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Efficiency analysis in the presence of uncertainty

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Abstract

In a stochastic decision environment, differences in information can lead rational decision makers facing the same stochastic technology and the same markets to make different production choices. Efficiency and productivity measurement in such a setting can be seriously and systematically biased by the manner in which the stochastic technology is represented. For example, conventional production frontiers implicitly impose the restriction that information differences have no effect on the way risk-neutral decision makers utilize the same input bundle. The result is that rational and efficient *ex ante* production choices can be mistakenly characterized as inefficient – informational differences are mistaken for differences in technical efficiency. This paper uses simulation methods to illustrate the type and magnitude of empirical errors that can emerge in efficiency analysis as a result of overly restrictive representations of production technologies.

1 Introduction

Efficiency studies typically specify an efficient frontier that is generated by either a primal or a dual representation of a nonstochastic technology. Efficiency is then measured relative to an estimated version of this nonstochastic frontier. The two most widely-used estimation approaches are data envelopment analysis (DEA) and stochastic frontier analysis (SFA).

DEA primarily involves the use of linear programming methods to identify a piecewise linear surface that envelops the observed data. Typically, it makes no allowance for measurement errors and other sources of statistical noise. All deviations from the estimated frontier are attributed to inefficiency (Färe, Grosskopf, Lovell, 1985, 1994; Coelli, Rao, O'Donnell and Battese, 2005).

SFA parametrizes the nonstochastic frontier and estimates the unknown parameters using econometric techniques. A symmetric random variable is usually included to account for statistical noise (hence the term *stochastic* frontier analysis), and either fixed parameters or one-sided random variables are used to account for inefficiency (Kumbhakar and Lovell, 2000).

Even though it is well recognized, particularly by SFA proponents, that a bad operating environment can make an efficient firm appear as though it were operating inside an estimated efficiency frontier, most of this literature lacks an explicit recognition that production invariably takes place under conditions of uncertainty. With limited exceptions (e.g., Land, Lovell, and Thore, 1994; Olesen and Petersen, 1995; Gong and Sun, 1995; Post, Cherchye, and Kuosamen, 2002), DEA models and estimation techniques are nonstochastic.[1] And although SFA models are stochastic, their stochastic elements arise primarily from econometric concerns (measurement error, missing variables) and not as a response to the stochastic decision environment in which firms actually operate. In the few models where risk is explicitly recognized, latent variables representing uncertainty are subsumed into the noise and/or inefficiency error terms. Convenient distributional assumptions are then used to ensure compliance with stylized facts concerning uncertain production (e.g.,

[1] More precisely, they are intended to represent nonstochastic frontiers. Banker (1993) has shown how proper distributional assumptions on inefficiency yields a statistical interpretation of DEA models.

Battese, Rambaldi and Wan, 1997; Kumbhakar, 2002).

This empirical practice thus ignores the fundamental role that a stochastic decision environment can play in framing and conditioning producer decisions, and thus observed outcomes. Rational producers who operate in a stochastic world choose to operate at points of tangency between their *ex ante* preferences and the frontier of the stochastic technology. In this paper, we show that failure to account properly for the stochastic elements of the producer decision environment can lead to spurious measures of inefficiency.

The paper is organized as follows. In Section 2, we specify a common stochastic technology for a population of firms that potentially have different risk attitudes and information sets, and potentially experience different *ex post* production environments. All firms are assumed to act rationally, meaning they maximize an *ex ante* preference function subject to a stochastic technology contraint. Thus, *they are both technically and allocatively efficient.*

In Section 3, we use a simple numerical example to show how different preferences and information sets can lead to large variations in optimal input-output choices. Thus, we show that the stochastic nature of the decision environment can give rise to the type of observed heterogeneity in input usage that economists often attribute to bounded rationality (for example, satisficing), agency and monitoring concerns, incomplete contracting, or hierarchical decision-making.

In Section 4, we construct, by simulation methods, several data sets on rational single-input single-output firms. Then we apply and evaluate the performance of conventional DEA and SFA estimators. Our rationality assumption ensures that there is no technical inefficiency to be measured in the data. However, when standard frontier methods (both DEA and SFA) are used, we obtain technical inefficiency estimates that are non-zero and similar in magnitude to those commonly reported in the empirical efficiency literature. When the same methods are used to estimate a representation of the technology that explicitly recognizes the inherently uncertain decision environment, all firms are found to be fully efficient. Thus, the problem lies with the restrictive specification of the technology, not with the DEA and SFA estimators.

In Section 5 we offer some concluding comments.

2 Model

In contrast to the efficiency-measurement literature, the decision theory and modern financial economics literatures, both of which have their antecedents in Savage (1954), Arrow (1953) and Debreu (1952), assume an inherently uncertain decision environment. Uncertainty is modeled by a set of states of 'Nature', and economic variables, such as production and consumption, are treated as acts that map that set of states to an outcome space (typically, the bounded reals). It was the genius of Arrow and Debreu to recognize that analysis developed in nonstochastic economics transfers directly, and with little or no change, into this stochastic decision environment once *ex ante* preferences and technologies are properly defined. Our goal is to investigate, in an Arrow-Debreu framework, the consequences of a stochastic decision environment for standard practice in empirical *ex post* efficiency measurement.

The two key components of the model are the representation of the production technology and the description of firm behaviour. The production technology defines nonstochastic input and stochastic output combinations that are technically feasible. Firms having access to this technology make production choices that reflect their risk preferences and beliefs concerning the relative probabilies of different states of Nature.

2.1 The Technology

All firms have access to a common stochastic production technology where a nonstochastic input, denoted by $x \in \Re_+$ (the positive reals), is used to produce a stochastic output, denoted by $\tilde{z}$. Uncertainty is resolved by Nature making a choice from a state space, Ω. For the sake of a simple exposition that corresponds to our simulation experiments, we take $\Omega = \{1, 2\}$. The arguments presented below, however, generalize to arbitrary Ω.[2] If Nature picks s from Ω, then the realized or *ex post* value of $\tilde{z}$ is denoted by z_s.

Production activities take place over two time periods: in period 0 the producer picks (*ex ante*) the nonstochastic input x; in period 1 Nature chooses from Ω to resolve uncertainty. For realization $s \in \Omega$, the *ex post* realization of stochastic output is given by the Cobb-

[2] In particular, Ω can be either finite or infinite. In practice, the only change in what follows is in terms of the derivative concepts used.

Douglas function

$$\ln z_s = b^{-1} [\ln x_s - \ln a_s], \tag{1}$$

where $b \geq 1$ is a parameter whose interpretation will be apparent shortly, and x_s is the amount of the nonstochastic scalar input that is committed in period 0 to production in state of Nature s. This technology, except in a limiting case treated below, is what Chambers and Quiggin (2000) have referred to as *state-allocable.*

The terms $a_s \geq 0$ are open to at least two distinct but related interpretations. First, they can be thought of as technical parameters that are specific to the production of output in state s. A second interpretation, and the one we give them in this paper, is that they are *ex post* realizations of an unobservable scalar random variable that is within Nature's control. This random variable is denoted by $\tilde{a}$. With this interpretation, it is important to emphasize that the a_s terms are pure uncertainty effects that emerge from Nature's role in the stochastic production process. They do not derive from measurement errors or the econometrician's lack of knowledge about functional form. At time 0 the producer chooses x_s for all values of $s \in \Omega$, and then faces the uncertainty of not knowing the realization of $\tilde{a}$ and, through (1), the realization of $\tilde{z}$ that Nature will choose.

Associated with (1) is the state-specific input requirement function

$$x_s = a_s z_s^b. \tag{2}$$

Although z_s and a_s are realizations of random variables, it is important to observe that, because of the timing of production, x_s is chosen nonstochastically. Thus, the proper interpretation of (2) is that $a_s z_s^b$ is the amount of the input that must be committed in period 0 if output z_s is to occur when Nature chooses s from Ω. To ensure that z_1 is produced when Nature picks $\{1\}$ from Ω and that z_2 is produced when Nature picks $\{2\}$, the producer must therefore commit an input in period 0 totalling at least

$$a_1 z_1^b + a_2 z_2^b \equiv g(z_1, z_2).$$

Given a total input level of x, the convex transformation function defining technically feasible production patterns is:

$$t(z_1, z_2, x) = g(z_1, z_2) - x.$$

Technically-feasible but inefficient production patterns are given by $\{(z_1, z_2) : t(z_1, z_2, x) < 0\}$, and technically-efficient patterns are $\{(z_1, z_2) : t(z_1, z_2, x) = 0\}$. Thus, the input distance function (the reciprocal of the input-oriented radial efficiency measure) for this stochastic technology is

$$D_I(x, z_1, z_2) = \frac{x}{g(z_1, z_2)},$$

while the output distance function (the reciprocal of the output-oriented measure) is of the CET form (Powell and Gruen, 1967):

$$D_o(z_1, z_2, x) = g(z_1, z_2)^{\frac{1}{b}} x^{-\frac{1}{b}}.$$

Given a positive normalized input price of $w > 0$, the *ex ante* minimal cost associated with producing the random variable $\tilde{z}$ is

$$c(w, z_1, z_2) = wg(z_1, z_2).$$

Thus, b is interpretable as the cost flexibility (the reciprocal of the elasticity of size) associated with production of output in state s. It is equal across all states of Nature. Furthermore, wba_s is the marginal cost of *ex post* output in state s in the neighborhood of the degenerate production plan $z_s = 1$ for all s. The parametric restriction $b \geq 1$ implies the technology always exhibits nonincreasing returns to scale.

This technology has several convenient properties. The marginal rate of transformation between *ex post* outputs is (for fixed x)

$$MRT = -\left(\frac{a_1}{a_2}\right)\left(\frac{z_1}{z_2}\right)^{b-1}.$$

Thus, the elasticity of transformation between any pair of *ex post* outputs is a constant:

$$\sigma = \left|\frac{d\ln(z_1/z_2)}{d\ln|MRT|}\right| = \frac{1}{1-b}.$$

Two limiting special cases are of interest. First, as $b \rightarrow 1$, the transformation function $t(z_1, z_2, x)$ converges to a linear, constant-returns-to-scale (CRS) transformation function that exhibits an infinite elasticity of transformation, meaning *ex post* output is perfectly substitutable between states. Second, as $b \rightarrow \infty$, the elasticity of transformation converges to zero, meaning no *ex post* output substitutability is possible. In this second case, the technology has a particularly familiar interpretation. To see this, renormalize $a_s \equiv e_s^{-b}$ for all $s \in \Omega$. Then application of a limiting argument originally due to Hardy, Littlewood, and Polya (1934) shows that as $b \to \infty$,

$$g\left(z_1, z_2\right)^{\frac{1}{b}} \to Max\left\{\frac{z_1}{e_1}, \frac{z_2}{e_2}\right\}.$$

This is the *ex ante* input requirement function for a stochastic production function of the form (Chambers and Quiggin, 1998)

$$z_s = xe_s, \qquad s \in \Omega,$$

where x is not allocable across states, and e_s can now be interpreted as a realization of a multiplicative error term. Thus, by a suitable parametric restriction, the technology can be made to be isomorphic (have the same transformation and isocost curves) to the stochastic production function with multiplicative errors, which has been a cornerstone of SFA efficiency measurement.

For illustrative purposes, the transformation curves $\{(z_1, z_2) : t\left(z_1, z_2, x\right) = 0\}$ corresponding to $a = (a_1, a_2) = (1.5, 5)$, $x = 1$, and $b = 1.1, 2$ and 11 are depicted in Figure 1. These settings for b correspond to high, moderate and low degrees of substitutability between *ex post* outputs.

3 Efficient Firm Behavior

There are as many ways to characterize efficient firm behavior as there are objective functions for firms. We seek an objective function that ensures that firms operate at technically efficient points – those satisfying $D_I\left(x, z_1, z_2\right) = D_o\left(z_1, z_2, x\right) = 1$. Given the smooth technology that we postulate, this is guaranteed if firms have preferences that are strictly increasing in both period 0 consumption (which is nonstochastic) and period 1 consumption (which is stochastic). Therefore, for concreteness and for easy comparison with existing studies, we assume that, subject to the constraint imposed by the technology, the firm seeks to maximize

$$W\left(y\right)$$

where $y = (y_1, y_2)$ and $y_s = z_s - wx$ is the *ex post* net return in state of Nature s. The function W, which gives the producer's joint evaluation of *ex post* net returns, is strictly increasing in y and is suitably smooth to allow differential changes in its arguments. This form is quite general, and includes, as a very restrictive special case, the expected utility of net returns (profit) class of objective functions.

The first-order conditions for efficient firm behavior can be written in the form

$$\frac{W_s(y)}{\sum_{m\in\Omega} W_m(y)} - bwa_s z_s^{b-1} \leq 0, \qquad s \in \Omega \tag{3}$$

where $W_s(y) \equiv \partial W(y)/\partial y_s > 0$. This form is particularly convenient analytically because the monotonicity of W ensures that

$$\pi_s(y) \equiv \frac{W_s(y)}{\sum_{m\in\Omega} W_m(y)} \in (0,1), \qquad s \in \Omega, \tag{4}$$

and $\sum_{s\in\Omega} \pi_s(y) = 1$. Hence, $\pi_s(y)$ can be viewed as a probability. In fact, the probabilities defined by (4) are what are referred to in the finance literature as *risk-neutral probabilities* – the subjective probabilities a risk-neutral firm would need to have if it were to select the same production plan as a rational firm with preferences $W(y)$. The importance of this result is that any efficient choice for a rational firm with an objective function defined over net-returns can be viewed *as though it were generated by a risk-neutral firm with subjective probabilities given by the risk-neutral probabilities.* Thus, we can investigate the behaviour of firms having any net returns preferences by studying the behaviour of risk-neutral firms with different probabilities.

Summing equation (3) across states implies that any rational (z_1, z_2) for a net-returns objective function must satisfy:[3]

$$1 - bw\left[\sum_{s\in\Omega} a_s z_s^{b-1}\right] \leq 0.$$

The set

$$\Xi(w) = \left\{(z_1, z_2) : 1 - bw\left[\sum_{s\in\Omega} a_s z_s^{b-1}\right] \leq 0\right\}$$

[3] As Chambers and Quiggin (2004) point out, this condition holds for any individual who strictly prefers more 0 period consumption to less, more period 1 consumption to less, and faces a financial market in which there exists a riskless asset. Therefore, as a practical matter it is much more general than the net-returns formulation might suggest.

is what Chambers and Quiggin (2000) have referred to as the *efficient set.* It contains all *ex post* outputs that could be rationally chosen by individuals facing this technology and having net-returns preferences. Its boundary,

$$\bar{\Xi}(w) = \left\{ (z_1, z_2) : 1 - bw \left[\sum_{s \in \Omega} a_s z_s^{b-1} \right] = 0 \right\},$$

is what Chambers and Quiggin (2000) have referred to as the *efficient frontier.* It represents all interior solutions to the first-order conditions above. Observe that no assumptions were made on attitudes towards risk in deriving the efficient set and the efficient frontier.

If $b > 1$ and $\pi_s(y) \propto a_s$, an interior solution to the first-order conditions (3) must be an equal-output production plan (i.e., a point lying on the ray bisecting $\Re_+^2$). As $b \to \infty$, this riskless plan converges to $(z_1, z_2) = (1, 1)$, and as $b \to 1$ it converges to $(z_1, z_2) = (e^{-1}, e^{-1})$.

In the special case where $b = 1$ (linear costs), it is evident that $\Xi(w) = \Re_+^2$ if and only if $1 - w \sum_{s \in \Omega} a_s \leq 0$. The *ex ante* marginal cost of raising $\tilde{z}$ nonstochastically in the direction of the equal-output ray (that is, increasing z_s by the same amount in both states) is $w \sum_{s \in \Omega} a_s$, while the marginal return is 1. Thus, $\Xi(w) = \Re_+^2$ provided the marginal net return from raising $\tilde{z}$ in the direction of the equal-output ray is nonpositive. If $1 - w \sum_{s \in \Omega} a_s = 0$ then any *ex post* output pair can be rationalized by choosing

$$\pi_s(y) = wa_s = \frac{a_s}{\sum_{m \in \Omega} a_m} \propto a_s.$$

If $1 - w \sum_{s \in \Omega} a_s < 0$ then any individual will obtain a marginal net profit by lowering $\tilde{z}$ in the direction of the equal-output ray, so that $z_s = 0$ for some s. If $1 - wa_s = 0$ then any point on the z_s axis can be rationalized by choosing $\pi_s(y) = 1$. If $1 - wa_s < 0$ for all $s \in \Omega$ then $\Xi(w) = \varnothing$.

Because $\pi_s(y)$ is defined by the preference functional, the point at which a rational individual locates on the efficient frontier is determined by a number of factors including his or her subjective beliefs about Ω (as reflected in his or her subjective probabilities), [4] his or her attitudes towards risk, as well as the confluence of these two factors. We illustrate by considering two special cases.

[4] This presumes, of course, that individuals formulate unique probability measures. It is well recognized that otherwise rational individuals can behave as though they are not probabilistically sophisticated.

First, consider a risk-neutral firm with subjective probabilities given by (π_1, π_2). Then $W_s(y) = \pi_s$ and an interior equilibrium occurs at

$$z_s = \left(\frac{\pi_s}{bwa_s}\right)^{\frac{1}{b-1}} \tag{5}$$

for $s = 1, 2$. This solution is trivially on the efficient frontier. However, depending upon the individual's beliefs and the parameters of the production technology, a rational individual may choose z_1 less than, equal to, or greater than z_2. The riskless output combination

$$z_1 = z_2 = \left(\frac{1}{bw(a_1 + a_2)}\right)^{\frac{1}{b-1}} \tag{6}$$

is rational when $\pi_s \propto a_s$. If $\pi_1/\pi_2 > a_1/a_2$ then a rational individual will choose $z_1 > z_2$ and commit $x \leq (a_1)^{\frac{1}{1-b}} (bw)^{\frac{b}{1-b}}$ to the production process. Conversely, if $\pi_1/\pi_2 < a_1/a_2$ then he or she will choose a production plan with $z_1 < z_2$ and $x \leq (a_2)^{\frac{1}{1-b}} (bw)^{\frac{b}{1-b}}$. As we illustrate in Section 3 below, it is trivial to extend the risk-neutral example to the expected utility of net returns case.

Second, consider a firm with CES preferences:

$$W(y) = (\delta y_1^r + (1 - \delta) y_2^r)^{\frac{1}{r}}.$$

These preferences are additively separable across Ω, but they are not expected utility preferences. Direct calculation establishes that

$$\pi_s(y) = \frac{\delta y_s^{r-1}}{\delta y_1^{r-1} + (1 - \delta) y_2^{r-1}}$$

for $s = 1, 2$. Moreover, by results due to Hardy, Littlewood, and Polya (1934)

$$\begin{aligned} \lim_{r \to -\infty} W(y) &= \min\{y_1, y_2\} \\ &= \min\{z_1 - wx, z_2 - wx\} \\ &= \min\{z_1, z_2\} - wx \end{aligned}$$

If $a_1 > 0$ and $a_2 > 0$, then it follows almost trivially that in this limiting case (Chambers and Quiggin, 2000), a rational individual chooses the riskless production plan given by (6). The corresponding risk-neutral probabilities are

$$\pi_s(y) = bwa_s \left(\frac{1}{bw[a_1 + a_2]}\right)^{\frac{1}{b-1}}$$

for $s = 1, 2$.

4 Numerical Example

Consider a risk-neutral producer with subjective probabilities given by (π_1, π_2). It is clear from the equilibrium condition (5) that, except in the limiting case where $b = 1$, z_s is nondecreasing in π_s. Thus, a rational producer who attaches a higher probability to the occurence of state s will allocate more input to production in that state. At one extreme, when $\pi_1 = 0$, the equilibrium point on the efficient frontier is given by $z_1 = 0$ and $z_2 = (bwa_2)^{1/(1-b)}$. At the other extreme, when $\pi_1 = 1$, the rational producer will choose $z_1 = (bwa_1)^{1/(1-b)}$ and $z_2 = 0$. The slope of the efficient frontier that connects these extreme points is given by

$$\frac{dz_2}{dz_1} = -\frac{a_1 z_1^{b-2}}{a_2 z_2^{b-2}} \leq 0.$$

To make these ideas more concrete, consider a stochastic technology with $a = (1.5, 5)$ and $b = 2$. These parameter settings were used to construct the dashed transformation curve in Figure 1. The first four columns in Table 1 report the production choices of efficient producers facing a normalized input price of $w = 0.5$ and having subjective probabilities that range over the unit interval. The first row reveals that when $(\pi_1, \pi_2) = (0, 1)$ the rational producer chooses $(z_1,\ z_2) = (0, 2)$; the last row reveals that when $(\pi_1, \pi_2) = (1, 0)$ the optimizing choice is $(z_1,\ z_2) = (0.667, 0)$. The efficient frontier connecting these two points is a straight line with slope $dz_2/dz_1 = -a_1/a_2 = -3$. Thus, the shape of the efficient frontier differs from the shape of the transformation function depicted in Figure 1, the latter being a strictly concave function depicting the trade-off between *ex post* outputs when the input level is fixed at $x = 1$.

For this particular cost structure, a rational (risk-averse or risk-neutral) producer will choose the riskless production plan $(z_1,\ z_2) = (0.5, 0.5)$ when his or her risk-neutral probabilities satisfy the condition $\pi_1/\pi_2 = a_1/a_2 = 3$ (that is, when $\pi_1 = \frac{a_1}{a_1+a_2} = 0.75$ and $\pi_2 = \frac{a_2}{a_1+a_2} = 0.25$). At this point, $(0.5, 0.5)$, and for these specific probabilities, $(.75, .25)$, the technology is, in the terminology of Chambers and Quiggin (2000), *not inherently risky.* That is, any stochastic output with expected value $\pi_1 z_1 + \pi_2 z_2 = 0.5$ is more costly to produce than the non-stochastic output $(0.5, 0.5)$, and similarly for any other expected output level. A technology is *inherently risky* in the sense of Chambers and Quiggin (2000) when the cost-minimising (z_1, z_2) for a given expected output is truly stochastic, so that $z_1 \neq z_2$. When this happens, there is always a trade-off between cost-minimisation and risk-reduction. However, when the technology is not inherently risky, the least risky (for the given probabilities) stochastic output is also the least costly.[5] It is important to recognize that the concepts of inherently risky and not inherently risky depend critically upon the producer's subjective probabilities. At the point where $(z_1, z_2) = (0.5, 0.5)$, this technology is inherently risky for any probability distribution other than $(\pi_1, \pi_2) = (.75, .25)$.

Observe that the non-stochastic production plan requires less input than any other production plan on the efficient frontier. This will be true for any technology of the general form (2). However, for arbitrary stochastic technologies, it is not generally true that a non-stochastic production plan is always the least costly on the efficient frontier. To see why it is true in this case, it suffices to minimise $a_1 z_1^b + a_2 z_2^b$ while restricting our choices to be on the efficient frontier, that is, subject to the constraint (derived above) that

$$a_1 \left(b z_1^{b-1}\right) + a_2 \left(b z_2^{b-1}\right) = 1.$$

It is easily verified that minimising $a_1 z_1^b + a_2 z_2^b$ subject to this constraint yields $z_1 = z_2$.

Finally, the results in Table 1 bear out our earlier observation that if $\pi_1/\pi_2 > a_1/a_2 = 3$ then a rational individual will choose $z_1 > z_2$ and $x \leq (a_1)^{\frac{1}{1-b}} (bw)^{\frac{b}{1-b}} = 0.667$; if $\pi_1/\pi_2 < 3$ then a rational individual will choose $z_2 > z_1$ and $x \leq 2$.

The input-output combinations reported in Table 1 are those chosen by rational risk-neutral producers having a range of information sets (subjective probabilities). However, they are also combinations that may be chosen by risk-averse producers having somewhat different information sets. To see this, consider a producer who attaches probability p_1 to state of Nature $\{1\}$, and who maximizes the expected utility of net returns. If the utility function is exponential then the welfare function is

$$W(y) = -p_1 \exp(-Ay_1) - (1 - p_1) \exp(-Ay_2)$$

where A denotes the coefficient of absolute risk aversion. The corresponding risk-neutral

[5] This condition may also be described by saying that the cost function is generalized Schur-concave with respect to the probability vector $(0.75, 0.25)$.

probability for state of Nature $\{1\}$ is

$$\pi_1 = \frac{p_1 \exp(-Az_s)}{p_1 \exp(-Az_1) + (1-p_1)\exp(-Az_2)}. \tag{7}$$

For any values of (π_1, z_1, z_2) and A there exists a value of p_1 that solves equation (7). For example, if $(\pi_1, z_1, z_2) = (0.5, 0.333, 1)$ and $A = 1$, the solution is $p_1 = 0.3392$. This means that any rational producer who i) maximizes the expected utility of net returns, ii) has an exponential utility function with coefficient of absolute risk aversion $A = 1$, and iii) attaches probability 0.3392 to state of Nature $\{1\}$, will make exactly the same production choice as a risk-neutral producer who regards both states of Nature as equally likely. Other solutions for p_1 corresponding to $A = 1$ and different values of (π_1, z_1, z_2) are reported in the second last column in Table 1. Solutions corresponding to $A = 10$ are reported in the last column. For this technology, where the mean expected output is approximately 0.5, these values of A imply coefficients of relative risk aversion ranging from approximately 0.5 (slightly risk-averse) to 5 (moderately risk-averse).[6]

Repeating this numerical exercise using $b = 11$ (and leaving the normalized input price and all other parameter settings unchanged) yields the optimal production choices reported in Table 2. This value of b was previously used to construct the near-square transformation function in Figure 1. The efficient frontier now connects the points $(z_1, z_2) = (0, 0.904)$ and $(z_1, z_2) = (0.801, 0)$, and has slope $dz_2/dz_1 = -3(z_1/z_2)^9$. Observe that the output pairs reported in Table 2 expose producers to no greater risk than the corresponding pairs reported in Table 1, and that all but the first and last production plans are relatively riskless. The two risky production plans are only chosen by rational risk-neutral producers when they attach a subjective probability of more than 0.95 to one or other state. Moderately risk-averse producers (those with exponential utility and $A = 10$) only choose these risky plans when they regard one or other state as a near-certainty (subjective probability greater than approximately 0.99). The results reported in Table 2 are consistent with our earlier observation that if $\pi_s(y) \propto a_s$ then as $b \to \infty$ the riskless production plan converges to $(z_1, z_2) = (1, 1)$. More generally, if $\pi_s(y) \neq 0$ for all $s \in \Omega$, then as $b \to \infty$ the riskless production

[6] We take 10 to be the highest plausible level of relative risk aversion. However, some researchers view levels of relative risk aversion as plausible if they lie in the range 1 to 4. These researchers would characterize 5 as extremely risk averse.

plans of *all* rational individuals converge to $(z_1, z_2) = (1, 1)$, irrespective of the distribution of the random variable $\tilde{a}$ (i.e., irrespective of the elements of the vector a).

Finally, when $b = 1.1$ (the value of b underpinning the almost linear transformation function depicted in Figure 1), the optimal production choices are those reported in Table 3. These production plans are at least as risky as the corresponding plans reported in Tables 1 and 2. The riskless production plan is consistent with our earlier observation that as $b \to 1$ riskless output converges to $z_1 = z_2 = e^{-1} = 0.3678$.

5 Properties of DEA and SFA Estimators

As we have just illustrated, rational firms with access to the same stochastic technology may make different *ex ante* production choices due to differences in beliefs and/or preferences. Firms allocate more input to states of Nature they believe to be more probable, and more risk-averse firms produce closer to the equal-output ray. Firms with identical production plans may also realize different outputs due to the role Nature plays in selecting different states. Unfortunately, conventional parametric and nonparametric efficiency measurement methods fail to recognize this multiplicity of sources of variation in inputs and outputs. Thus, unless all firms select riskless production plans, conventional estimators[7] of efficiency will be biased. In this section, we use simulation methods to investigate the nature of this bias.

To conserve space, and to facilitate interpretation of our findings, all our empirical work was underpinned by a fixed set of risk-neutral probabilities. Rather than choose values for π_1 that are equally spaced across the unit interval, such as those used to construct Tables 1 to 3, we used 23 values randomly drawn from a standard triangular distribution. We also included two values of special interest, namely the mode of the triangular distribution ($\pi_1 = 0.5$) and the value that would cause rational firms to adopt riskless production plans (for the stochastic technologies we consider, $\pi_1 = 0.75$). Associated with each value is a

[7] Strictly speaking, the term 'predictor' should be used if inferences are being drawn using a random effects stochastic frontier model. In that case, the object of inference (firm efficiency) is a random variable, not a parameter. However, for simplicity, we use the term 'estimator' in both contexts.

nonstochastic input-output combination that will potentially be chosen by a rational efficient firm.

The 25 selected probabilities are reported in ascending order in the second column of Table 4, and the associated production choices of firms facing a technology with $w = 0.5, a = (1.5, 0.5)$ and $b = 2$ are reported in columns 3 to 5. Recall that these parameter settings, which permit a moderate degree of substitutability between *ex post* outputs, were also used to prepare Table 1. The role of Nature is to choose realizations of the random variable $\tilde{a}$, and thereby assign firms to potentially different states. We perform this role by randomly assigning each firm to state $\{1\}$ with probability 0.5. Column 6 of Table 4 reports one sample of realized states, and column 7 lists the associated realized outputs.

The last two columns in Table 4 report efficiency scores obtained using variable returns to scale (VRS) DEA and SFA estimators (and the inputs and realized outputs reported in columns 3 and 7). The DEA specification was an input-oriented model. The SFA specification was a half-normal random effects frontier with a Cobb-Douglas functional form. The DEA methodology implicitly imposes a monotonicity constraint on the estimated production frontier. To ensure the estimated SFA frontier also satisfies this property, we restricted the slope coefficient in the Cobb-Douglas function to be nonnegative.[8]

It is useful to interpret the results reported in Table 4 with the aid of Figure 2. Panel (a) in this figure plots the observed data points and the loci of all rational and efficient input-output combinations that are possible using the technology. Panel (b) plots the data points and the estimated DEA and SFA frontiers.

The DEA results reported in Table 4 are unsurprising in several respects. First, as noted above, the riskless production plan $z_1 = z_2 = 0.5$ requires an input commitment of $x = 0.5$, which is less than the input requirement of any other production plan. Again, this is a property of the particular technology that we have used to generate our numerical results. DEA methodology guarantees that the firm in the sample that uses the least amount of input will be placed on the estimated VRS frontier, irrespective of its output level or the outputs of other firms. Thus, the estimated efficiency of a rational firm choosing the riskless production plan is guaranteed to be 1 (Firm 21 in this sample). Second, because there is one

[8] All results were generated using the SHAZAM software.

firm in the sample that chooses the riskless production plan (and also because of our choice of technology – there exist well-behaved stochastic technologies for which the following is not true), any rational efficient firm that chooses a risky production plan and experiences an unfavourable state of Nature has an input-oriented DEA efficiency score of $0.5x^{-1} < 1$. Pictorially, firms experiencing unfavourable states of Nature will be those located below the horizontal line passing through the point of riskless production ($z_1 = z_2 = 0.5$). The input-oriented DEA efficiency scores for these firms are ratio measures of the distances between the data points and the vertical segment of the DEA frontier at $x = 0.5$. Finally, observe from Figure 2 and Table 1 that $z_2 > z_1$ for any firm in the sample that uses $x \geq 0.667$. Thus, firms operating on this scale (all firms with $\pi_1 \leq 0.5$) will be found to be fully efficient (using the input-oriented DEA estimator) if and only if they experience state of Nature $\{2\}$.

When properly understood, the SFA results are equally unsurprising. For a start, the estimated SFA frontier in Figure 2 is horizontal at $z_s = 1.5283$ because, for this sample, the correlation between observed inputs and outputs is plausibly negative. Hence, the monotonicity constraint is binding and the maximum likelihood estimate of the output elasticity is exactly zero.[9] Second, the SFA efficiency estimates are uniformly lower than the DEA estimates. This is largely because the input-oriented DEA efficiency scores have a theoretical lower bound of 0.25, while the output-oriented SFA efficiency scores have a theoretical lower bound of zero. Third, with a horizontal SFA frontier, rational firms will be found to be relatively efficient if and only if they have $\pi_2 \approx 1$ and experience state of Nature $\{2\}$. Few of these firms will be found to be precisely 100% efficient owing to the way the SFA methodology makes adjustments for statistical noise. There is no measurement error in this data set, so the only source of statistical noise is misspecification of the stochastic technology (i.e., using a single-valued function to approximate the deterministic input-output loci depicted in Figure 2). Finally, unlike the DEA estimator, the SFA estimator does not guarantee that the firm choosing the riskless production plan will be found to be fully, or even highly, efficient. In this sample, Firm 21 is estimated to be only 34% efficient

[9] Sampling theory estimators have the undesirable property that binding inequality constraints lead to estimates that lie on the constraint boundary, with estimated standard errors of zero. The problem can be overcome using Bayesian methodology (Coelli *et al.* 2006).

($\approx 0.5/1.5283$). The estimated efficiency of this firm is governed by the position of the SFA frontier. In turn, this is largely determined by the output of the firm that has π_2 closest to 1 and experiences state of Nature $\{2\}$. In this sample, this lucky risk-taking firm is Firm 1.

We conducted a simulation experiment that involved replicating this exercise $N = 500$ times. The production choices reported in columns 3 to 5 of Table 4 were held fixed across replications. Thus, the only stochastics in our experiment are those associated with the role of Nature – in each replication we again assigned each firm to state $\{1\}$ with probability 0.5. Table 5 reports descriptive statistics on the efficiency scores obtained from this experiment. These statistics can be viewed as estimates of the means, standard deviations, maxima and minima of the sampling distributions of the DEA and SFA estimators. For example, the first row of Table 5 reveals that the SFA estimator of the efficiency of Firm 1 has an estimated sampling distribution that is centred at 0.489 with a standard deviation of 0.418. It is important to remember that there is no technical inefficiency or measurement error in any of the 500 simulated data sets. Thus, any mean values that are less than 1 indicate that the conventional DEA and SFA estimators are biased.

Predictably, the results reported in Table 5 reveal that the DEA and SFA estimators are both seriously biased, although the bias is much smaller when DEA is used to estimate the efficiencies of firms adopting riskless or near-riskless production plans ($\pi_1 \approx 0.75$). However, the performance of the DEA estimator in these cases is an artifact of the estimation methodology. As we observed earlier, the DEA linear program will ensure that, in every replication of the experiment, the firm that adopts the riskless production plan will be placed on the efficient frontier, irrespective of its output level. Firms that adopt near-riskless production plans will be estimated to be hightly efficient (using the input-oriented DEA estimator) because they use an input level close to $x = 0.5$.

The results reported in Table 5 give a very incomplete, if not misleading, picture of estimator performance. More informative summary measures of performance are presented in Figure 4, where we depict the estimated sampling distributions of our estimators for the cases $\pi_1 \in \{.061, .5, .75, .943\}$. These histograms are representative of all estimated sampling distributions obtained in our experiment, and reflect the systematic estimation errors identified in our earlier discussion of Table 4 and Figure 2. In the case of the DEA estimator, the fact that states have been assigned with equal probability means that, irrespective of input or realized output levels, the estimator takes the value $0.5x^{-1}$ with probability no less than 0.5. Indeed, for any firm with $\pi_1 \leq 0.5$ (i.e., any firm using $x \geq 0.667$), the DEA estimator takes the values $0.5x^{-1}$ and 1 with equal probability. For any firm with $\pi_1 > 0.75$, the DEA estimator takes discrete values in the interval $[0.5x^{-1}, 1]$ with probabilities no greater than 0.5. For example, Firm 25, with subjective probability $\pi_1 = 0.943$, is placed on the DEA frontier if and only if it experiences state $\{1\}$ and Firms 2 to 19 also experience state $\{1\}$. The probability of this joint event is so small that it did not occur once in 500 replications of our experiment.

Matters are slightly less clear cut in the case of the SFA estimator, owing to the concessions made for statistical noise. However, for any firm operating on a relatively large scale (e.g., $x \geq 1$), the SFA estimator takes a relatively high value with probability 0.5 and a relatively low value with the same probability (see the estimated sampling distribution for Firm 1). For any firm operating on a relatively small scale (at or near the point of riskless production), the SFA estimator can take any value greater than about 0.3 (see the histogram for Firm 21). This lower bound is the approximate efficiency score in the worst-case scenario where the estimated SFA frontier is horizontal (i.e., the monotonicity constraint is binding) and some firms with high π_2 experience state $\{2\}$ (thus causing the estimated frontier to be at or near the level of the maximum output that is feasible using the technology).

It is evident that the DEA and SFA estimators are seriously and systematically biased. The sampling distributions of the estimators have upper bounds of 1 (the true value), but they have lower bounds that are inversely related to the riskiness of firm production plans (equivalently, scale of operations). This finding is robust to changes in the parameter settings, as evidenced by Figures 3, 5 and 6.

Panel (a) in Figure 3 depicts rational production plans for the case where the stochastic technology exhibits a high degree of substitution between *ex post* outputs ($b = 1.1$). The plans depicted in this panel are, in fact, only the subset of plans that would be optimally selected by firms with risk-neutral probabilities in the range $\pi_1 \geq 0.6$ (the remaining plans extend the two loci in the directions indicated). Panel (b) in Figure 3 traces out the set of rational production plans when the degree of subsitution between outputs is low ($b = 11$). Observe from Table 3 that most firms facing this technology adopt riskless or near-riskless production plans.

We repeated our simulation experiment using these two alternative stochastic technologies, and the estimated sampling distributions of the DEA and SFA estimators are depicted in Figures 5 and 6. These sampling distributions exhibit all the features identified above. Observe that estimator performance deteriorates (improves) when the parameter settings used in our simulation experiment are adjusted so that efficient firms optimally choose more (less) risky production plans.

Finally, to demonstrate that the unreliability of the DEA and SFA estimators derives from misspecification of the stochastic technology, and not from the linear programming or maximum likelihood estimation methodologies, we used the data reported in columns 3 to 5 of Table 4 (i.e., data on both realized and unrealized *ex post* outputs) to estimate the stochastic technology within a traditional multiple-output framework, where we now treat (z_1, z_2) as two separate outputs. The SFA model was a stochastic output distance function (e.g., Coelli and Perelman, 1999). The two estimators placed all firms almost exactly on the VRS frontier (all estimated technical efficiency scores were equal to 1 when rounded to 3 decimal places). Unfortunately, this multiple-output estimation approach is unavailable in any practical setting because it requires data on the complete *ex ante* output choice, not just data on the *ex post* realization of the random variable $\tilde{z}$. An estimation approach that offers some promise of being able to identify flexible stochastic technologies is the finite mixtures approach of O'Donnell and Griffiths (2006).

6 Concluding comments

Conventional efficiency analysis, and most of modern production economics, is concerned with estimating nonstochastic behaviour and nonstochastic technologies. Stochastic elements of the decision environment are typically only recognized when and if it is econometrically convenient. This practice places strong and as yet untested *a priori* restrictions on stochastic technologies. This paper shows that if the restrictions are invalid then the application of standard methods of efficiency analysis to data arising from production under uncertainty may give rise to spurious findings of efficiency differences between firms.

In considering our results, one should remember that we have intentionally considered a case where all firms face exactly the same decision environment. They face a common stochastic technology and the same input and output markets. They only differ in their subjective beliefs about that decision environment, and those differences in beliefs lead them to prepare *rationally* for that stochastic world in different ways. The empirical outcome is seriously biased efficiency scores as estimated by either standard input-oriented DEA or output-oriented SFA methods. And although we have not considered output-oriented DEA or input-oriented SFA efficiency measures in this paper, it is apparent, however, that seriously biased results can also occur for them. Moreover, this remains true even if all firms have exactly the same belief structure. Suppose, for example, that all firms faced the technology with moderate output substitutability and had a subjective probability of $\pi_1 = .05$. If they were all risk neutral, they would all rationally choose the same input level (1.807) and produce $(z_1, z_2) = (.033, 1.9)$. If Nature then chooses realizations of $\tilde{a}$ by assigning outcomes to state $\{1\}$ with a probability .5, then roughly half the firms in any representative sample would be judged as inefficient using output-oriented DEA methods, even though all firms operate efficiently.

When it is realized that real-world data typically reflect multiple sources of behavioural differences across firms, then it is apparent that, in attempts to estimate or approximate a supposed common frontier technology, serious problems likely can emerge from the practice of ignoring the interplay of these other sources with the truly stochastic nature of many production technologies. Therefore, an important implication of our results is that it is necessary to reconsider the findings of virtually all previous empirical studies of efficiency to determine whether the results may have been affected by a failure to take appropriate account of uncertainty. In some cases, it may be necessary to qualify policy recommendations derived from findings of widespread inefficiency. More importantly, perhaps, it is necessary to develop robust techniques that will make it possible to disentangle differences in technical efficiency from differences caused by the stochastic nature of production.

The major finding of the current study is that the incorporation of an appropriate representation of uncertainty is a matter of major urgency if empirical efficiency analysis is to remain relevant to a fundamentally uncertain economic world. Results derived from a non-stochastic approximation to that stochastic world clearly cannot be regarded as reliable.

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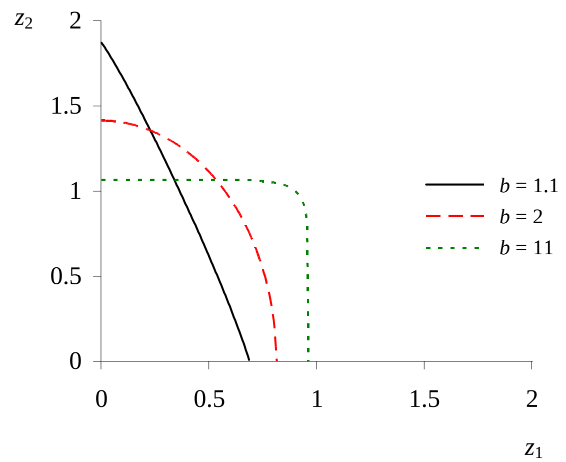


Figure 1: Transformation Curves for Input Level $x = 1$.

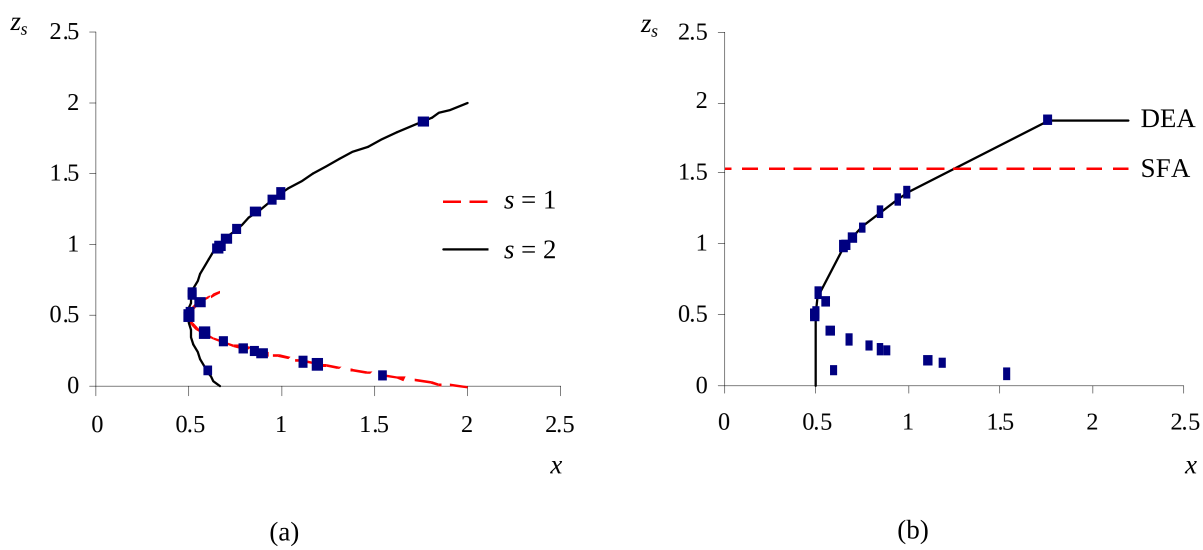
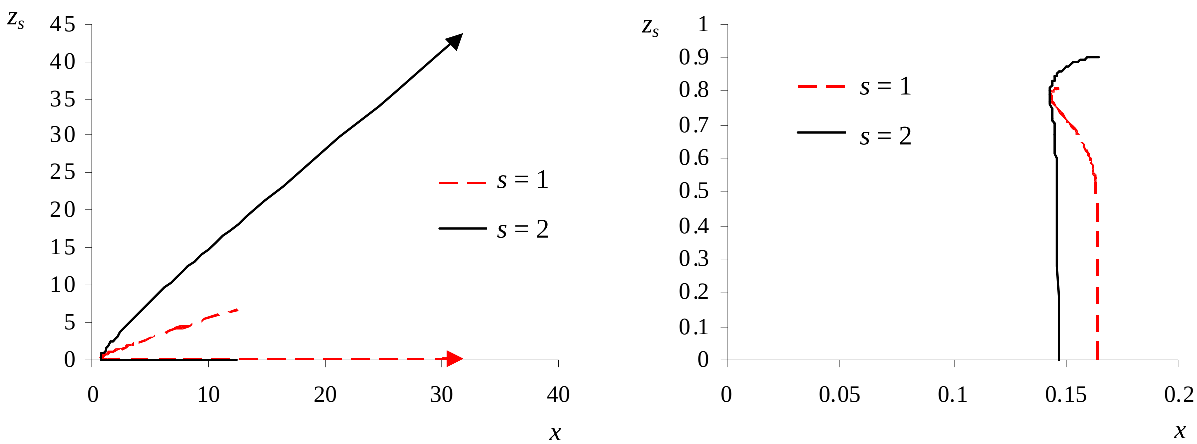


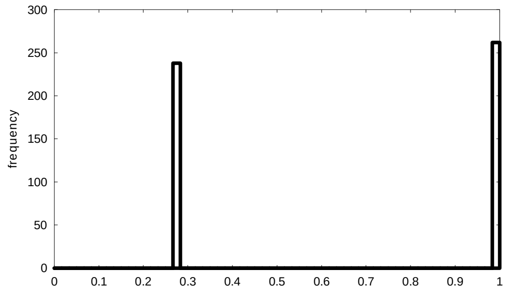
Figure 2: Efficient Production Plans and Frontiers: Moderate Output Substitutability



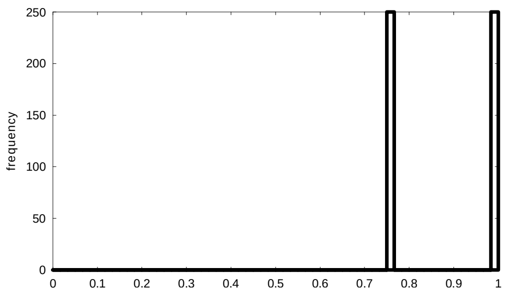
(a) High Substitutability (b = 1.1)

(b) Low Substitutability (b = 11)

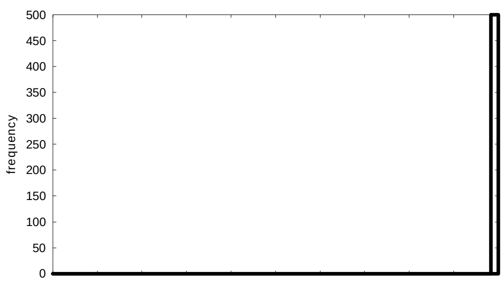
Figure 3: Efficient Production Plans: High and Low Output Substitutability



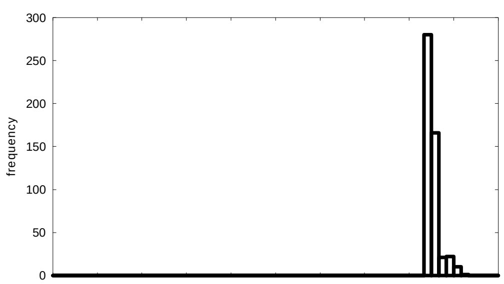
(a) DEA: Firm 1 (π_1 = 0.061)



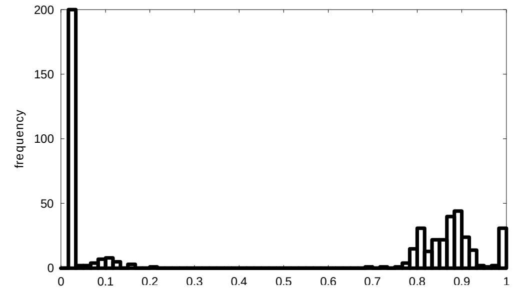
(b) DEA: Firm 14 (π_1 = 0.5)



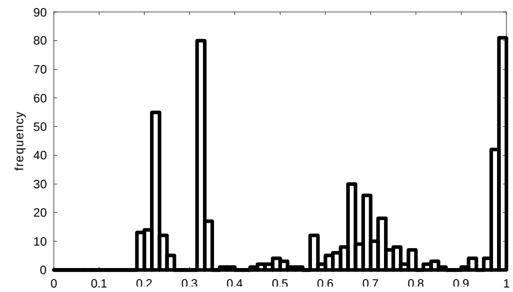
(c) DEA: Firm 21 (π_1 = 0.75)



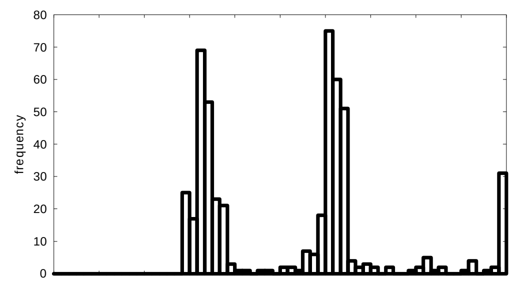
(d) DEA: Firm 25 (π_1 = .943)



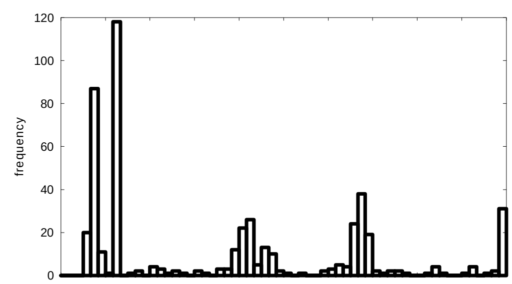
(e) SFA: Firm 1 (π_1 = 0.061)



(f) SFA: Firm 14 (π_1 = 0.5)

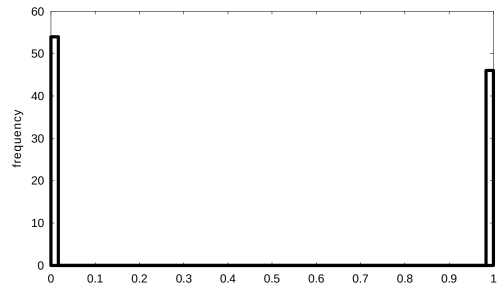


(g) SFA: Firm 21 (π_1 = 0.75)

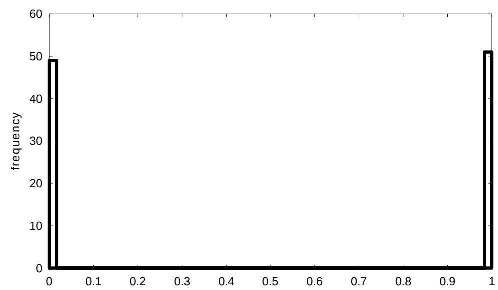


(h) SFA: Firm 25 (π_1 = .943)

Figure 4: Estimated Sampling Distributions: Moderate Output Substitutability



(a) DEA: Firm 1 (π_1 = 0.061)



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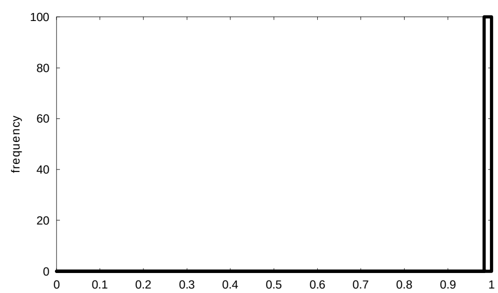
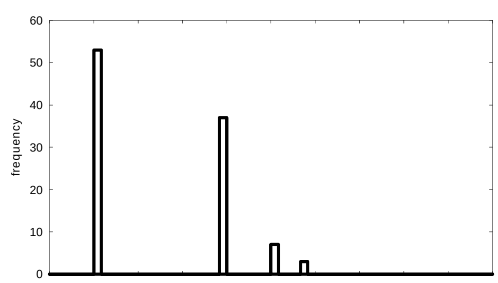
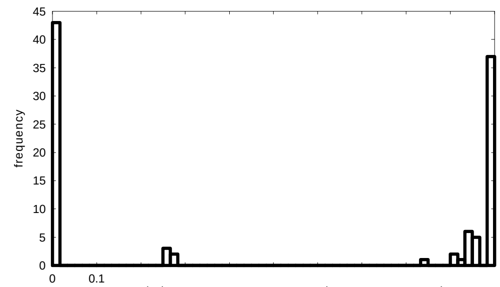


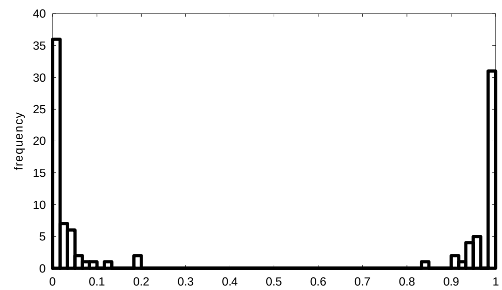
(c) DEA: Firm 21 (π_1 = 0.75)



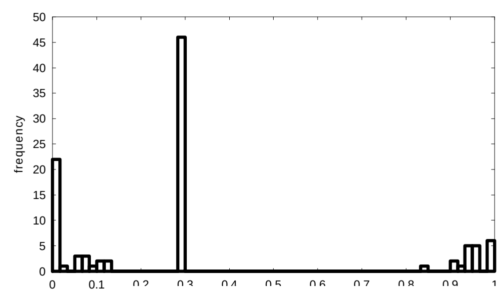
(d) DEA: Firm 25 (π_1 = .943)



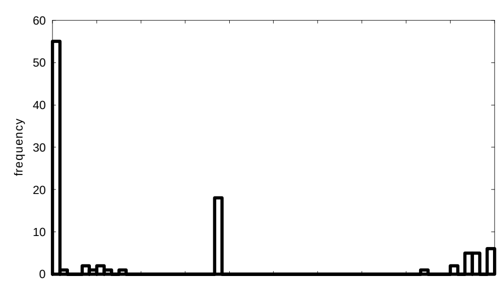
(e) SFA: Firm 1 (π_1 = 0.061)



(f) SFA: Firm 14 (π_1 = 0.5)

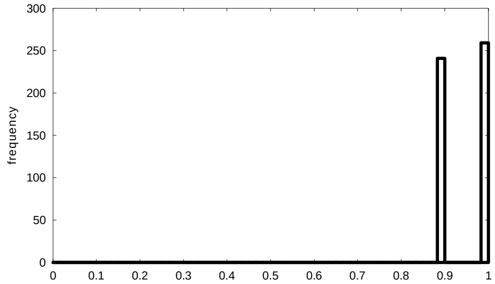


(g) SFA: Firm 21 (π_1 = 0.75)

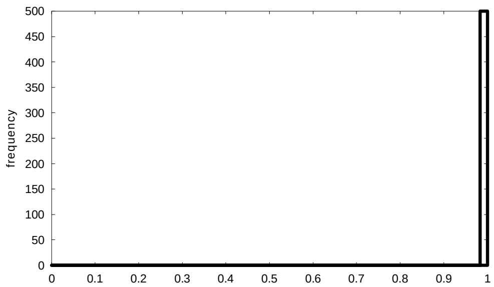


(h) SFA: Firm 25 (π_1 = .943)

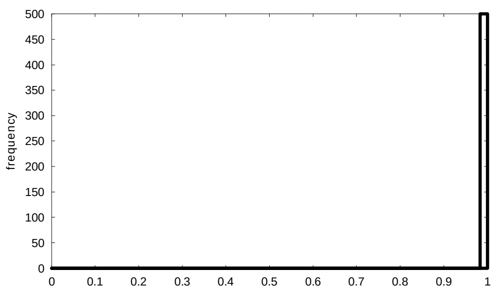
Figure 5: Estimated Sampling Distributions: High Output Substitutability



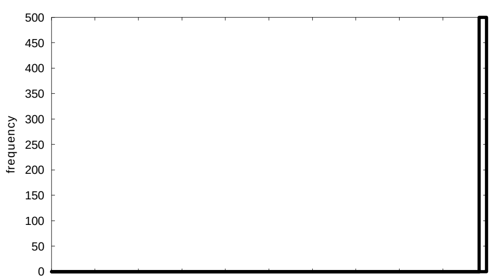
(a) DEA: Firm 1 (π_1 = 0.061)



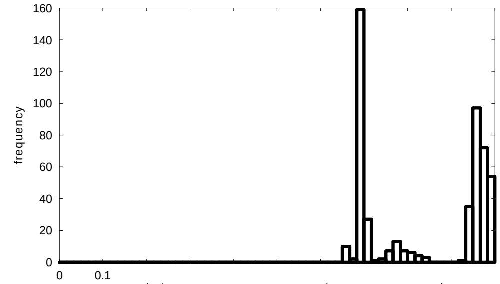
(b) DEA: Firm 14 (π_1 = 0.5)



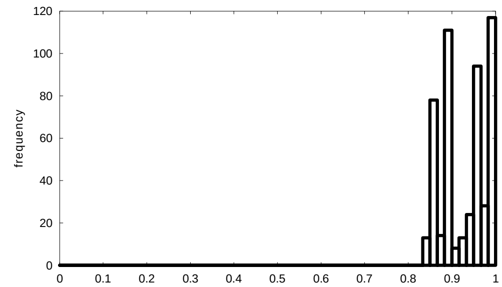
(c) DEA: Firm 21 (π_1 = 0.75)



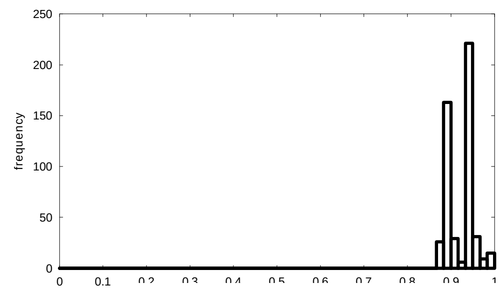
(d) DEA: Firm 25 (π_1 = .943)



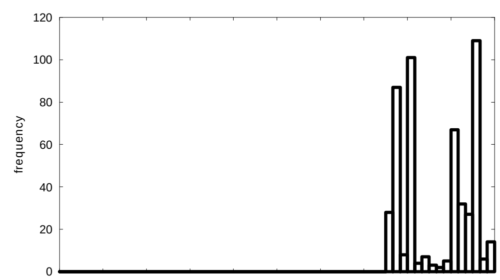
(e) SFA: Firm 1 (π_1 = 0.061)



(f) SFA: Firm 14 (π_1 = 0.5)



(g) SFA: Firm 21 (π_1 = 0.75)



(h) SFA: Firm 25 (π_1 = .943)

Figure 6: Estimated Sampling Distributions: Low Output Substitutability

Table 1: Production Choices: Moderate Output Substitutability

π_1	z_1	z_2	x	$A = 1$	$A = 10$
0.00	0.000	2.000	2.000	0.0000	0.0000
0.05	0.033	1.900	1.807	0.0081	0.0000
0.10	0.067	1.800	1.627	0.0193	0.0000
0.15	0.100	1.700	1.460	0.0344	0.0000
0.20	0.133	1.600	1.307	0.0545	0.0000
0.25	0.167	1.500	1.167	0.0808	0.0000
0.30	0.200	1.400	1.040	0.1143	0.0000
0.35	0.233	1.300	0.927	0.1563	0.0000
0.40	0.267	1.200	0.827	0.2077	0.0001
0.45	0.300	1.100	0.740	0.2688	0.0003
0.50	0.333	1.000	0.667	0.3392	0.0013
0.55	0.367	0.900	0.607	0.4176	0.0059
0.60	0.400	0.800	0.560	0.5014	0.0267
0.65	0.433	0.700	0.527	0.5872	0.1143
0.70	0.467	0.600	0.507	0.6713	0.3808
0.75	0.500	0.500	0.500	0.7500	0.7500
0.80	0.533	0.400	0.507	0.8205	0.9382
0.85	0.567	0.300	0.527	0.8809	0.9879
0.90	0.600	0.200	0.560	0.9307	0.9980
0.95	0.633	0.100	0.607	0.9700	0.9997
1.00	0.667	0.000	0.667	1.0000	1.0000

Table 2: Production Choices: Near-Zero Output Substitutability

π_1	z_1	z_2	x	$A=1$	$A=10$
0.00	0.000	0.904	0.164	0.0000	0.0000
0.05	0.600	0.899	0.161	0.0376	0.0026
0.10	0.643	0.894	0.158	0.0796	0.0089
0.15	0.670	0.889	0.156	0.1241	0.0193
0.20	0.689	0.884	0.154	0.1707	0.0345
0.25	0.705	0.878	0.152	0.2189	0.0557
0.30	0.718	0.872	0.150	0.2686	0.0840
0.35	0.729	0.866	0.149	0.3196	0.1208
0.40	0.739	0.859	0.147	0.3716	0.1673
0.45	0.748	0.851	0.146	0.4245	0.2248
0.50	0.756	0.843	0.145	0.4781	0.2937
0.55	0.763	0.834	0.145	0.5322	0.3738
0.60	0.769	0.825	0.144	0.5867	0.4634
0.65	0.776	0.814	0.143	0.6413	0.5592
0.70	0.781	0.801	0.143	0.6958	0.6567
0.75	0.787	0.787	0.143	0.7500	0.7500
0.80	0.792	0.769	0.143	0.8036	0.8335
0.85	0.797	0.748	0.144	0.8562	0.9025
0.90	0.801	0.718	0.144	0.9073	0.9539
0.95	0.806	0.670	0.145	0.9561	0.9866
1.00	0.810	0.000	0.147	1.0000	1.0000

Table 3: Production Choices: High Output Substitutability

π_1	z_1	z_2	x	$A = 1$	$A = 10$
0.00	0.000	40E+4	73E+4	0.0000	0.0000
0.05	0.000	24E+4	42E+4	0.0000	0.0000
0.10	0.000	14E+4	23E+4	0.0000	0.0000
0.15	0.000	79590	12E+4	0.0000	0.0000
0.20	0.000	43408	63139	0.0000	0.0000
0.25	0.000	22766	31044	0.0000	0.0000
0.30	0.000	11420	14534	0.0000	0.0000
0.35	0.000	5543	6432	0.0000	0.0000
0.40	0.001	2444	2666	0.0000	0.0000
0.45	0.002	1024	1024	0.0000	0.0000
0.50	0.007	394.8	358.9	0.0000	0.0000
0.55	0.017	137.7	112.6	0.0000	0.0000
0.60	0.041	42.39	30.88	0.0000	0.0000
0.65	0.092	11.15	7.206	0.0000	0.0000
0.70	0.193	2.387	1.548	0.2064	0.0000
0.75	0.386	0.386	0.701	0.7500	0.7500
0.80	0.735	0.041	1.084	0.8889	0.9998
0.85	1.348	0.002	2.084	0.9561	1.0000
0.90	2.387	0.000	3.906	0.9899	1.0000
0.95	4.099	0.000	7.080	0.9991	1.0000
1.00	6.846	0.000	12.49	1.0000	1.0000

Table 4: Inputs, Outputs, States of Nature, and Efficiency Estimates

Firm	π_1	x	z_1	z_2	s	z_s	DEA	SFA
1	0.061	1.767	0.041	1.878	2	1.878	1.000	0.901
2	0.125	1.541	0.083	1.750	1	0.083	0.324	0.060
3	0.240	1.194	0.160	1.520	1	0.160	0.419	0.112
4	0.270	1.115	0.180	1.461	1	0.180	0.448	0.126
5	0.318	0.999	0.212	1.365	2	1.365	1.000	0.819
6	0.341	0.947	0.227	1.319	2	1.319	1.000	0.805
7	0.367	0.891	0.245	1.265	1	0.245	0.561	0.170
8	0.384	0.856	0.256	1.231	2	1.231	1.000	0.775
9	0.387	0.851	0.258	1.225	1	0.258	0.588	0.180
10	0.418	0.793	0.279	1.163	1	0.279	0.630	0.194
11	0.441	0.755	0.294	1.118	2	1.118	1.000	0.727
12	0.475	0.701	0.317	1.049	2	1.049	1.000	0.693
13	0.489	0.682	0.326	1.022	1	0.326	0.733	0.225
14	0.500	0.667	0.333	1.000	2	1.000	1.000	0.666
15	0.501	0.665	0.334	0.997	2	0.997	1.000	0.664
16	0.508	0.656	0.339	0.983	2	0.983	1.000	0.656
17	0.573	0.584	0.382	0.854	1	0.382	0.857	0.263
18	0.574	0.583	0.382	0.853	1	0.382	0.858	0.264
19	0.668	0.518	0.446	0.663	2	0.663	1.000	0.452
20	0.673	0.516	0.449	0.654	2	0.654	1.000	0.445
21	0.750	0.500	0.500	0.500	2	0.500	1.000	0.343
22	0.755	0.500	0.504	0.489	1	0.504	1.000	0.354
23	0.790	0.504	0.527	0.420	1	0.527	0.997	0.360
24	0.902	0.561	0.601	0.197	1	0.601	0.909	0.410
25	0.943	0.599	0.629	0.114	2	0.114	0.834	0.081
Mean	0.498	0.797	0.332	1.004	1.520	0.672	0.846	0.429

Table 5: Characteristics of Sampling Distributions: Moderate Output Substitutability

		DEA				SFA			
Firm	π_1	Mean	St.Dev.	Min	Max	Mean	St.Dev.	Min	Max
1	0.061	0.659	0.358	0.283	1.000	0.487	0.422	0.018	1.000
2	0.125	0.638	0.337	0.324	1.000	0.489	0.418	0.041	1.000
3	0.240	0.705	0.291	0.419	1.000	0.572	0.401	0.095	1.000
4	0.270	0.746	0.275	0.448	1.000	0.606	0.391	0.103	1.000
5	0.318	0.738	0.250	0.501	1.000	0.570	0.380	0.121	1.000
6	0.341	0.763	0.236	0.528	1.000	0.596	0.369	0.130	1.000
7	0.367	0.777	0.219	0.561	1.000	0.595	0.358	0.140	1.000
8	0.384	0.788	0.208	0.584	1.000	0.594	0.349	0.146	1.000
9	0.387	0.794	0.206	0.588	1.000	0.598	0.344	0.148	1.000
10	0.418	0.808	0.185	0.630	1.000	0.585	0.333	0.159	1.000
11	0.441	0.835	0.169	0.662	1.000	0.595	0.317	0.168	1.000
12	0.475	0.853	0.143	0.713	1.000	0.593	0.302	0.181	1.000
13	0.489	0.867	0.134	0.733	1.000	0.598	0.292	0.186	1.000
14	0.500	0.875	0.125	0.750	1.000	0.598	0.295	0.191	1.000
15	0.501	0.884	0.124	0.752	1.000	0.616	0.295	0.191	1.000
16	0.508	0.881	0.119	0.763	1.000	0.603	0.293	0.194	1.000
17	0.573	0.932	0.072	0.857	1.000	0.601	0.256	0.218	1.000
18	0.574	0.932	0.071	0.858	1.000	0.600	0.249	0.219	1.000
19	0.668	0.982	0.017	0.966	1.000	0.564	0.208	0.255	1.000
20	0.673	0.984	0.105	0.969	1.000	0.561	0.211	0.256	1.000
21	0.750	1.000	0.000	1.000	1.000	0.531	0.198	0.286	1.000
22	0.755	1.000	0.000	1.000	1.000	0.527	0.198	0.280	1.000
23	0.790	0.995	0.003	0.992	1.000	0.508	0.203	0.240	1.000
24	0.902	0.902	0.015	0.891	0.952	0.411	0.264	0.112	1.000
25	0.943	0.848	0.018	0.834	0.927	0.365	0.297	0.065	1.000