayment penalties. As such, the value of agricultural mortgage-backed securities with yield maintenance have more value and, therefore, lower yields. The yield reduction can be interpreted as the minimum interest rate break a Farm Credit System or commercial bank could offer potential mortgagors to induce them to accept a loan with a specific prepayment penalty imposed by Farmer Mac. In a similar way, default risk is quantified and the option to default is determined to be generally of limited value to the mortgagor in the case of agricultural real estate. It was also demonstrated that prepayment penalties, while offering investors a natural shield against prepayment risk, are an imperfect means of accomplishing such an objective. Profit maximizing mortgagors can still find situations where prepayment is advantageous even after bearing the cost of the prepayment penalty and nominal refinancing costs.

[13] It should be noted that the null hypothesis being tested is whether $r^* \leq r_t + \xi + \zeta + \gamma_t$ with the probabilities reported in Table 6 being the probability that the null hypothesis is rejected. Consequently, the hypothesis itself presupposes that ζ adequately covers the mortgagor's cost of refinancing and that any degree of inequality above induces prepayment.

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