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Supermodularity and the comparative statics of risk

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Abstract

In this paper, it is shown that a wide range of comparative statics results from expected utility theory can be extended to generalized expected utility models using the tools of supermodularity theory.

1 Introduction

There is a large literature on the comparative-statics of choice under uncertainty. The archetypal result is Pratt's demonstration, for the portfolio problem with one safe asset and one risky asset, that an expected-utility maximizer displays decreasing absolute risk aversion if and only if the amount of the risky asset in the optimal portfolio increases with an increase in base wealth. There is a sense in which this result may be viewed as trivial. If the concepts 'risk aversion declining in wealth' and ' riskiness' have been defined correctly, an increase in wealth must, by definition, lead to the choice of a less risky portfolio.

A natural way of rephrasing the notion 'risk aversion declining in wealth' is to say that mean wealth and risk-taking are complements. This immediately suggests the possibility of analyzing the problem using the tools of supermodularity (Milgrom 1994; Topkis 1998), which have revolutionized the study of comparative static problems in economics in recent years.

It is natural therefore, to consider whether the tools of supermodularity theory can be applied to derive general comparative static results without dependence on a particular functional form. The object of this paper is to provide an affirmative answer to this question.

2 Model

Let $Y \subseteq \Re^S$ be a space of state-contingent vectors, with a partial ordering in which the relationship $\mathbf{y} \preceq \mathbf{y}'$ is interpreted as '$\mathbf{y}$ is less risky than $\mathbf{y}'$'. We assume that there exists a known probability vector π over S and define the associated expected-value function μ,

$$\mu(\mathbf{y}) = \sum \pi_s y_s$$

Assume that the risk ordering $\preceq$ embodies the minimal notion of riskiness

$$\mu(\mathbf{y})\,\mathbf{1} \preceq \mathbf{y} \quad \forall \mathbf{y} \tag{1}$$

and, in addition, that $\preceq$ is quasi-concave.

Preference orderings P over Y are represented by certainty equivalent functionals of the form

$$e(\mathbf{y}) = \inf\{c : c\mathbf{1}P\mathbf{y}\}.$$

A preference ordering, and the associated certainty equivalent, will be described as risk-averse, with respect to $\preceq$ if

$$\mathbf{y} \preceq \mathbf{y}' \& \mu(\mathbf{y}) = \mu(\mathbf{y}') \Rightarrow e(\mathbf{y}) \geq e(\mathbf{y}')$$

Let E be a space of risk-averse, quasi-concave certainty equivalent preference functionals, also with a partial ordering $\preceq^*$, in which the relationship $e \preceq^* \tilde{e}$ is interpreted as 'e is less risk-averse than $\tilde{e}$'.[1]

Let $r : E \times Y \rightarrow \Re$ be the risk premium operator

$$r(e, \mathbf{y}) = \mu(\mathbf{y}) - e(\mathbf{y}).$$

The explicit specification of r as a function of e facilitates the application of supermodularity concepts in what follows. Since both E and Y are ordered spaces, it makes sense to say that r is increasing or decreasing, and supermodular or submodular in its arguments. We begin by considering increasingness.

Assuming that r is increasing in both its arguments, the partial orders $\preceq$ and $\preceq^*$ are consistent with r in the sense that

$$\begin{aligned} e &\preceq {}^*\tilde{e} \Rightarrow r(e, \mathbf{y}) \leq r(\tilde{e}, \mathbf{y}) \quad \forall \mathbf{y} \\ &\Leftrightarrow \ e(\mathbf{y}) \geq \tilde{e}(\mathbf{y})\text{; and} \end{aligned}$$

$$\begin{aligned} \mathbf{y} &\preceq \mathbf{y}' \Rightarrow r(e, \mathbf{y}) \leq r(e, \mathbf{y}') \quad \forall e \\ &\Leftrightarrow \ e(\mathbf{y}) \geq e(\mathbf{y}') \text{ if } \mu(\mathbf{y}) = \mu(\mathbf{y}') \end{aligned}$$

We may observe, that, for all $e, \mathbf{y}, c$,

$$\begin{aligned} e(c\mathbf{1}) &= c\text{; and} \\ r(e, c\mathbf{1}) &= 0. \end{aligned}$$

Since r is increasing in both its arguments, the relationship $\mu(\mathbf{y})\mathbf{1} \preceq \mathbf{y}$ now implies $r(e, \mathbf{y}) \geq 0 \ \forall \mathbf{y}$ so that preferences are risk averse in the sense

$$e(\mathbf{y}) \leq \mu(\mathbf{y}) \ \forall \mathbf{y} \tag{2}$$

By the assumption that preferences are quasi-concave, this implies that, for any $\mathbf{y}$ and $\lambda \in [0, 1]$,

$$e(\mathbf{y}) \leq e(\lambda\mathbf{y} + (1 - \lambda)\mu(\mathbf{y})\mathbf{1}).$$

We will therefore assume that the risk ordering $\preceq$ satisfies

$$\lambda\mathbf{y} + (1 - \lambda)\mu(\mathbf{y})\mathbf{1} \preceq \mathbf{y} \tag{3}$$

[1]Note that, associated with any risk ordering $\preceq$ there is a natural dual notion $\preceq^*$ of more risk-averse [Quiggin, J. (1992), 'Efficient sets with and without the expected utility hypothesis: a generalization', Journal of Mathematical Economics 21, 395–99.] so that the use of the star notation makes sense. However, the duality relationship will not be developed in detail in the present paper.

2.0.1 Examples

Some examples will prove useful in what follows. We first define a series of risk orderings (partial orderings of Y). They are arranged in increasing order of the number of pairs $(\mathbf{y}, \mathbf{y}')$ ordered in terms of increasing risk. We will say that a risk ordering $\preceq^o$ is stronger than another ordering $\preceq'$ if

$$\mathbf{y} \preceq' \mathbf{y}' \Rightarrow \mathbf{y} \preceq^o \mathbf{y}'$$

so that any pair $(\mathbf{y}, \mathbf{y}')$ ordered by $\preceq'$ is also ordered by $\preceq^o$.

Example 1 *The minimal risk ordering consistent with risk aversion, requiring that receipt of mean income with certainty is preferred to the corresponding risky state-contingent income vector is denoted* $\preceq_0$ *. The only risk-ordering relationships implied by* $\preceq_0$ *are of the form* $\mu(\mathbf{y})\mathbf{1} \preceq_0 \mathbf{y}$*.*

Example 2 *Consider the multiplicative-spread risk ordering* $\preceq_1$*,described by*

$$\lambda\mathbf{y} + (1-\lambda)\mu(\mathbf{y})\mathbf{1} \preceq_1 \mathbf{y}$$

where $0 \leq \lambda \leq 1$*.*

For many purposes (essentially those where the outcome is a linear function of a choice variable) it is sufficient to consider the ordering $\preceq_1$. However, stronger orderings are frequently useful. We consider two such orderings.

Example 3 *Suppose that* $\mathbf{y}$*,* $\mathbf{y}'$ *and* $\varepsilon = \mathbf{y} - \mathbf{y}'$ *are comonotonic, in the sense that* $(y_s - y_t)(y'_s - y'_t) \geq 0 \ \forall s, t$*, and that* $\mu(\varepsilon) = 0$*. Then we write* $\mathbf{y} \preceq_m \mathbf{y}'$*, stated as* $\mathbf{y}'$ *is derived from* $\mathbf{y}$ *by a monotone spread (Quiggin 1991; Chateauneuf and Cohen 1994, Chateauneuf, Cohen and Meilijson 2004). Observe that if* $\mathbf{y} \preceq_1 \mathbf{y}'$, $\mathbf{y} - \mathbf{y}' = (1-\lambda)(\mathbf{y} - \mu(\mathbf{y})\ \mathbf{1})$ *which satisfies the stated conditions.*

Example 4 *We will use the notation* $\mathbf{y} \preceq_\pi \mathbf{y}'$ *to mean '*$\mathbf{y}'$ *is derived from* $\mathbf{y}$ *by a mean-preserving increase in risk in the sense of Rothschild and Stiglitz (1970).*[2] *Quiggin (1991) shows that* $\mathbf{y} \preceq_m \mathbf{y}' \Rightarrow \mathbf{y} \preceq_\pi \mathbf{y}'$*, but the reverse implication does not hold.*

Next, we define the certainty-equivalent for two commonly-used preference orderings

Example 5 *For expected utility (EU) preferences, the certainty equivalent is given by*

$$e(\mathbf{y}) = u^{-1}\left(\sum_s \pi_s u(y_s)\right)$$

where u *is a concave, twice-differentiable von Neumann-Morgenstern utility function*

[2]See also Hadar and Russell (1969) and Hanoch and Levy (1969).

Example 6 *For rank-dependent expected utility (RDEU) preferences, the certainty equivalent is given by*

$$e(\mathbf{y}) = u^{-1}\left(\sum_s w_s u(y_s)\right)$$

where u is a concave, twice-differentiable von Neumann-Morgenstern utility function as before and the w_s are probability weights, defined as

$$w_s = f\left(\sum_{j=1}^{s} \pi_j\right) - f\left(\sum_{j=1}^{s-1} \pi_j\right)$$

where f is a probability weighting function applied to the cumulative distribution function

$$F_s = \sum_{j=1}^{s} \pi_j.$$

2.1 Supermodularity and optimization

The notion of supermodularity has been widely recognized as a crucial tool in comparative static analysis and as the most useful formalization of the intuitive concept of complementarity (Topkis 1998). Supermodularity is defined with respect to a partial order. Let X be a lattice, that is a partially ordered set with ordering $\preceq$ with the property that for any x, x', there exists a least upper bound (or join) $x \vee x' \in X$ and a greatest lower bound (or meet) $x \wedge x' \in X$. The join $x \vee x'$ satisfies

$$\begin{aligned} x, x' &\preceq x \vee x' \\ x, x' &\preceq x'' \Rightarrow x \vee x' \preceq x'' \end{aligned}$$

Similarly,

$$\begin{aligned} x \wedge x' &\preceq x, x' \\ x'' &\preceq x, x' \Rightarrow x'' \preceq x \vee x' \end{aligned}$$

Two examples of lattices will prove particularly important in what follows. First, a chain (that is, a totally ordered set) is a lattice.Second, if X and Z are lattices with partial orderings $\preceq_X$ and $\preceq_Z$, so is $X \times Z$ with the partial ordering $(x, z) \preceq (x', z')$ defined by $x \preceq_X x'$ and $z \preceq_Z z'$.

A mapping $f : X \to \Re$ is *supermodular* [3] if for all x,x'

$$f(x \vee x') + f(x \wedge x') \geq f(x) + f(x').$$

[3] Supermodularity is sometimes referred to as L-superadditivity, where L is mnemonic for lattice. See, for example, Marshall and Olkin (1979).

$f : X \to \Re$ is submodular if $-f$ is supermodular.

The ordering $\preceq$ induces an ordering on $2^X/\emptyset$, the set of non-empty subsets of X, that is of particular interest in characterizing the solutions to optimization problems. If X', X'' are non-empty subsets of X, the ordering $X' \sqsubseteq X''$ means that for any $x' \in X'$, $x'' \in X''$, $x' \vee x'' \in X''$, and $x' \wedge x'' \in X'$. In particular, if $X' = \{x'\}$ and $X' = \{x''\}$ are singletons, as is commonly the case for well-behaved optimization problems, $X' \sqsubseteq X''$ if and only if $x' \preceq x''$.

Now let $\phi : T \to 2^X/\emptyset$ be a correspondence from the partially ordered set T (with ordering $\preceq$) whose range consists of non-empty subsets of X. Then ϕ is *increasing* as a function of t, if $t \preceq t' \Rightarrow \phi(t) \sqsubseteq \phi(t')$.

We will rely primarily on Lemma 1, which is a slight simplification of Topkis (Theorem 2.8.2, p77).

Lemma 1 *If X and T are lattices and $f : X \times T \to \Re$ is supermodular on (x,t), then t $\phi(t) = \arg\max_x \{f(x,t)\}$ is increasing on the set*

$$\left\{t : \arg\max_x \{f(x,t)\} \text{ is non-empty}\right\}.$$

3 Choice problems and comparative static problems

A typical choice problem begins with a set A of choice variables, with typical element α, a vector $\boldsymbol{\theta} \in \Theta$ of exogenous variables, and a mapping $\mathbf{f} : A \times \Theta \to Y$ .The mapping $\mathbf{f}$ yields, for each value $\boldsymbol{\theta}$ of the exogenous variables and each value α of the choice variable, a state-contingent outcome vector $\mathbf{f}(\alpha; \boldsymbol{\theta})$. The choice problem is then to choose α to solve

$$\max_\alpha \{e(\mathbf{f}(\alpha; \boldsymbol{\theta}))\}.$$

Because we are interested in conducting comparative static analysis over the space of certainty equivalent functionals as well as over Θ, it is informative to rewrite this optimization problem in terms of the equivalent representation

$$\max_\alpha \{\mu(\mathbf{f}(\alpha; \boldsymbol{\theta})) - r(e, \mathbf{f}(\alpha; \boldsymbol{\theta}))\}$$

Denote the set of optimizers for this problem by.

$$\alpha(e, \theta) = \arg\max_\alpha \{\mu(\mathbf{f}(\alpha; \boldsymbol{\theta})) - r(e, \mathbf{f}(\alpha; \boldsymbol{\theta}))\}. \quad (4)$$

For any given value of $\boldsymbol{\theta}$, A inherits an ordering $\preceq^A$ from the partial ordering of Y by way of the mapping $\mathbf{f}$. This inherited ordering is defined by

$$\alpha \preceq^A \alpha' \Leftrightarrow \mathbf{f}(\alpha; \boldsymbol{\theta}) \preceq \mathbf{f}(\alpha'; \boldsymbol{\theta}).$$

Conversely, for any given α, Θ inherits an ordering $\preceq^{\Theta}$ from the partial ordering of Y, which is defined by

$$\boldsymbol{\theta} \preceq^{\Theta} \boldsymbol{\theta}' \Leftrightarrow \mathbf{f}(\alpha; \boldsymbol{\theta}) \preceq \mathbf{f}(\alpha; \boldsymbol{\theta}'). \tag{5}$$

The results of supermodularity theory are directly applicable when the ordering $\preceq^{A}$ is independent of the particular choices of $\boldsymbol{\theta}$, and A and Θ are lattices under $\preceq^{A}$ and $\preceq^{\Theta}$, respectively.

Both A and Θ may also be equipped with orderings not derived from $\preceq$. In many settings, for example, α is a scalar variable with the natural ordering, though this will typically coincide with the induced ordering $\preceq^{A}$. Except where otherwise stated, whenever α is a scalar, we will assume that the orderings $\preceq^{A}$ and $\preceq^{\Theta}$coincide with the natural ordering.

Example 7 *The portfolio problem: Let $w \in \Re_{+}$ represent initial wealth, $\alpha \in \Re$ the amount of the risky asset held, $\mathbf{r} \in \Re_{+}^{S}$ the vector of state-contingent returns on the risky asset and $p \in \Re_{+}$ the price of the asset. Thus $\boldsymbol{\theta} = (w, \mathbf{r}, p)$ is the vector of exogenous parameters. Define the vector of state-contingent returns contingent upon the choice of α by:*

$$\mathbf{f}(\alpha, \boldsymbol{\theta}) = \alpha\mathbf{r} + (w - p\alpha)\mathbf{1}.$$

So, for example, for almost any plausible risk ordering, $\preceq$, $(\alpha = 0) \preceq^{A} (\alpha = 1)$ because the former implies a riskless position. More generally, for plausible risk orderings, higher choices of α represent increases in risk. Hence the two comparative static problems are to determine conditions on r under which less risk averse individuals will choose higher levels of α and under which changes in θ will lead an individual with given preferences to choose a higher level of α.

Two main classes of comparative static problems are of interest in what follows. First, we first consider the influence that different preference structures have on optimal choices by considering under what conditions will $\alpha(e, \theta)$ be increasing in e. Second, considering changes in the exogenous variables θ, under what conditions will $\alpha(e, \theta)$ be increasing in θ.

3.1 Changes in preferences

If we consider only changes from e to $\tilde{e}$ with $e \preceq^{*} \tilde{e}$, the set E can always be restricted to ensure that it is totally ordered (that is, a chain) with respect to $\preceq^{*}$. Hence, the substantive requirement for the application of supermodularity theory is that A is a lattice with respect to $\preceq^{A}$. Under the assumption that A is a lattice, the comparative statics of changes in preferences may be derived directly from Lemma 1

Proposition 1 *In the class of problems (4) consider changes from e to $\tilde{e}$, with θ held constant, implying an associated shift from $\alpha(e,\theta)$ to $\alpha(\tilde{e},\theta)$. For the given θ, suppose A is a lattice under $\preceq^A$. Then if r is supermodular in e and $\mathbf{y}, \alpha(e,\theta)$ is decreasing in e.*
Proof: By (5), if r is supermodular in e and $\mathbf{y} = \mathbf{f}(\alpha;\boldsymbol{\theta})$ for given θ, then r is supermodular in e and α, so

$$\mathbf{f}(\alpha;\boldsymbol{\theta}) = \mu(\mathbf{f}(\alpha;\boldsymbol{\theta})) - r(e,\mathbf{f}(\alpha;\boldsymbol{\theta}))$$

is submodular in (e,α) and Lemma 1 applies.

If we relax the assumption that A is a lattice under $\preceq^A$, we obtain:

Proposition 2 *In the class of problems (4) consider changes from e to $\tilde{e}$ with an associated shift from $\alpha(e,\theta)$ to $\alpha(\tilde{e},\theta)$. Then if r is supermodular in E and Y, there exists no choice problem of the form (4), and no pair $e,\tilde{e}$, such that $\tilde{e} \preceq^* e$ and $\alpha(\tilde{e},\theta) \preceq^A \alpha(e,\theta)$. That is, a reduction in risk aversion cannot lead to a reduction in risk-taking.*
Proof: Let $\widehat{\alpha} = \alpha(e,\theta)$ and denote $r(\mathbf{f}(\alpha,\theta),e)$ by $\hat{r}(e,\alpha)$. For any α such that $\alpha \preceq^A \widehat{\alpha} \Leftrightarrow \mathbf{f}(\alpha,\theta) \preceq \mathbf{f}(\widehat{\alpha},\theta)$. Supermodularity of the risk premium for the partial ordering defined by $\left(\preceq^*, \preceq^A\right)$ requires

$$\hat{r}(e \vee \tilde{e}, \alpha \vee \hat{\alpha}) + \hat{r}(e \wedge \tilde{e}, \alpha \wedge \hat{\alpha}) \geq \hat{r}(e,\alpha) + \hat{r}(\tilde{e},\hat{\alpha}),$$

Taking $\alpha \preceq^A \widehat{\alpha}$ and $\tilde{e} \preceq^* e$, supermodularity then implies

$$\hat{r}(e,\hat{\alpha}) + \hat{r}(\tilde{e},\alpha) \geq \hat{r}(e,\alpha) + \hat{r}(\tilde{e},\hat{\alpha}),$$

whence

$$\hat{r}(e,\hat{\alpha}) - \hat{r}(e,\alpha) \geq \hat{r}(\tilde{e},\hat{\alpha}) - \hat{r}(\tilde{e},\alpha).$$

Rewriting in terms of certainty equivalents then gives:

$$\begin{aligned}\tilde{e}(\mathbf{f}(\widehat{\alpha},\theta)) - \tilde{e}(\mathbf{f}(\alpha,\theta)) &\geq e(\mathbf{f}(\widehat{\alpha},\theta)) - e(\mathbf{f}(\alpha,\theta)) \\ &\geq 0\end{aligned}$$

by the optimality of $\widehat{\alpha}$ for e. Hence α cannot be optimal for $\tilde{e}$.

The negative form of Proposition 2 reflects the fact that $\preceq$ and $\preceq^A$ are partial orders. Hence, it is not, in general, possible to exclude the possibility that $\alpha(e,\theta)$ and $\alpha(\tilde{e},\theta)$ will be unrelated by $\preceq^A$, and will not have a least upper bound.

The results above may be restated in essentially equivalent terms by allowing E to be an arbitrary partially ordered set, rather than a chain, and replacing the supermodularity condition with one of increasing differences. That is, for $\alpha \preceq^A \alpha'$, $e \preceq \tilde{e}$

$$\hat{r}\left(e,\alpha'\right) - \hat{r}(e,\alpha) \geq \hat{r}\left(\tilde{e},\alpha'\right) - \hat{r}(\tilde{e},\alpha),$$

3.2 Changes in parameters

Now consider the problem of changes in θ. Since

$$e\left(\mathbf{f}(\alpha,\boldsymbol{\theta})\right)=\mu\left(\mathbf{f}(\alpha,\boldsymbol{\theta})\right)-r\left(\mathbf{f}(\alpha,\boldsymbol{\theta}),e\right)$$

it is natural to consider the supermodularity and submodularity properties of μ and r considered as functions of α and θ.

As in the previous section, if we consider only changes from θ to θ' with $\theta \preceq^{\Theta} \theta'$, the set Θ can always be restricted to ensure that it is totally ordered (hence, a lattice) with respect to $\preceq^{\Theta}$. Hence, the substantive requirement for the application of supermodularity theory is that A is a lattice with respect to $\preceq^{A}$.

Clearly, if μ is supermodular (submodular) in α,θ and r is submodular (supermodular) in α,θ, then e is supermodular (submodular) in α,θ with respect to the orderings $\preceq^{A}$, $\preceq^{\Theta}$. A sufficient condition for μ to be supermodular (submodular) in α,θ is that each y_s should be supermodular (submodular) in α,θ for all s. Hence, we obtain

Proposition 3 *The following are sufficient conditions for $\alpha\left(e,\theta\right)$ to be increasing (decreasing) in θ :*
(i)μ is supermodular (submodular) in α,θ; and
(ii)r is submodular (supermodular) in α,θ.

Proposition 3 yields the corollary:

Corollary 1 *The following are sufficient conditions for $\alpha\left(e,\theta\right)$ to be increasing (decreasing) in θ*
(i) y_s is supermodular (submodular) in α,θ for all s.
(ii) r is submodular (supermodular) in α,θ

In most cases of interest, it is fairly easy to determine the supermodularity properties of μ and often of each y_s. In the portfolio problem, for example, μ is supermodular in $\mathbf{r}$ and α, submodular in p and α and a valuation (both supermodular and submodular) in w and α. Hence, most interest is in determining the supermodularity properties of the risk premium r with respect to α and θ.

As in the previous section, these results may be restated in terms of increasing differences.

4 Wealth and risk aversion

As was noted in the introduction, the archetypal result in the comparative statics of choice under uncertainty is Pratt's demonstration, for the portfolio problem with one safe asset and one risky asset, that an expected-utility maximizer displays decreasing absolute risk aversion if and only if the amount of the risky asset in the optimal portfolio increases with an increase in base wealth. Moreover, decreasing absolute risk aversion may be reinterpreted as a supermodularity condition

Definition 1 *Preferences display decreasing (increasing, constant) absolute risk aversion (for a given risk ordering* $\preceq$ *on* Y*) if the function* $\check{e} : Y \times \Re \to \Re$

$$\check{e}(\mathbf{y},\delta) = e(\mathbf{y}+\delta\mathbf{1})$$

is supermodular (submodular, a valuation) in $\mathbf{y}$ *and* δ *where* Y *is partially ordered by* $\preceq$ *.*

Since $\mu(\mathbf{y}+\delta\mathbf{1}) = \mu(\mathbf{y}) + \delta$ is a valuation in $\mathbf{y}$ and δ, the induced risk premium $\hat{r}$ is submodular if and only if $\check{e}$ is supermodular and *vice versa.*

Quiggin and Chambers (1998) define constant (decreasing, increasing) absolute risk aversion by the requirement $e(\mathbf{y}+\delta\ \mathbf{1}) = (\geq, \leq)\, e(\mathbf{y}) + \delta$.

We have

Lemma 2 *If preferences display decreasing (increasing, constant) absolute risk aversion according to Definition 1, then for all* $\mathbf{y}$ *and* $\delta \geq 0$,

$$\begin{aligned} e(\mathbf{y}+\delta\mathbf{1}) &\geq (\leq, =)\, e(\mathbf{y}) + \delta \\ r(\mathbf{y}+\delta\mathbf{1}) &\leq (\geq, =)\, r(\mathbf{y}) . \end{aligned}$$

Proof: Since $\mu(\mathbf{y})\,\mathbf{1} \preceq \mathbf{y}, 0 \leq \delta$*, submodularity of* $\check{e}$ *in* $\mathbf{y}$ *and* δ *requires*

$$\check{e}(\mathbf{y},0) + \check{e}(\mu(\mathbf{y})\,\mathbf{1},\delta) \leq \check{e}(\mathbf{y},\delta) + \check{e}(\mu(\mathbf{y})\,\mathbf{1},0)$$

or

$$e(\mathbf{y}) + \mu(\mathbf{y}) + \delta \leq e(\mathbf{y}+\delta\mathbf{1}) + \mu(\mathbf{y}) .$$

For the case of constant absolute risk aversion we can prove the reverse implication for all $\preceq$

Lemma 3 *Preferences display constant risk aversion if and only if*

$$e(\mathbf{y}+\delta\mathbf{1}) = e(\mathbf{y}) + \delta,\ \forall \mathbf{y},\delta.$$

Proof: By Theorem 2.6.4 of Topkis, $\check{e}(\mathbf{y},\delta)$ *is a valuation if and only if*

$$\check{e}(\mathbf{y},\delta) \equiv e(\mathbf{y}+\delta\mathbf{1}) = v(\mathbf{y}) + m(\delta)$$

for all $\mathbf{y}$ *and* δ*. Set* $\delta = 0$ *to obtain*

$$v(\mathbf{y}) = e(\mathbf{y}) - m(0) ,$$

whence

$$e(\mathbf{y}+\delta\mathbf{1}) = e(\mathbf{y}) + m(\delta) - m(0).$$

Set $\mathbf{y}$ *equal to zero to obtain by the agreement property*

$$\delta = 0 + m(\delta) - m(0),$$

whence

$$\check{e}(\mathbf{y},\delta) \equiv e(\mathbf{y}+\delta\mathbf{1}) = e(\mathbf{y}) + \delta,$$

as claimed.

More generally, the converse holds for $\preceq_0$.

Lemma 4 *Preferences display decreasing (increasing) absolute risk aversion for* $\preceq_0$*if and only if*

$$e(\mathbf{y}+\delta\mathbf{1}) \geq (\leq, =)\, e(\mathbf{y}) + \delta,\ \forall \mathbf{y},\delta \geq 0$$

Proof: "only if" has already been shown. Supermodularity for $\preceq_0$ *requires only that if* $\delta \geq \delta'$,

$$\begin{aligned} & e(\mathbf{y}+\delta\mathbf{1}) - e\left(\mathbf{y}+\delta'\mathbf{1}\right) \geq e(\mu(\mathbf{y})\mathbf{1}+\delta\mathbf{1}) - e\left(\mu(\mathbf{y})\mathbf{1}+\delta'\mathbf{1}\right) \\ = \ & \delta - \delta' \end{aligned}$$

and the result is obtained by setting $\delta' = 0$.

Thus, the definition of constant absolute risk aversion is equivalent to that proposed by Quiggin and Chambers (1998) regardless of the choice of risk ordering $\preceq$. By contrast, the definitions of decreasing and increasing absolute risk aversion proposed will, in general, coincide with that of Quiggin and Chambers (1998) only for $\preceq_0$. As will be shown below, a stronger result holds for expected-utility preferences.

Definition 1 and Proposition 1 immediately yield a general form of the Pratt result. Consider a choice problem that includes wealth w as an exogenous variable with the property that for all α, θ_{-w} (where θ_{-w} denotes the vector of exogenous variables other than w), w, δ

$$\mathbf{y}(\alpha; \theta_{-w}, w+\delta) = \mathbf{y}(\alpha; \theta_{-w}, w) + \delta\mathbf{1} \tag{6}$$

It is immediate that for problems satisfying (6), $\mu(\mathbf{y}(\alpha;\theta))$ is a valuation in α and w. Moreover, $r(\mathbf{y}(\alpha;\theta))$ is submodular in α and w if and only if preferences display DARA.

Proposition 4 *Suppose that preferences display DARA. Then for any choice problem satisfying (6),* α^* *is increasing in* δ.

It is straightforward to extend the approach adopted here to other concepts of constant, increasing and decreasing relative risk aversion. The most popular is the relative risk aversion concept, relating $e(\mathbf{y})$ to $e(t\mathbf{y}')$ for $t > 0$. However, it is also possible to consider movements in more general directions, such as an increase in the endowment of some general asset with payoff $\mathbf{g}$.

4.1 Comparative statics under DARA

More generally, partition the vector θ as $\theta = (\theta_I, \theta_{-I})$ where θ_I is a subvector of exogenous parameters subject to change, while the elements of θ_{-I} are held constant. Suppose that, for some $\theta_I(0)$, and all α,

$$\mathbf{y}(\alpha;\theta) = \mathbf{y}(\alpha;\theta_{-I},\theta_I(0)) + g(\alpha,\theta_I)\,\mathbf{1} \tag{7}$$

That is, changes in the parameters in the subvector θ_I only produce translations in the outcome distribution $\mathbf{y}(\alpha;\theta)$, and therefore change the risk premium only through wealth effects, while parameters in the subvector θ_{-I} may affect the riskiness of $\mathbf{y}(\alpha;\theta)$ more generally. Say that α is risk-increasing if

$$\alpha < \alpha' \Rightarrow \mathbf{y}(\alpha;\theta_{-I},\theta_I(0)) \preceq \mathbf{y}\left(\alpha';\theta_{-I},\theta_I(0)\right)$$

It is straightforward to derive

Proposition 5 *Suppose that preferences display DARA and that α is risk-increasing. Then for any choice problem satisfying (7), and for which $g(\alpha,\theta_I)$ is supermodular in α and θ_I,* $\arg\max_\alpha \{e(\mathbf{y}(\alpha;\boldsymbol{\theta}))\}$ *is increasing in θ_I.*

Proof: Under the stated assumptions $e(\mathbf{y}(\alpha;\boldsymbol{\theta}))$ is supermodular in α and θ_I. Hence, Lemma 1 applies.

Noting the assumption (3) that $\lambda\mathbf{y} + (1-\lambda)\mu(\mathbf{y})\mathbf{1} \preceq \mathbf{y}$, we can now consider the case when

$$\mathbf{y}(\alpha;\theta) = f(\alpha,\theta)(\mathbf{y}(1,1) - \mu(\mathbf{y}(1,1))\mathbf{1}) + g(\alpha,\theta)\mathbf{1} \tag{8}$$

That is, changes in α and θ affect the riskiness of $\mathbf{y}$ through $f(\alpha,\theta)$ and the mean through $g(\alpha,\theta)$. Provided f is increasing in α, the set A is totally ordered by $\preceq^A$, which corresponds with the usual ordering. We may derive:

Proposition 6 *Suppose that preferences display DARA. Then for any choice problem satisfying (8), and for which $f(\alpha,\theta_I)$ is supermodular in α and θ, and $g(\alpha,\theta)$ is submodular in α and θ,* $\arg\max\{e(\mathbf{y}(\alpha;\boldsymbol{\theta}))\}$ *is decreasing in θ.*

Proof: Under the stated assumptions $e(\mathbf{y}(\alpha;\boldsymbol{\theta}))$ is submodular in α and θ_I. Hence, Lemma 1 applies.

A larger class of problems may be addressed using the monotone spread ordering. In particular, consider the case when $\alpha \in \Re_+$ and $\theta \in \Re_+$ are non-negative scalars and let $\mathbf{y}^1 = f(1,1)$. Suppose that f is differentiable in both arguments and that there exists a mapping

$$y_s(\alpha,\theta) = g\left(\alpha,\theta,y_s^1\right) \tag{9}$$

We have

Lemma 5 *Suppose (9) holds and g is supermodular in α and y_s^1 for all θ. Then, for all θ, $\alpha < \alpha' \Rightarrow \mathbf{y}(\alpha;\boldsymbol{\theta}) \preceq_m \mathbf{y}(\alpha';\boldsymbol{\theta})$. Hence, the ordering $\preceq_m$ on Y induces a total ordering $\preceq^A$ on A which is independent of θ and coincides with the usual scalar ordering.*

This yields

Proposition 7 *Suppose that preferences display DARA with respect to $\preceq_m$ that 9 holds and that $\mu(\mathbf{y})$ is submodular with respect to α and θ. Then* $\arg\max\{e(\mathbf{y}(\alpha;\boldsymbol{\theta}))\}$ *is decreasing in θ.*

Proof: Under the stated assumptions $e(\mathbf{y}(\alpha;\boldsymbol{\theta}))$ is submodular in α and θ_I. Hence, Lemma 1 applies.

4.2 Characterizing DARA

To achieve substantial generalizations of results previously obtained for expected utility, it is necessary to give conditions under which generalized expected utility preferences display DARA with respect to some risk ordering, such as $\preceq_m$. We will simplify the analysis of the EU and RDEU cases by assuming that the utility function is twice differentiable.

Proposition 8 *(i) EU preferences display DARA with respect to $\preceq_m$ if and only if they satisfy DARA in the Arrow–Pratt sense: $-u''(y)/u'(y)$ is a decreasing function of y, that is, only if they display DARA with respect to $\preceq_0$*
(ii) RDEU preferences display DARA with respect to $\preceq_m$ if and only if $-u''(y)/u'(y)$ is a decreasing function of y , that is only if they display DARA with respect to $\preceq_0$

Proof:
(i) Only if is trivial
Choose $\mathbf{y} \preceq_m \mathbf{y}'$. Then $\mathbf{y}' = \mathbf{y} + \boldsymbol{\theta}$ where, for all s,t,

$$(y_s - y_t)(\theta_s - \theta_t) \geq 0$$

Hence there exists $\delta_m \geq 0$, such that

$$\sum \pi_s u'(y_s)\,\varepsilon_s \approx 0$$

where $\varepsilon = \boldsymbol{\theta} + \delta_m \mathbf{1}$.For supermodularity of of $\check{e}$ in $\mathbf{y}$ and δ , we wish to determine the sign of

$$\begin{aligned} -\frac{\partial}{\partial\delta}\left(\sum \pi_s u'(y_s + \delta\mathbf{1})\,\varepsilon_s\right) &= \sum -\pi_s u''(y_s)\,\varepsilon_s \\ &= \sum \pi_s \frac{-u''(y_s)}{u'(y_s)} u'(y_s)\,\varepsilon_s. \end{aligned}$$

Now there exists t such that

$$\varepsilon_{t-1} < 0 < \varepsilon_t$$

and, since $\frac{-u''(y_s)}{u'(y_s)}$ is decreasing, we can choose $\overline{a}$,

$$\frac{-u''(y_t)}{u'(y_t)} < \overline{a} < \frac{-u''(y_{t-1})}{u'(y_{t-1})}.$$

Now

$$\begin{aligned} \sum_{s=1}^{t-1} \pi_s \frac{-u''(y_s)}{u'(y_s)} u'(y_s)\,\varepsilon_s + \sum_{s=t}^{S} \pi_s \frac{-u''(y_s)}{u'(y_s)} u'(y_s)\,\varepsilon_s &< \sum_{s=1}^{t-1} \pi_s \overline{a} u'(y_s)\,\varepsilon_s + \sum_{s=t}^{S} \pi_s \overline{a} u'(y_s)\,\varepsilon_s \\ &= \overline{a}\left(\sum_{s=1}^{t-1} \pi_s u'(y_s)\,\varepsilon_s + \sum_{s=t}^{S} \pi_s u'(y_s)\,\varepsilon_s\right) \\ &= 0, \end{aligned}$$

which is sufficient for supermodularity since it implies

$$\frac{\partial}{\partial\delta}\left(\sum \pi_s u'(y_s + \delta\mathbf{1})\,\varepsilon_s\right) > 0.$$

(ii)If $\mathbf{y} \preceq_m \mathbf{y}'$, then for any δ, δ', $\mathbf{y} + \boldsymbol{\delta}\mathbf{1}$ and $\mathbf{y}' + \boldsymbol{\delta}'\mathbf{1}$ are comonotonic. The result now follows from the proof of (i), replacing the probability distribution π and cumulative distribution function F with the associated transformed probability distribution $f(F)$ where f is the RDEU probability transformation function. ■

The following trivial corollary is useful in a range of problems involving multiplicative changes in risk

Corollary 2 *Proposition 8 holds if* $\preceq_m$ *is replaced by* $\preceq_1$.

We now consider an extension to the case of invariant preferences, considered by Quiggin and Chambers (2004). Preferences are invariant and only if they display both translation-invariance and radial invariance on equal-mean sets. That is if, for all $\delta \in \Re$, $t \in \Re_{+}$, and $\mathbf{y}, \mathbf{y}'$ such that $\mu(\mathbf{y}) = \mu(\mathbf{y}')$,

$$e(\mathbf{y}) \geq e(\mathbf{y}') \Rightarrow e(\mathbf{y} + \delta\mathbf{1}) \geq e(\mathbf{y}' + \delta\mathbf{1}), e(t\mathbf{y}) \geq e(t\mathbf{y}').$$

Quiggin and Chambers show that preferences are invariant if and only if they can be given a two-parameter represention of the form

$$e(\mathbf{y}) = \phi(\mu(\mathbf{y}), \rho(\mathbf{y})),$$

with ϕ increasing in its first argument and decreasing in its second and $\phi(c,0) = c$, where $\rho(\mathbf{y})$ is a risk index, that is, a positively linear homogeneous, convex and translation invariant function such that $\rho(c\mathbf{1}) = 0 \forall c$. The paradigmatic case of such a risk index is that of the standard deviation.

The following result follows immediately from the linear homogeneity and translation invariance of ρ.

Proposition 9 *Invariant preferences display DARA with respect to $\preceq_1$ if and only if ϕ is submodular in μ and ρ.*

Proof: Since the combination of linear homogeneity and translation invariance implies linearity on equal mean sets,

$$\rho(\lambda\mathbf{y} + (1-\lambda)\mu(\mathbf{y})\mathbf{1}) = \lambda\rho(\mathbf{y})$$

for $\lambda \geq 0$.

Observe that, for any $\mathbf{y} \in \Re^S$, $\delta \geq 0$, $\lambda \in (0,1)$

$$\mathbf{y} + \delta\mathbf{1} = \frac{1}{\lambda}(\lambda\mathbf{y} + \lambda\delta\mathbf{1})$$

In particular,we can choose λ so that $\lambda\delta = (1-\lambda)\mu(\mathbf{y})$, yielding

$$\mathbf{y} + \delta\mathbf{1} = \frac{1}{\lambda}(\lambda\mathbf{y} + (1-\lambda)\mu(\mathbf{y})\mathbf{1})$$

so, by linear homogeneity

$$\begin{aligned} \rho(\mathbf{y} + \delta\mathbf{1}) &= \frac{1}{\lambda}\rho(\lambda\mathbf{y} + (1-\lambda)\mu(\mathbf{y})\mathbf{1}) \\ &= \rho(\mathbf{y}) \end{aligned}$$

Hence, if $\mathbf{y} = (\lambda\mathbf{y}' + (1-\lambda)\mu(\mathbf{y})\mathbf{1})$, then

$$\rho(\mathbf{y}' + \delta\mathbf{1}) - \rho(\mathbf{y} + \delta\mathbf{1}) = (1-\lambda)\rho(\mathbf{y}'), \forall\delta$$

It follows that$e(\mathbf{y} + \delta\mathbf{1}) -$ $e(\mathbf{y} + \delta\mathbf{1}) -$ is increasing in δ, $\lambda \in (0,1)$ if and only if

Example 8 *(cont)The results derived above show that the standard results from the EU theory of portfolio choice can be extended to generalized expected utility whenever preferences display DARA with respect to* $\preceq_1$. *In particular*

(i) An increase in wealth will lead to a larger holding of the risky asset;

(ii) A reduction in the price of the risky asset will lead to a larger holding of the risky asset; and

(iii) A multiplicative increase in the riskiness of returns will lead to a smaller holding of the risky asset.

All these results can be extended to the large class of comparative static problems considered by Feder (1977) using the risk ordering $\preceq_m$.

5 Concluding comments

The tools of supermodularity theory provide methods to simplify and generalize comparative static analyses based on the manipulation of first-order conditions. As this paper shows, the analysis of choice under certainty offers substantial scope for the use of such tools. For the case of expected utility, the analysis above simplifies the derivation of a wide range of comparative static results that are already known, and permits the relaxation of assumptions about differentiability, uniqueness and so on. More substantively, the results derived here do not depend on the expected utility hypothesis, but only on the characterization of behavioral conditions which may be expressed in terms of the supermodularity properties of the risk premium.

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