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Dual structures for the sole-proprietorship firm

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Dual Structures for the Sole-Proprietorship Firm

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"Duality is about the choice of independent variables in terms of which one defines a theory". That quote from Gorman (1976) is approaching its thirtieth anniversary. In the interim, dual methods have percolated throughout economics. Almost universally, they have helped clarify fundamental aspects of individual behavior, while replacing manipulation of first and second-order conditions with more elegant, and transparent, representations of equilibrium behavior.

Some areas remain relatively untouched. One such area is the economics of (risk-averse) individual decisionmaking under conditions of risk or uncertainty. This is particularly true for firms that face incomplete markets. The conventional wisdom seems to be that, in the absence of complete markets and unique state-claim prices, the fundamental nonlinearities of the firm's preference sets and its choice sets are inextricably entangled. Thus, most treatments of financial market equilibrium and, more generally, of market equilibrium for incomplete markets are in primal terms.

This paper presents a dual representation of firm-level and market-level equilibrium behavior for a sole proprietorship economy with competitive and frictionless financial markets and stochastic production opportunities in a two-period setting. Apart from the slightly richer specification of current period input space used here and some resulting notational differences, the world envisioned is the same as that modeled in Milne (1976, 1995) and Magill and Quinzii's (1995) sole proprietorship models.

A sole proprietorship economy is, of course, unrealistic. Nevertheless, it represents an important benchmark that captures the essence of many of the problems faced by producing firms in a stochastic world. In particular, the link between asset pricing and real resource allocation is clearest in this simple setting. Thus, a sound understanding of its nature is essential to comprehension of more realistic market settings.

The sole proprietorship model fuses two frequently distinct models, that of an investor facing frictionless markets and that of the firm facing a stochastic production technology. Detailed comparative-static analysis has proved particularly elusive in this framework even under the implausible assumption of expected-utility preferences and strong restrictions on the firm's stochastic technology. As Cochrane (2001, p.43) has noted:

> We routinely think of betas and factor risk prices ... as determining expected

returns. But the whole consumption process, discount factor and factor risk premia change when the production technology changes. Similarly, we are on thin ice if we say anything about the effects of policy interventions, new markets and so on.

Our conceptual approach is closely related to dual general-equilibrium trade models (Dixit and Norman 1980; Woodland 1982). It differs from Milne's (1976, 1995) elegant induced preference approach in that we consider dual representations of the producer's choice set and preferences over the space of present-value prices rather than indirect representations of technologies and preferences induced over asset space. Because the firm's present value prices (risk-neutral probabilities, stochastic discount factor, virtual state-claim prices, state-claim densities) are the essential building blocks of most asset pricing theories, they are a natural choice of dependent variables in terms of which to define a dual theory of financial market equilibrium.

The treatment of firm-level equilibrium present here is essentially equivalent to the dual treatment of autarkic equilibrium and nontraded goods equilibrium in Woodland (1982). Our market-level equilibrium is equivalent to the dual treatment of a trading equilibrium between several large countries. That said, our results do not simply translate trade results to finance and production theory. The behavioral implications of the structure of the firm's choice set implied by the coexistence of a stochastic, but convex, production technology and a frictionless financial market are quite different, and in some ways considerably richer, than what emerges from basic trade theory.

In what follows, we first introduce some basic notation and concepts. Then we introduce the stochastic technologies and the frictionless asset market. Three distinct cost functions and their dual present-value profit functions are discussed next. One pair, the derivative-cost function and its dual present-value profit function, are relatively new to the literature on stochastic choice and thus are of interest in their own right. Each representation, however, proves useful at different points. We then formulate a dual representation of firm-level equilibrium for firms taking current period input and asset prices as given. The key component is the depiction of the firm's equilibrium present-value price vector as the fixed point of the subdifferential of one of the dual cost structures. This representation clarifies the nexus between resource allocation and asset pricing for the sole proprietorship firm. A consequence of its simplicity is that comparative-static analysis proves straightforward. We illustrate by first developing firm-level endowment comparative-static results that appear to generalize existing results. Then, firm-level comparative statics of input price change and technical change are considered. A dual economy equilibrium is then formulated. The dual equilibrium model is then used to state conditions for the firms' production choices to be independent of their risk preferences in equilibrium. These conditions entail Pareto optimality, but do not require *either* that the firm's consumption choices lie within the span of financial markets or the assumption of an extreme version of linear risk tolerance. Illustrative general-equilibrium comparative static results are stated, and the substantive part of the paper concludes with a demonstration of a dual and probability-free decomposition of the economy-level cost of idiosyncratic risk. Concluding comments are then offered.

1 Notation and Preliminaries

Denote the unit vector by $\mathbf{1} \in \Re^S_+$. For a convex function [1] $f : \Re^S \to \Re$, its *subdifferential*[2] at $\mathbf{m}$ is the closed, convex set:

$$\partial f(\mathbf{m}) = \left\{ \mathbf{k} \in \Re^S : f(\mathbf{m}) + \mathbf{k}(\mathbf{m}' - \mathbf{m}) \leq f(\mathbf{m}') \text{ for all } \mathbf{m}' \right\}. \quad (1)$$

The elements of $\partial f(\mathbf{m})$ are referred to as *subgradients.* If f is differentiable at $\mathbf{m}$, $\partial f(\mathbf{m})$ is a singleton and corresponds to the usual gradient. Conversely, if $\partial f(\mathbf{m})$ is a singleton, f is differentiable at $\mathbf{m}$.

For f convex, its convex conjugate is denoted

$$f^*(\mathbf{k}) = \sup_{\mathbf{m}} \{\mathbf{km} - f(\mathbf{m})\}.$$

[1]These results on convex functions are all drawn directly from Rockafellar (1970).

[2]We shall use the same notation to denote superdifferentials of concave functions.

If f is proper and closed,[3] then f^* is also a proper and closed convex function and

$$f(\mathbf{m}) = \sup_{\mathbf{k}} \{\mathbf{km} - f^*(\mathbf{k})\}, \tag{2}$$

and, on the relative interior of their domains,

$$\mathbf{k} \in \partial f(\mathbf{m}) \Leftrightarrow \mathbf{m} \in \partial f^*(\mathbf{k}). \tag{3}$$

2 State-Contingent Technologies and the Asset Structure

We model K sole-proprietorship firms facing a stochastic environment in a two-period setting. The current period, 0, is certain, but the future period, 1, is uncertain. Uncertainty is resolved by 'Nature' making a choice from $\Omega = \{1, 2, ..., S\}$. Although we do not use probability measures in what follows,our results may be interpreted in these terms, with the space of random variables given by $\Re^{\Omega} = \Re^{S}$. Each element of Ω is referred to as a state of nature.

Firms have access to a stochastic production technology which transforms current period inputs into stochastic period 1 output. They also have access to competitive financial markets which transform asset purchases made in the current period into stochastic payoffs (denominated in the units of the output) in period 1.

Each firm's technology is represented by a single-product, state-contingent input correspondence.[4] To make this explicit, let $\mathbf{x} \in \Re_{+}^{N}$ be a vector of inputs committed prior to the resolution of uncertainty (period 0), and let $\mathbf{z} \in \Re_{+}^{S}$ be a vector of *ex ante* or state-contingent outputs also chosen in period 0. If state $s \in \Omega$ is realized (picked by 'Nature'), and the producer has chosen the *ex ante* input–output combination $(\mathbf{x}, \mathbf{z})$, then the realized or *ex post* output in period 1 is z_s.

[3] f is proper if $f(\mathbf{x}) < \infty$ for at least one $\mathbf{x}$, and $f(\mathbf{x}) > -\infty$ for all $\mathbf{x}$. A proper convex function is closed if it is lower-semicontinuous.

[4] For a generalization to the multiple-output case, see Chambers and Quiggin (2000, Chapter 4). Our results extend straightforwardly to that case.

The continuous input correspondence,[5] $X : \Re_+^S \rightarrow \Re_+^N$, which maps state-contingent output vectors into input sets that are capable of producing that state-contingent output vector, is defined by

$$X(\mathbf{z}) = \{\mathbf{x} \in \Re_+^N : \mathbf{x} \text{ can produce } \mathbf{z}\}.$$

We impose the following properties on $X(\mathbf{z})$:

X.1 $X(\mathbf{0}_{MxS}) = \Re_+^N$ (no fixed costs), and $\mathbf{0} \notin X(\mathbf{z})$ for $\mathbf{z} \geq \mathbf{0}$ and $\mathbf{z} \neq \mathbf{0}$ (no free lunch).

X.2 $\mathbf{z}' \leq \mathbf{z} \Rightarrow X(\mathbf{z}) \subseteq X(\mathbf{z}')$.

X.3 $\lambda X(\mathbf{z}) + (1-\lambda)X(\mathbf{z}') \subseteq X(\lambda\mathbf{z} + (1-\lambda)\mathbf{z}') \qquad 0 \leq \lambda \leq 1.$

X.4 X is continuous.

The first part of X.1 says that doing nothing is always feasible, while its second part says that realizing a positive output in any state of nature requires the commitment of some inputs. X.2 says that if an input combination can produce a particular mix of state-contingent outputs then it can produce a smaller mix of state-contingent outputs. X.3 ensures that the graph of the input correspondence

$$T = \{(\mathbf{x}, \mathbf{z}) : \mathbf{x} \in X(\mathbf{z})\},$$

is convex and thus exhibits diminishing returns.

Period 0 prices of inputs of $\mathbf{x}$ are denoted by $\mathbf{w} \in \Re_+^N$ and are non-stochastic. The firm's current period (nonstochastic) wealth is m_0 and its (stochastic) endowment of period 1 wealth is denoted by $\mathbf{m} \in \Re^S$.

Financial markets are frictionless, and the *ex ante* financial security payoffs are given by the $S \times J$ non-negative matrix $\mathbf{A}$. Without any true loss of generality, we shall assume that $\mathbf{A}$ contains no redundant assets.[6] The vector of state-contingent payoffs on the jth financial asset is denoted $\mathbf{A}_j \in \Re_+^S$, and its price is denoted v_j. Denote the span of the matrix $\mathbf{A}$ by M. The firm's portfolio vector, corresponding to the period 0 purchases of the financial assets, is denoted $\mathbf{h} \in \Re^J$.

[5]Each of the $k = (1, ..., K)$ firms potentially have access to a unique production technology, so that X should be subscripted by k, but where there can be no confusion we shall suppress that subscript.

[6]This is for notational convenience. Allowing for redundant assets would not affect the analysis, but it would force us to restate several of our results in terms of the basic assets.

Firm preferences over consumption in the current period y_0 and period 1 consumption, $\mathbf{y} \in \Re_+^S$, are given by the *ex ante preference* function $u(y_0, \mathbf{y})$. We take u to be strictly increasing in all arguments and quasi-concave. Each firm's problem, therefore, is to

$$\max_{\mathbf{x},\mathbf{z},\mathbf{h},\mathbf{y}} \left\{ u(y_0, \mathbf{y}) : y_0 = m_0 - \mathbf{wx} - \mathbf{vh}, \mathbf{y} \leq \mathbf{m} + \mathbf{z} + \mathbf{Ah}, \mathbf{x} \in X(\mathbf{z}) \right\}. \tag{4}$$

3 Three Cost and Three Present-value Profit Functions

In this section we consider three cost functions. Two, the production cost function and the arbitrage cost function, are well-known from the production literature and the asset pricing literature, respectively. The third, which is derived from the first two, and is referred to as the derivative-cost function, is less well-known but proves crucial in developing our arguments. To motivate interest in these cost functions, note that any firm solving (4) also solves:

$$\max_{\mathbf{y}} \left\{ u\left(m_0 - C(\mathbf{w}, \mathbf{v}, \hat{\mathbf{y}}), \hat{\mathbf{y}} + \mathbf{m}\right) \right\},$$

where

$$C(\mathbf{w}, \mathbf{v}, \hat{\mathbf{y}}) = \min_{\mathbf{h},\mathbf{z}} \left\{ \mathbf{wx} + \mathbf{vh} : \mathbf{Ah} + \mathbf{z} \geq \hat{\mathbf{y}}, \mathbf{x} \in X(\mathbf{z}) \right\} \tag{5}$$

is the *derivative-cost function.* If the firm behaved otherwise, at the optimal $\hat{\mathbf{y}} + \mathbf{m}$, there would exist a technically feasible $(\mathbf{x}', \mathbf{z}', \mathbf{h}')$ capable of generating $\hat{\mathbf{y}}$ but for which

$$\mathbf{wx}' + \mathbf{vh}' < C(\mathbf{w}, \mathbf{v}, \hat{\mathbf{y}}).$$

This implies the existence of an arbitrage, which is inconsistent with equilibrium.

As later developments imply, a complete treatment of equilibrium could be developed directly in terms of C. However, a closer examination of the structure of C illuminates firm equilibrium behavior. Therefore, we now consider constituent parts of C.

3.1 The production cost and present-value profit functions

Dual to $X(\mathbf{z})$ is the production cost function, $c : \Re_{+}^{N} \times \Re_{+}^{S} \rightarrow \Re_{+}$,

$$c(\mathbf{w}, \mathbf{z}) = \min_{\mathbf{x}} \{\mathbf{wx} : \mathbf{x} \in X(\mathbf{z})\} \quad \mathbf{w} \in \Re_{+}^{N}$$

if there exists an $\mathbf{x} \in X(\mathbf{z})$ and ∞ otherwise. Mathematically, $c(\mathbf{w}, \mathbf{z})$ is equivalent to a multi-product cost function familiar from non-stochastic production theory (F äre 1988). If the input correspondence satisfies properties X, $c(\mathbf{w}, \mathbf{z})$ satisfies (Chambers and Quiggin, 2000): $c(\mathbf{w}, \mathbf{z}) \geq 0$, $c(\mathbf{w}, 0_S) = 0$, and $c(\mathbf{w}, \mathbf{z}) > 0$ for $\mathbf{z} \geq \mathbf{0}, \mathbf{z} \neq \mathbf{0}$; $\mathbf{z}^o \geq \mathbf{z} \Rightarrow c(\mathbf{w}, \mathbf{z}^o) \geq c(\mathbf{w}, \mathbf{z})$; and $c(\mathbf{w}, \mathbf{z})$ is convex on $\Re_{+}^{S}$ and continuous on the interior of the region where it is finite. Moreover, c is superlinear (positively linearly homogeneous and concave) in $\mathbf{w}$ and satisfies Shephard's lemma, which in terms of superdifferentials can be expressed as

$$\mathbf{x} \in \arg\min_{\mathbf{x}} \{\mathbf{wx} : \mathbf{x} \in X(\mathbf{z})\} \Leftrightarrow \mathbf{x} \in \partial_{\mathbf{w}} c(\mathbf{w}, \mathbf{z}), \tag{6}$$

where subscripts on superdifferentials or subdifferentials give the argument with respect to which the differential is taken.

Consider a vector of present-value prices for period 1 consumption, $\mathbf{q} \in \Re_{+}^{S}$. The convex conjugate of c,

$$c^*(\mathbf{w}, \mathbf{q}) = \sup_{\mathbf{z}} \{\mathbf{qz} - c(\mathbf{w}, \mathbf{z})\},$$

is the present-value profit-function for the present-value 'price' vector $\mathbf{q}$. Let $\mathbf{z}' \in \arg\sup \{\mathbf{qz} - c(\mathbf{v}$ then

$$\mathbf{qz}' - c(\mathbf{w}, \mathbf{z}') \geq \mathbf{qz} - c(\mathbf{w}, \mathbf{z})$$

for all $\mathbf{z}$, and hence $\mathbf{q} \in \partial_{\mathbf{z}} c(\mathbf{w}, \mathbf{z}')$. Thus, by (3) $\mathbf{z}' \in \partial_{\mathbf{q}} c^*(\mathbf{w}, \mathbf{q})$, which restates the first part of Hotelling's lemma in terms of subdifferentials. Now note that (6) then implies the second part

$$\mathbf{x} \in \partial_{\mathbf{w}} c(\mathbf{w}, \mathbf{z}'), \quad \mathbf{z}' \in \partial_{\mathbf{q}} c^*(\mathbf{w}, \mathbf{q}) \Leftrightarrow -\mathbf{x} \in \partial_{\mathbf{w}} c^*(\mathbf{w}, \mathbf{q}) \tag{7}$$

so that present-value profit maximizing demands can be obtained directly from $c^*(\mathbf{w}, \mathbf{q})$.

Denote the set of present-value prices for which present-value profit is finite by

$$P(\mathbf{w}) = \left\{\mathbf{q} \in \Re_{+}^{S} : c^*(\mathbf{w}, \mathbf{q}) < \infty\right\}.$$

3.2 The arbitrage cost and present-value profit functions

Dual to the financial asset structure is the minimal valuation (for example, Prisman 1986; Ross 1987) $p : \Re^J_+ \times \Re^S_+ \rightarrow \Re$ defined by the linear program

$$p(\mathbf{v}, \mathbf{r}) = \min \{\mathbf{vh} : \mathbf{Ah} \geq \mathbf{r}\},$$

if $\{\mathbf{h} : \mathbf{Ah} \geq \mathbf{y}\}$ is nonempty and ∞ otherwise.[7] Here the notation is meant to remind the reader that $p(\mathbf{v}, \mathbf{r})$ is the price of the cheapest portfolio that dominates $\mathbf{r}$.

$p(\mathbf{v}, \mathbf{r})$ is mathematically equivalent to a multiple-output cost function for a linear production technology. Thus, we refer to it as the *arbitrage-cost function.* In financial applications, p is usually viewed exclusively as a pricing or a valuation function, and its functional dependence upon $\mathbf{v}$ is, therefore, suppressed. However, this dependence proves important in characterizing equilibrium. Because it is a cost function, its properties in $\mathbf{v}$ are well-known. It is superlinear in $\mathbf{v}$ and moreover, the optimal asset holding $\mathbf{h}(\mathbf{v}, \mathbf{r})$ for a desired $\mathbf{r}$ satisfies

$$\mathbf{h}(\mathbf{v}, \mathbf{r}) \in \partial_{\mathbf{v}} p(\mathbf{v}, \mathbf{r}). \tag{8}$$

Its basic properties in $\mathbf{r}$ are also well-known. It is sublinear [8] in $\mathbf{r}$ and $p(\mathbf{v}, \mathbf{0}) \leq 0$. In addition if $\mathbf{r}$ is translated in the direction of any of the basic financial assets, its value increases by exactly the asset price times the length of the translation. More formally,

$$p(\mathbf{v}, \mathbf{r}+\delta \mathbf{A}_j) = p(\mathbf{v}, \mathbf{r}) + \delta v_j, \quad \delta \in \Re. \tag{9}$$

An important consequence of this property for our future analysis is detailed in the following lemma:

Lemma 1 $\partial_{\mathbf{r}} p(\mathbf{v}, \mathbf{r} + \hat{\mathbf{r}}) = \partial_{\mathbf{r}} p(\mathbf{v}, \mathbf{r})$ *for* $\hat{\mathbf{r}} \in M$.

[7] By the fundamental duality theorem of linear programming:

$$p(\mathbf{v}, \mathbf{r}) = \max_{\mathbf{q}} \{\mathbf{qr} : \mathbf{qA} \leq \mathbf{v}\},$$

so that it is also interpretable as the upper arbitrage bound on $\mathbf{r}$. Similarly, in the absence of arbitrage $-p(\mathbf{v}, -\mathbf{r})$ is interpretable as the lower arbitrage bound on $\mathbf{r}$.

[8] Sublinearity is trivially established from the dual linear program for p, which shows that p is the support function for the convex set $\{\mathbf{q} : \mathbf{qA} \leq \mathbf{v}\}$.

Proof By definition

$$\begin{aligned}\partial p(\mathbf{v},\mathbf{r}+\delta\mathbf{A}_j) &= \{\mathbf{q}: p(\mathbf{v},\mathbf{r}+\delta\mathbf{A}_j)+\mathbf{q}(\mathbf{r}'+\delta\mathbf{A}_j-[\mathbf{r}+\delta\mathbf{A}_j])\leq p(\mathbf{v},\mathbf{r}'+\delta\mathbf{A}_j) \text{ for all } \mathbf{r}'+\delta A \\ &= \{\mathbf{q}: p(\mathbf{v},\mathbf{r})+\mathbf{q}(\mathbf{r}'-\mathbf{r})\leq p(\mathbf{v},\mathbf{r}') \text{ for all } \mathbf{r}'\} = \partial p(\mathbf{v},\mathbf{r}),\end{aligned}$$

where the second equality follows by (9). For any $\hat{\mathbf{r}} \in M$ there must exist an unique $\hat{\mathbf{h}}$ such that $\hat{\mathbf{r}} = \mathbf{A}\hat{\mathbf{h}}$ (recall $\mathbf{A}$ contains no redundant assets by assumption). Apply the above recursively to obtain the result.■

The absence of arbitrage can be defined formally in terms of $p(\mathbf{v},\mathbf{r})$ (Prisman 1986; Ross 1987). There is an arbitrage if there exists either a portfolio priced at zero for which $\mathbf{r} \geq \mathbf{0}$ but $\mathbf{r} \neq \mathbf{0}$, or if there exists a negatively priced portfolio for which $\mathbf{r} = \mathbf{0}$ (Ross 1976; Prisman 1986; Ross 1987; Magill and Quinzii 1995; LeRoy and Werner 2000). Thus, the absence of an arbitrage requires $p(\mathbf{v},\mathbf{r}) > 0$ for $\mathbf{r} \geq \mathbf{0}$ with $\mathbf{r} \neq \mathbf{0}$ and $p(\mathbf{v},\mathbf{0}) \geq 0$. By the latter and the basic property that a portfolio yielding the zero asset cannot have a strictly positive price, $p(\mathbf{v},\mathbf{0}) \leq 0$, the absence of an arbitrage implies that $p(\mathbf{v},\mathbf{0}) = 0$.

Dual to p is the present-value arbitrage profit function defined as the convex conjugate of the arbitrage cost function

$$p^*(\mathbf{v},\mathbf{q}) = \sup_{\mathbf{r}}\{\mathbf{q}\mathbf{r} - p(\mathbf{v},\mathbf{r})\}.$$

Because p is sublinear over $\mathbf{r}$, it equals either 0 or ∞. By conjugacy, therefore,

$$\begin{aligned}p(\mathbf{v},\mathbf{0}) &= \sup_{\mathbf{q}}\{-p^*(\mathbf{v},\mathbf{q})\} \\ &= -\inf_{\mathbf{q}}\{p^*(\mathbf{v},\mathbf{q})\}.\end{aligned}$$

The absence of an arbitrage, thus, requires that there exist a set of present-value prices, $\mathcal{N}(\mathbf{v})$, such that

$$\mathcal{N}(\mathbf{v}) = \{\mathbf{q}: p^*(\mathbf{v},\mathbf{q}) = 0\}.$$

$\mathcal{N}(\mathbf{v})$ is the *set of no-arbitrage prices* implied by the asset structure $\mathbf{A}$ and $\mathbf{v}$. Alternatively, they can be characterized by

$$\begin{aligned}\mathbf{q} &\in \partial_{\mathbf{r}} p(\mathbf{v},\mathbf{0}) \\ &= \left\{\mathbf{q} \in \Re^S_+ : \mathbf{q}\mathbf{A} = \mathbf{v}\right\}.\end{aligned}$$

By (9), it follows for any $\hat{\mathbf{r}} \in M$ that

$$\begin{aligned} p(\mathbf{v}, \hat{\mathbf{r}}) &= p(\mathbf{v}, \mathbf{0} + \hat{\mathbf{r}}) \\ &= p\left(\mathbf{v}, \mathbf{0} + \mathbf{A}\hat{\mathbf{h}}\right) \\ &= p(\mathbf{v}, \mathbf{0}) + \sum_j \hat{h}_j v_j. \end{aligned}$$

The absence of arbitrage requires $p(\mathbf{v}, \mathbf{0}) = 0$, so that for any $\hat{\mathbf{r}} \in M$

$$p(\mathbf{v}, \hat{\mathbf{r}}) = \mathbf{v}\hat{\mathbf{h}}.$$

Hence, in the absence of an arbitrage p is linear on M (Prisman 1986; Ross 1987; Clark 1993). Because p is linear on M, the Riesz representation theorem implies that there is an unique element of M, $\bar{\mathbf{q}}(\mathbf{v})$, such that

$$p(\mathbf{v}, \mathbf{r}) = \bar{\mathbf{q}}(\mathbf{v})\mathbf{r}, \quad \mathbf{r} \in M.$$

We refer to $\bar{\mathbf{q}}(\mathbf{v})$ as the *pricing kernel* for M. Concretely, $\bar{\mathbf{q}}(\mathbf{v})$ is the orthogonal projection of $\mathcal{N}(\mathbf{v})$ onto M and is given by

$$\bar{\mathbf{q}}(\mathbf{v}) = \mathbf{v}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'.$$

3.3 The derivative-cost and present-value profit functions

Now return to the cost structure defined by (5), which for convenience we repeat here

$$C(\mathbf{w}, \mathbf{v}, \mathbf{y}) = \min_{\mathbf{h},\mathbf{z}} \{\mathbf{w}\mathbf{x} + \mathbf{v}\mathbf{h} : \mathbf{A}\mathbf{h} + \mathbf{z} \geq \mathbf{y}, \mathbf{x} \in X(\mathbf{z})\}.$$

C is the lowest cost that the firm must incur to raise at least $\mathbf{y}$. Although we suppress it notationally for the moment, it is good to recall that C is firm-specific because it depends on the firm's technology.

The derivative cost problem can be conveniently rewrittten

$$\begin{aligned} C(\mathbf{w}, \mathbf{v}, \mathbf{y}) &= \min_{\mathbf{h},\mathbf{z}} \{c(\mathbf{w}, \mathbf{z}) + \mathbf{v}\mathbf{h} : \mathbf{z} + \mathbf{A}\mathbf{h} \geq \mathbf{y}\} \\ &= \min_{\mathbf{r},\mathbf{z}} \{c(\mathbf{w}, \mathbf{z}) + p(\mathbf{v}, \mathbf{r}) : \mathbf{z} + \mathbf{r} \geq \mathbf{y}\}. \end{aligned}$$

C is a cost function. Therefore, it is superlinear in $(\mathbf{w}, \mathbf{v})$ and satisfies versions of Shephard's Lemma. We now detail its properties in $\mathbf{y}$. (A proof with further properties is presented in Chambers and Quiggin 2002.)

Lemma 2 *C satisfies: 1)* $C(\mathbf{w},\mathbf{v},\mathbf{y})$ *is a nondecreasing, convex function of* $\mathbf{y}$ *that is continuous on the interior of the region where it is finite, 2)* $C(\mathbf{w},\mathbf{v},\mathbf{0}) \leq 0$, *and 3)* $C\left(\mathbf{w},\mathbf{v},\mathbf{y}+\boldsymbol{\delta}\mathbf{A}_j\right) = C(\mathbf{w},\mathbf{v},\mathbf{y}) + \delta v_j$.

Lemma 2.1 ensures that standard methods from convex analysis can be invoked in analyzing C. The firm's *present-value function* is the convex conjugate of C. More formally, for $\mathbf{q} \in \Re^S_+$,

$$\begin{aligned}
C^*(\mathbf{w},\mathbf{v},\mathbf{q}) &= \sup_{\mathbf{y}} \{\mathbf{q}\mathbf{y} - C(\mathbf{w},\mathbf{v},\mathbf{y})\} \\
&= \sup_{\mathbf{y}} \left\{\mathbf{q}\mathbf{y} - \min_{\mathbf{r},\mathbf{z}} \{c(\mathbf{w},\mathbf{z}) + p(\mathbf{v},\mathbf{r}) : \mathbf{r}+\mathbf{z} \geq \mathbf{y}\}\right\} \\
&= \sup_{\mathbf{y},\mathbf{r},\mathbf{z}} \{\mathbf{q}\mathbf{y} - c(\mathbf{w},\mathbf{z}) - p(\mathbf{v},\mathbf{r}) : \mathbf{r}+\mathbf{z} \geq \mathbf{y}\} \\
&= \sup_{\mathbf{r},\mathbf{z}} \{\mathbf{q}\mathbf{r} - \mathbf{p}(\mathbf{v},\mathbf{r}) + \mathbf{q}\mathbf{z} - c(\mathbf{w},\mathbf{z})\} \\
&= \begin{cases} \infty & \mathbf{q} \notin \mathcal{N}(\mathbf{v}) \\ c^*(\mathbf{w},\mathbf{q}) & \mathbf{q} \in \mathcal{N}(\mathbf{v}) \end{cases}.
\end{aligned}$$

By conjugacy, therefore,

Theorem 3 *If* $C(\mathbf{w},\mathbf{v},\mathbf{y}) > -\infty$, *then* $C(\mathbf{w},\mathbf{v},\mathbf{y}) = \sup_{\mathbf{q}} \{\mathbf{q}\mathbf{y} - c^*(\mathbf{w},\mathbf{q}) : \mathbf{q} \in P(\mathbf{w}) \cap \mathcal{N}(\mathbf{v})\}$.

As long as C is finite, it equals the maximal value, taken over $P(\mathbf{w}) \cap \mathcal{N}(\mathbf{v})$, of the difference between the firm's valuation of $\mathbf{y}$ and its present-value profit. Put another way, it is the upper bound that the firm attaches to its portfolio and production cost given that its preferences lead it to a choice of $\mathbf{y}$. Because the firm's shadow price of $\mathbf{z}$ is $c(\mathbf{w},\mathbf{z})$, $C(\mathbf{w},\mathbf{v},\mathbf{y})$ thus places an upper bound on the firm's valuation of $\mathbf{y}$. The corresponding lower bound is

$$\inf_{\mathbf{q}} \{\mathbf{q}\mathbf{y} - c^*(\mathbf{w},\mathbf{q}) : \mathbf{q} \in P(\mathbf{w}) \cap \mathcal{N}(\mathbf{v})\}.$$

These bounds are interesting in their own right. They bound the firm willingness to pay for and willingness to sell $\mathbf{y}$. Thus, they are crucial in asset valuation. Because they restrict attention to present values belonging to both and $\mathcal{N}(\mathbf{v})$ *and* $P(\mathbf{w})$ instead of just $\mathcal{N}(\mathbf{v})$, they provide tighter bounds on nonreplicable assets than the more familiar no-arbitrage bounds. Because the technology is firm-specific, the bounds are potentially firm-specific. These bounds can be extended to develop even tighter bounds by incorporating further restrictions on the acceptable volatility of $\mathbf{q}$.as in Bernardo and Ledoit (2000) and Cochrane and Saá-Requejo (2000).

Theorem 3 has other implications. First, for C to be consistent with a reasonable equilibrium, the firm's subjective present-value prices can only assume values that permit neither a financial arbitrage nor a production arbitrage, that is, $\mathbf{q} \in P(\mathbf{w}) \cap \mathcal{N}(\mathbf{v})$. For strictly convex technologies, this restriction is not problematic. But if the technology is not strictly convex, then *at the firm level* problems can arise if $P(\mathbf{w}) \cap \mathcal{N}(\mathbf{v}) = \emptyset$. In that case, arbitrages exist between the production technology and the financial structure. Such arbitrages cannot prevail in a market equilibrium. Either no market equilibrium would exist or $(\mathbf{w}, \mathbf{v})$ would adjust to eliminate them.

Arbitrages between the production technology and financial markets can be illustrated by two polar cases. Suppose that the production technology has isocost curves which exhibit perfect substitutability between state-contingent outputs of the form[9]

$$c(\mathbf{w}, \mathbf{z}) = \boldsymbol{\mu}(\mathbf{w})\,\mathbf{z}.$$

Then $P(\mathbf{w}) = \{\mathbf{q} : \mathbf{q} \leq \boldsymbol{\mu}(\mathbf{w})\}$, and a strictly positive output can emerge in state s (and be consistent with zero profit) if and only if $q_s = \mu_s(\mathbf{w})$. Suppose also that there exists only a single financial asset, $\mathbf{A}_1$, so that $\mathcal{N}(\mathbf{v}) = \{\mathbf{q} : \mathbf{q}\mathbf{A}_1 = v_1\}$. Unless $P(\mathbf{w})$ intersects the hyperplane $\mathcal{N}(\mathbf{v})$, arbitrarily large present-value profit opportunities exist for the production technology at $\mathbf{q} \in \mathcal{N}(\mathbf{v})$.

[9]This technology is a special case of what Chambers and Quiggin (2000) refer to as a 'state-allocable' technology.

Maintaining the same asset structure, now suppose that[10]

$$c(\mathbf{w}, \mathbf{z}) = \gamma(\mathbf{w}) \max\{c_1 z_1, ..., c_S z_s\}.$$

Present-value profit maximization for $\mathbf{q} \in \Re^S_{++}$ requires that firms be technically efficient so that in the optimum

$$z_s = \frac{c_1}{c_s} z_1$$

for all s. Thus, firms facing $\mathbf{q} \in \Re^S_{++}$, this technology, and this asset structure solve

$$\max_{h_1, z_1} \{h_1(\mathbf{q}\mathbf{A}_1 - v_1) + (\mathbf{q}\mathbf{c}^* - c_1\gamma(\mathbf{w})) z_1\},$$

with $z_1 \geq 0$, where $\mathbf{c}^* = \left(1, \frac{c_1}{c_2}, ..., \frac{c_1}{c_S}\right)$.[11] There exists a finite present-value profit (equalling zero) for some $\mathbf{q}$ if and only if

$$\{\mathbf{q} : \mathbf{q}\mathbf{A}_1 = v_1\} \cap \{\mathbf{q} : \mathbf{q}\mathbf{c}^* \leq c_1\gamma(\mathbf{w})\} \neq \emptyset.$$

Suppose that the firm operates its technology at the level

$$z^* = \min\{z > 0 : z\mathbf{c}^* \geq \mathbf{A}_1\}$$

to produce a financial claim that dominates $\mathbf{A}_1$. The cost to the firm $z^* c_1 \gamma(\mathbf{w})$. Because this resulting financial claim dominates $\mathbf{A}_1$, it must fetch a per-unit price of at least v_1. Otherwise, an arbitrage exists. Thus, if $\frac{v_1}{c_1\gamma(\mathbf{w})} > z^*$, an arbitrage opportunity exists which offers an unboundedly large profit to the producer.

When $P(\mathbf{w}) \cap \mathcal{N}(\mathbf{v}) \neq \emptyset$, Theorem 3 reconfirms the Fisher separation theorem (Milne 1995; Magill and Quinzii 1995). Firms maximize present-value profit from both financial

[10]This technology is dual to a stochastic production function with a multiplicative productivity shock, whose input correspondence is

$$X(\mathbf{z}) = \left\{\mathbf{x} : \frac{f(\mathbf{x})}{c_s} \geq z_s, s \in \Omega\right\}.$$

This specification has played an important role in some empirical studies of asset pricing (Jermann, 1998; Tallarini, 2000). The lack of substitutability manifested here between state-contingent outputs is characteristic of *all* stochastic production function representations (Chambers and Quiggin 2000).

[11]In essence, this production structure adds another financial asset, nonlinearly priced, to the asset structure. But unlike the other asset, this asset is subject to short selling restrictions since $z_1 \geq 0$.

and production operations *given* the present-value prices determined subjectively by their equilibrium consumption patterns. Thus, the sole-proprietorship firm can be viewed as though it operates with a separate finance division and a separate production division. It controls practices in each not by providing individual production targets but *by providing a price signal in terms of* $\mathbf{q} \in P(\mathbf{w}) \cap \mathcal{N}(\mathbf{v})$. It is this basic observation, which provides the fundamental rationale for this paper.

4 The Dual Representation of the Sole-Proprietorship Firm Equilibrium

Define the *present-value expenditure function,* $E : \Re^S_+ \times \Re \to \Re$ for the firm's preference structure by

$$E(\mathbf{q},u) = \min_{c_0,\mathbf{c}} \{c_0 + \mathbf{qc} : u(c_0, \mathbf{c}) \geq u\}, \quad \mathbf{q} \in \Re^S_+$$

if there exists $(c_0, \mathbf{c})$ such that $u(c_0, \mathbf{c}) \geq u$ and ∞ otherwise. The properties of E are well-known: it is nondecreasing in u and nondecreasing and concave in $\mathbf{q}$. Moreover, if

$$(c_0, \mathbf{c}) \in \arg\min \{c_0 + \mathbf{qc} : u(c_0, \mathbf{c}) \geq u\},$$

then

$$\begin{aligned} \mathbf{c} &\in \partial_{\mathbf{q}} E(\mathbf{q},u), \\ c_0 &= E(\mathbf{q},u) - \mathbf{q}\partial_{\mathbf{q}} E(\mathbf{q},u). \end{aligned}$$

By Theorem 3, the properties of $c^*(\mathbf{w}, \mathbf{q})$ and $p(\mathbf{v}, \mathbf{r})$, firm equilibrium for any competitive firm facing prices $(\mathbf{w}, \mathbf{v})$ is characterized by $(\mathbf{r}, \mathbf{q},u) \in \Re^S \times \{P(\mathbf{w}) \cap \mathcal{N}(\mathbf{v})\} \times \Re$ solving:

$$\begin{aligned} E(\mathbf{q},u) &= m_o + c^*(\mathbf{w}, \mathbf{q}) + \mathbf{qm}, \\ \mathbf{r} &\in \partial_{\mathbf{q}} E(\mathbf{q},u) - \partial_{\mathbf{q}} c^*(\mathbf{w}, \mathbf{q}) - \mathbf{m}, \\ \mathbf{q} &\in \partial_{\mathbf{r}} p(\mathbf{v}, \mathbf{r}). \end{aligned}$$

First a word about notation. Subdifferentials and superdifferentials are generally sets, and these sets are singletons if and only if the relevant function is differentiable at the point in question. For the expenditure function and for most reasonably smooth technologies,[12] imposing differentiability does not seem particularly onerous or to involve any true loss of generality. Thus, with little true loss of generality, these subdifferentials and superdifferentials can be read as gradients. But this is not the case for p, which routinely exhibits important subdifferentials (for example, $\mathcal{N}(\mathbf{v})$), which are not singletons. Thus, expressions of the form

$$\mathbf{r} \in \partial_{\mathbf{q}} E(\mathbf{q},u) - \partial_{\mathbf{q}} c^*(\mathbf{w},\mathbf{q}) - \mathbf{m},$$

should formally be read as '$\mathbf{r}$ belongs to the set defined by subtracting each element of the set $\partial_{\mathbf{q}} c^*(\mathbf{w},\mathbf{q})$ from each element of the set $\partial_{\mathbf{q}} E(\mathbf{q},u)$ and translating the resulting set by $-\mathbf{m}$'. A more proper, but also more cumbersome, notation would be

$$\mathbf{r} = \mathbf{x} - \mathbf{z} - \mathbf{m}, \quad \mathbf{x} \in \partial_{\mathbf{q}} E(\mathbf{q},u), \mathbf{z} \in \partial_{\mathbf{q}} c^*(\mathbf{w},\mathbf{q}).$$

Keeping this notation in mind, the explanation is straightforward. The first condition requires that the firm's expenditure in present-value terms as a consumer equal its present value income. The second S conditions require that the firm's demand for $\mathbf{y}$ should equal its period 1 supply. The last S conditions require that $\mathbf{r}$ be chosen so that the marginal cost of raising each element of $\mathbf{r}$ is its present value.

By substitution of the S expressions into the last S, equilibrium requires that $(\mathbf{q},u) \in \{P(\mathbf{w}) \cap \mathcal{N}(\mathbf{v})\} \times \Re$ solve

$$\begin{aligned} E(\mathbf{q},u) &= m_o + c^*(\mathbf{w},\mathbf{q}) + \mathbf{qm} \\ \mathbf{q} &\in \partial_{\mathbf{r}} p\left(\mathbf{v}, \partial_{\mathbf{q}} E(\mathbf{q},\mathbf{u}) - \partial_{\mathbf{q}} c^*(\mathbf{w},\mathbf{q}) - \mathbf{m}\right). \end{aligned} \tag{10}$$

Given u, the firm's equilibrium present-value price vector is determined as a fixed point of the subdifferential of the arbitrage-cost function, as evaluated at the individual's period

[12]Notice, however, that the most familiar form of a stochastic technology, the stochastic production function, is *always nondifferentiable* at all economically efficient points (Chambers and Quiggin, 2000).

1 *excess demand.* [13] Standard fixed-point theorems familiar from the theory of general equilibrium can be used to determine the existence and uniqueness of such solutions.

By (10), $(\bar{\mathbf{q}}(\mathbf{v}), u)$ represents a firm equilibrium only if

$$\partial_{\mathbf{q}} E(\bar{\mathbf{q}}(\mathbf{v}), u) - \partial_{\mathbf{q}} c^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) - \mathbf{m} \cap M \neq \emptyset.$$

Such an equilibrium is referred to as a *separating firm equilibrium.* In such equilibria, the firm uses the pricing kernel to value its state 1 consumption and production decisions even though neither they or $\mathbf{y}$ are necessarily perfectly replicable in financial markets.. A separating firm equilibrium, therefore, satisfies the necessary and sufficient conditions for 'local separation' derived by Chambers and Quiggin (2003).

In a separating firm equilibrium, the *market value of the firm,* which is defined as

$$\bar{\mathbf{q}}(\mathbf{v}) \partial_{\mathbf{q}} c^*(\mathbf{w}, \mathbf{q}) + \mathbf{w} \partial_{\mathbf{w}} c^*(\mathbf{w}, \mathbf{q}),$$

corresponds to its present value, $c^*(\mathbf{w}, \mathbf{q})$, because $\mathbf{q} = \bar{\mathbf{q}}(\mathbf{v})$. More generally, the market value of the firm and its present value $c^*(\mathbf{w}, \mathbf{q})$ are not equal. Using Hotelling's Lemma, the market value of the firm can be written as,

$$c^*(\mathbf{w}, \mathbf{q}) + (\bar{\mathbf{q}}(\mathbf{v}) - \mathbf{q}) \partial_{\mathbf{q}} c^*(\mathbf{w}, \mathbf{q}),$$

which is the sum of the firm's present value, $c^*(\mathbf{w}, \mathbf{q})$, and what, following Magill and Quinzii (1995), one might term the firm's *entrepreneurial risk,* $(\bar{\mathbf{q}}(\mathbf{v}) - \mathbf{q}) \partial_{\mathbf{q}} c^*(\mathbf{w}, \mathbf{q})$. Because $\bar{\mathbf{q}}(\mathbf{v})$ is the orthogonal projection of $\mathcal{N}(\mathbf{v})$ onto M, in equilibrium $\bar{\mathbf{q}}(\mathbf{v}) - \mathbf{q} \perp M$. Hence, if $\partial_{\mathbf{q}} c^*(\mathbf{w}, \mathbf{q}) \subset M$, the firm's market value and its present value coincide. Thus, a sufficient condition for the firm's present value and market value to coincide is that

$$\cup_{\mathbf{q} \in P(\mathbf{w}) \cap \mathcal{N}(\mathbf{v})} \{\partial_{\mathbf{q}} c^*(\mathbf{w}, \mathbf{q})\} \subset M.$$

[13] One could represent equilibrium in terms of the firm's indirect utility function defined as

$$V(\mathbf{q}, e) = \sup\{\mathbf{u} : E(\mathbf{q}, u) \leq e\},$$

to eliminate the first equation in the equilibrium condition and then to use Roy's identity in the second S conditions. We have avoided this for two reasons: because V is only quasi-convex, it necessitates introducing another derivative concept. And two, most of the restrictions on preferences used in financial and consumer analysis are most clearly represented in terms of the expenditure function. Of course, identical results would be obtained using this approach. Milne (1995) presents such a treatment in the context of complete markets.

Chambers and Quiggin (2003) refer to this as *efficient-set spanning* to distinguish it from the more usual notion of spanning of production opportunities which requires that feasible state contingent outputs fall in M (Magill and Quinzii, 1995). Efficient-set spanning does not imply that the firm prices its consumption activities using the pricing kernel.

4.1 Comparative statics for linear risk tolerant preferences

The equilibrium representation provides a simple platform from which to conduct comparative-static analysis of the firm's equilibrium choices. To illustrate, consider the class of linear risk tolerant preferences. Linear risk tolerant (quasi-homothetic) ordinal preferences can be characterized by expenditure functions of the Gorman polar form (Brennan and Kraus, 1976; Chambers and Quiggin, 2002):

$$E(\mathbf{q},u) = E^0(\mathbf{q}) + E^1(\mathbf{q})\,u, \tag{11}$$

where E^0 and $E^1 > 0$ are expenditure functions. Denote

$$R(\mathbf{w},\mathbf{v}) = \cup_{\mathbf{q}\in P(\mathbf{w})\cap\mathcal{N}(\mathbf{v})} \left\{\mathbf{y} : \mathbf{y} \in \partial_{\mathbf{q}} E^1(\mathbf{q})\right\}.$$

We have:

Theorem 4 *Assume that firm preferences are characterized by (11). If* $R(\mathbf{w},\mathbf{v}) \subset M$, *then firm equilibrium* $\mathbf{q}$ *is independent of* m_0. *If* $\mathbf{m} \in M$, *and* $R(\mathbf{w},\mathbf{v}) \subset M$, *then firm equilibrium* $\mathbf{q}$ *is independent of* $(m_0,\mathbf{m})$.

Proof By linear risk tolerance

$$\partial_{\mathbf{q}} E(\mathbf{q},u) = \partial_{\mathbf{q}} E^0(\mathbf{q}) + \partial_{\mathbf{q}} E^1(\mathbf{q})\,u,$$

with

$$u = \frac{m_o + c^*(\mathbf{w},\mathbf{q}) + \mathbf{qm} - E^0(\mathbf{q})}{E^1(\mathbf{q})}.$$

Thus,using Lemma 1 in the firm equilibrium conditions gives

$$\begin{aligned}\mathbf{q} &\in \partial_{\mathbf{r}} p\left(\mathbf{v}, \partial_{\mathbf{q}} E^0(\mathbf{q}) + \partial_{\mathbf{q}} E^1(\mathbf{q})\,u - \partial_{\mathbf{q}} c^*(\mathbf{w},\mathbf{q}) - \mathbf{m}\right) \\ &= \partial_{\mathbf{r}} p\left(\mathbf{v}, \partial_{\mathbf{q}} E^0(\mathbf{q}) - \partial_{\mathbf{q}} c^*(\mathbf{w},\mathbf{q}) - \mathbf{m}\right),\end{aligned}$$

if $R(\mathbf{w},\mathbf{v}) \subset M$, and

$$\begin{aligned}\mathbf{q} &\in \partial_{\mathbf{r}} p\left(\mathbf{v}, \partial_{\mathbf{q}} E^0(\mathbf{q}) + \partial_{\mathbf{q}} E^1(\mathbf{q}) u - \partial_{\mathbf{q}} c^*(\mathbf{w},\mathbf{q}) - \mathbf{m}\right) \\ &= \partial_{\mathbf{r}} p\left(\mathbf{v}, \partial_{\mathbf{q}} E^0(\mathbf{q}) - \partial_{\mathbf{q}} c^*(\mathbf{w},\mathbf{q})\right),\end{aligned}$$

if $\mathbf{m} \in M$, and $R(\mathbf{w},\mathbf{v}) \subset M$.■

Preferences exhibiting linear risk tolerance generate state-contingent demands that are linear in real income, u. The condition $R(\mathbf{w},\mathbf{v}) \subset M$ requires that the Hicksian real-income effect always lie in the market span. Changes in real wealth, therefore, only create risk that can be priced in the market. Hence, the firm's equilibrium present values are independent of things that only shift u. When $\mathbf{m} \in M$, the period 1 endowment also plays no role in determining equilibrium $\mathbf{q}$ for the same intuitive reason. The endowment then carries no idiosyncratic risk, and therefore, it too can be priced accurately in the market.

Alternatively, one might think in terms of two-fund portfolio separation theory. Under linear risk tolerance, excess state-contingent demands consist of a component, $\partial_{\mathbf{q}} E^0(\mathbf{q}) - \partial_{\mathbf{q}} c^*(\mathbf{w},\mathbf{q})$, which is independent of real income, and a component, $u\partial_{\mathbf{q}} E^1(\mathbf{q})$, which is linear in real income. If the latter is in M, it can be priced accurately there by the principle of perfect replicability. That leaves only the first component to play a role in determining the firm's present-value prices.

Requiring that $R(\mathbf{w},\mathbf{v}) \subset M$ is a strong restriction. It considerably narrows the classes of preferences within the general linear risk tolerant class. But many such preferences are routinely used in both the finance and consumer demand literatures. One such important class of preferences are the preferences translation homothetic in a direction spanned by the market. More precisely, Chambers and Färe (1998) have shown that preferences are *translation homothetic in the direction of* $\mathbf{g}$ if and only if they exhibit linear risk tolerance and

$$E^1(\mathbf{q}) = \mathbf{q}\mathbf{g}.$$

Intuitively, translation homothetic preferences are the linear risk tolerant preferences for which the real income effect is independent of prices. [14] We have:

[14] Blackorby, Boyce, and Russell (1978) refer to translation homotheticity as *homotheticity to minus infinity.* Dickinson (1980) refers to it as a *linear parallel preference structure.*

Corollary 5 *If preferences are translation homothetic in the direction* $\mathbf{g} \in M$, *then equilibrium* $\mathbf{q}$ *is independent of* m_0. *If* $\mathbf{m} \in M$ *and preferences are translation homothetic in the direction of* $\mathbf{g} \in M$, *then equilibrium* $\mathbf{q}$ *is independent of* $(m_0, \mathbf{m})$.

The most familiar member of the translation homothetic class in the finance literature is the constant absolute risk averse (CARA) preference structure. In the standard portfolio problem with expected-utility preferences, CARA ensures that the optimal holdings of the risky asset are independent of the individual's wealth. Our next corollary generalizes that basic result to the case of producing firms with non-expected utility preferences:

Corollary 6 *If the firm's preferences are CARA and* $\mathbf{1} \in M$, *then equilibrium* $\mathbf{q}$ *is independent of* m_0. *If preferences are constant absolute risk averse,* $\mathbf{1} \in M$, *and* $\mathbf{m} \in M$, *then equilibrium* $\mathbf{q}$ *is independent of* $(m_0, \mathbf{m})$.

Proof CARA preferences are characterized by $E^1(\mathbf{q}) = \mathbf{q1}$ (Chambers and Quiggin, 2002).■

For CARA preferences, excess demands also consist of two parts. The first, $\partial_{\mathbf{q}} E^0(\mathbf{q}) - \partial_{\mathbf{q}} c^*(\mathbf{w}, \mathbf{q})$, which is independent of real income can be interpreted as the demand for the risky portfolio, while the second, $u\mathbf{1}$, is the demand for the traditionally riskless asset. If the riskless asset is in M, as it is in the standard portfolio problem, only the demand for the risky portfolio affects the determination of equilibrium $\mathbf{q}$.

Constant relative risk averse (CRRA) preferences are the subclass of linear risk tolerant prefrences given by $E^0 = 0$.(Chambers and Quiggin, 2002). We conclude:

Corollary 7 *If preferences are constant relative risk averse and* $R(\mathbf{w}, \mathbf{v}) \subset M$, *then equilibrium* $\mathbf{q}$ *is determined by the technology and the period-1 endowment.*

Proof If preferences are constant relative risk averse, $E^0 = 0$, thus under the conditions of the corollary.

$$\mathbf{q} \in \partial_{\mathbf{r}} p(\mathbf{v}, -\partial_{\mathbf{q}} c^*(\mathbf{w}, \mathbf{q}) - \mathbf{m}).■$$

Constant absolute risk aversion and constant relative risk aversion are both special cases of linear risk tolerance. Safra and Segal (1998) and Quiggin and Chambers (1998) have studied the further special class of preferences that exhibit both constant absolute risk aversion and constant relative risk aversion. Safra and Segal (1998) name this class of preferences *constant risk averse.* Particularly important members of this subclass of preferences include the risk-neutral preferences, Yaari's dual linear preference model, linear mean-standard deviation preferences, completely risk averse or maxmin preferences ($u(c_0, \mathbf{c}) = \hat{u}(c_0, \min\{c_1, ..., c_S\})$), and the maximin expected value preference structure that is a special case of Gilboa and Schmeidler's (1989) ambiguity-averse maximin expected utility class. Yaari has shown dual linear preferences lead to plunging behavior in the simple portfolio selection problem. Chambers and Quiggin (2002) have shown that plunging behavior is characteristic of the entire constant risk averse class of preferences when confronted with a convex choice problem. In the current context, we have that this type of plunging behavior on the firm's demand side leads to the firm determining its present value prices by the optimal adjustment of its production portfolio to its period 1 endowment.

Corollary 8 *If preferences are constant risk averse and* $\mathbf{1} \in M$*, then equilibrium* $\mathbf{q}$ *is determined by the technology and the period-1 endowment.*

Proof If preferences are constant risk averse, then (Chambers and Quiggin 2002)

$$E(\mathbf{q}, u) = \begin{cases} u\mathbf{q}\mathbf{1} & \mathbf{q} \in P^* \\ -\infty & \text{otherwise} \end{cases},$$

where P^* is a closed convex set. Thus, if $\partial E(\mathbf{q}, u)$ exists, it involves only riskless consumption.■

Chambers, Färe, and Quiggin (2003) have recently studied the class of preferences which are simultateously translation homothetic in the direction of $\mathbf{g}$ and exhibit constant relative risk aversion. These preferences include, among others, the class of preferences which are Leontief around $\mathbf{g}$ ($u(c_0, \mathbf{c}) = \hat{u}(c_0, \min\{g_1 c_1, ..., g_S c_S\})$). We have:

Corollary 9 *If preferences are translation homothetic in the direction of* $\mathbf{g} \in M$ *and exhibit CRRA, then equilibrium* $\mathbf{q}$ *is determined by the technology and the period-1 endowment..*

Proof If preferences are translation homothetic in the direction of $\mathbf{g}$ and exhibit constant risk aversion, they are $\mathbf{g}-$ generalized constant risk averse in the sense of Chambers, Färe, and Quiggin (2003), whence

$$E(\mathbf{q},u) = \begin{cases} u\mathbf{qg} & \mathbf{q} \in \tilde{P} \\ -\infty & \text{otherwise} \end{cases},$$

where $\tilde{P}$ is a closed convex set.■

Theorem 4 and its corollaries encompass both familiar results as well as what appear to be entirely new results. To this point, no structural assumption has been made on preferences other than linear risk tolerance. Thus, these results established extend beyond the expected utility preference class. These results apply for general technologies. Placing functional restrictions on the technology potentially yields further comparative-static results.

4.2 Changes in the technology and input prices

Perhaps the most salient characteristic of the sole-proprietorship literature is its relative paucity of comparative static results. A similar paucity is apparent in the literature on financial markets. But there, the tight focus on asset pricing makes it more understandable. However, in the sole-proprietorship literature, *even a complete focus on asset pricing does not justify ignoring comparative static effects.*

Two examples illustrate. Changes in current period input prices evoke changes in the firm's equilibrium behavior that are manifested dually in changes in equilibrium $\mathbf{q}$ and primally in its consumption and production choices. These changes in $\mathbf{q}$ evoke changes in asset valuation.

Similarly, exogenous changes in the firm's operating environment, be it in the form of technical advances or changes in the tax code, alter the firm's valuation and resource allocation behavior. Thus, any asset-pricing theory, to be complete, must be able to capture and evaluate these comparative-static effects.

We address these issues by providing benchmark cases for which the presence of an asset market washes out such effects by enabling the firm to use that market to accurately price the exogenous changes. For concreteness, we speak in terms of factor prices and technical change, but the basic results and approach are more general. We do not aim for a complete taxonomy. Instead, the aim is to illustrate the general approach. Therefore, a specific technology is used. Extending these methods to more general technologies is straightforward.

Consider present-value profit functions of the form

$$c^{*}(\mathbf{w},\mathbf{q},t)=a(\mathbf{q})+b(\mathbf{w},\mathbf{q},t), \tag{12}$$

where t now indexes the state of the technology. Define

$$R^{X}(\mathbf{w},\mathbf{v})=\cup_{\mathbf{q}\in P(\mathbf{w})\cap\mathcal{N}(\mathbf{v})}\{\mathbf{y}:\mathbf{y}\in\partial_{\mathbf{q}}b(\mathbf{w},\mathbf{q},t)\}.$$

We have:

Theorem 10 *Suppose that the firm's technology is characterized by (12) and its preferences by (11). If* $R^{X}(\mathbf{w},\mathbf{v})\subset M$ *and* $R(\mathbf{w},\mathbf{v})\subset M$*, then equilibrium* $\mathbf{q}$ *is independent of* $(m_0,\mathbf{w},t)$. *If* $R^{X}(\mathbf{w},\mathbf{v})\subset M$, $R(\mathbf{w},\mathbf{v})\subset M$, *and* $\mathbf{m}\in M$,*then equilibrium* $\mathbf{q}$ *is independent of* $(m_0,\mathbf{m},\mathbf{w},t)$.

Theorem 10 has corollaries paralleling exactly those derived for Theorem 4. Each of these appear to be new. However, because they follow from a straightforward rewriting of earlier results, we leave their exact statement, proof, and intuitive motivation to the interested reader. A number of further comparative-static results are available by considering particular profit structures that satisfy $R^{X}(\mathbf{w},\mathbf{v})\subset M$. That, too, is left to the reader.

5 Dual Market Equilibrium

Borrowing the terminology of Magill and Quinzii (1995), we define $(\mathbf{w}, \mathbf{v}, \mathbf{q}_1, , , , \mathbf{q}_K, u_1, ..., u_K) \in \Re^N_+ \times \Re^J_+ \times \{P_k(\mathbf{w}) \cap \mathcal{N}(\mathbf{v})\}^K \times \Re^K$ to be a *dual entrepreneurial equilibrium* if it solves[15]

$$
\begin{aligned}
E_k(\mathbf{q}_k, \mathbf{u}_k) &= m_{ok} + c^*_k(\mathbf{w}, \mathbf{q}_k) + \mathbf{q}_k \mathbf{m}_k, \quad k = 1, .., K \\
\mathbf{q}_k &\in \partial_{\mathbf{r}} p\left(\mathbf{v}, \partial_{\mathbf{q}} E_k(\mathbf{q}_k, u_k) - \partial_{\mathbf{q}} c^*_k(\mathbf{w}, \mathbf{q}_k) - \mathbf{m}_k\right), \\
k &= 1, .., K \\
\sum_{k=1}^K \partial_{\mathbf{w}} c^*_k(\mathbf{w}, \mathbf{q}_k) &= -\bar{\mathbf{x}}, \\
\sum_{k=1}^K \partial_{\mathbf{v}} p\left(\mathbf{v}, \partial_{\mathbf{q}} E_k(\mathbf{q}_k, u_k) - \partial_{\mathbf{q}} c^*_k(\mathbf{w}, \mathbf{q}_k) - \mathbf{m}_k\right) &= \mathbf{0}, \qquad (13)
\end{aligned}
$$

where $\bar{\mathbf{x}}$ is the current period endowment of inputs. The first $K(S+1)$ expressions repeat the dual-firm equilibrium conditions. The next N conditions, via Hotelling's lemma (7), require that the economy-wide input demand equal its supply. The last J equations, via Shephard's lemma (8), require that financial portfolios balance one another.

Comparative static analysis is straightforward. For example,

Theorem 11 *Let firms have preferences translation homothetic in the direction* $\mathbf{g}_k \in M$ $(k = 1, ..., K)$*. In dual entrepreneurial equilibrium* $(\mathbf{w}, \mathbf{v}, \mathbf{q}_1, , , , \mathbf{q}_K)$ *is independent of* (m_{0k}) $k = 1, .., K$*. If* $\mathbf{m}_k \subset M$ *for all* k*, then in dual entrepreneurial equilibrium* $(\mathbf{w}, \mathbf{v}, \mathbf{q}_1, , , , \mathbf{q}_K)$ *is independent of* $(m_{0k}, \mathbf{m}_k)$ $k = 1, ..., K$*.*

Proof Under the conditions of the theorem, $(\mathbf{w}, \mathbf{v}, \mathbf{q}_1, , , , \mathbf{q}_K)$ are determined by

$$
\begin{aligned}
\mathbf{q}_k &\in \partial_{\mathbf{r}} p\left(\mathbf{v}, \partial_{\mathbf{q}} E^0_k(\mathbf{q}_k) - \partial_{\mathbf{q}} c^*_k(\mathbf{w}, \mathbf{q}_k) - \mathbf{m}_k\right) \\
k &= 1, .., K \\
\sum_{k=1}^K \partial_{\mathbf{w}} c^*_k(\mathbf{w}, \mathbf{q}_k) &= -\bar{\mathbf{x}}, \\
\sum_{k=1}^K \partial_{\mathbf{v}} p\left(\mathbf{v}, \partial_{\mathbf{q}} E^0_k(\mathbf{q}_k) - \partial_{\mathbf{q}} c^*_k(\mathbf{w}, \mathbf{q}_k)\right) &= \mathbf{0}. \blacksquare
\end{aligned}
$$

[15] Here, for the sake of simplicity, we have assumed that each firm owns and operates its own technology. By introducing a slightly more complex notation that distinguishes between owners of firms and non-owners, we could make the model correspond exactly to the primal model formulated in, for example, Milne (1995).

Theorem 11 is a general statement that encompasses a number of well-known results on both financial-market and sole-proprietorship equlibria as special cases (for example, see Magill and Quinzii (1995)) as well as some apparently new results. It is straightforward to derive corollaries that parallel the corollaries associated with Theorem 4. We leave the derivation and intuitive motivation of these results to the reader. The arguments parallel earlier ones.

Comparative-static results for t are also straightforward:

Theorem 12 *Assume that firms have preferences translation homothetic in the direction* $\mathbf{g}_k \in M$ *$(k = 1, ..., K)$ and that their technologies satisfy*

$$c_k^* (\mathbf{w}, \mathbf{q}) = a_k (\mathbf{q}, \mathbf{w}) + \mathbf{q}\mathbf{b}_k (\mathbf{t})$$

with $\mathbf{b}_k (\mathbf{t}) \subset M$ *for all* k*, then in dual entrepreneurial equilibrium* $(\mathbf{w}, \mathbf{v}, \mathbf{q}_1, , , , \mathbf{q}_K)$ *is independent of* t*.*

It is well-known (Milne, 1995; Magill and Quinzii, 1995; LeRoy and Werner, 2000) that placing enough structure on linear risk tolerant preferences to satisfy Gorman's (1953) conditions for the existence of an aggregate indifference curve can produce instances where equilibrium in incomplete markets is Pareto optimal because of identical income effects across agents. Such results can apply here as well. Because they represent a rewriting of well-known results in dual terms, we leave their exact statement to the interested reader. Instead, we now turn our attention to alternative conditions consistent with a Paretian equilibrium in the presence of incomplete markets.

Define $(\mathbf{w}, \mathbf{v}, \bar{\mathbf{q}} (\mathbf{v}), u_1, ..., u_K)$ to be a *separating dual entrepreneurial equilibrium* if $(\bar{\mathbf{q}} (\mathbf{v}), u_k)$ is a dual firm equilibrium for each k, and the N material balance and J financial balance conditions in (13) are satisfied. Pareto optimality is straightforward since Theorem 3 implies that $\bar{\mathbf{q}} (\mathbf{v}) \in \partial_{\mathbf{y}} C_k (\mathbf{w}, \mathbf{v}, \mathbf{y}_k)$ in such an equilibrium. We have:

Theorem 13 *There exists a separating dual entreprenuerial equilibrium at* $(\mathbf{w}, \mathbf{v}, \bar{\mathbf{q}} (\mathbf{v}), u_1, ..., u_I$

if and only if

$$\begin{aligned} E_k\left(\bar{\mathbf{q}}\left(\mathbf{v}\right), u_k\right) &= m_{ok} + c_k^*\left(\mathbf{w}, \bar{\mathbf{q}}\left(\mathbf{v}\right)\right) + \bar{\mathbf{q}}\left(\mathbf{v}\right)\mathbf{m}_k \\ \partial_{\mathbf{q}} E_k\left(\bar{\mathbf{q}}\left(\mathbf{v}\right), u_k\right) - \partial_{\mathbf{q}} c_k^*\left(\mathbf{w}, \bar{\mathbf{q}}\left(\mathbf{v}\right)\right) - \mathbf{m}_k &\in M, \quad k = 1, ..., K, \\ \sum_{k=1}^{K} \partial_{\mathbf{w}} c_k^*\left(\mathbf{w}, \bar{\mathbf{q}}\left(\mathbf{v}\right)\right) &= -\bar{\mathbf{x}}, \\ \sum_{k=1}^{K} \partial_{\mathbf{v}} p\left(\mathbf{v}, \partial_{\mathbf{q}} E_k\left(\bar{\mathbf{q}}\left(\mathbf{v}\right), u_k\right) - \partial_{\mathbf{q}} c_k^*\left(\mathbf{w}, \bar{\mathbf{q}}\left(\mathbf{v}\right)\right) - \mathbf{m}_k\right) &= \mathbf{0}. \end{aligned}$$

Theorem 13 requires that the firm's excess demands for state-contingent incomes lie in M. *It does not require any of the following*: the firm's net consumption choices, $\partial_{\mathbf{q}} E_k\left(\bar{\mathbf{q}}\left(\mathbf{v}\right), u_k\right) - \mathbf{m}_k$, lie in M; its feasible production sets are spanned by M; or that its optimal production choices fall in M. Rather, each firm's production activities allow the firms to extend pricing via the pricing kernel to encompass assets, $\partial_{\mathbf{q}} E_k\left(\bar{\mathbf{q}}\left(\mathbf{v}\right), u_k\right) - \mathbf{m}_k$ and $\partial_{\mathbf{q}} c_k^*\left(\mathbf{w}, \bar{\mathbf{q}}\left(\mathbf{v}\right)\right)$ that are not perfectly replicable. Thus, it generalizes the *effectively complete market* conditions for Pareto optimality developed, for example, by LeRoy and Werner (2000) and others.

Chambers and Quiggin (2003) show that the ability of the firm's technology to extend the pricing kernel in this fashion to nonreplicable assets depends critically upon the flexibility of the firm's production technology. In particular, they show that the stochastic production function representation, which is the cornerstone of much analysis of the risk-averse firm, is unlikely to permit this extension without severe restrictions upon firm preferences.

5.1 Measuring market-level idiosyncratic risk

One characteristic of a separating dual entrepreneurial equilibrium is that state-contingent production plans maximize economy-wide market value. In symbolic terms,

$$\begin{aligned} \max\left\{\bar{\mathbf{q}}\left(\mathbf{v}\right)\sum_{k=1}^{K}\mathbf{z}_k - \mathbf{w}\sum_{k=1}^{K}\mathbf{x}_k\colon \mathbf{x}_k \in X_k\left(\mathbf{z}_k\right), k = 1, ..., K\right\} &= \sum_{k=1}^{K}\max\left\{\bar{\mathbf{q}}\left(\mathbf{v}\right)\mathbf{z}_k - \mathbf{w}\mathbf{x}_k\colon \mathbf{x}_k \in X_k \right. \\ &= \sum_{k=1}^{K} c_k^*\left(\mathbf{w}, \bar{\mathbf{q}}\left(\mathbf{v}\right)\right). \end{aligned}$$

Thus, the market is allocatively efficient. Given $(\bar{\mathbf{q}}(\mathbf{v}), \mathbf{w})$, there is no possible reallocation of production activities across firms that will increase the economy-wide market value.

If financial markets are not complete, equilibrium is not generally Pareto optimal. In equilibrium, because of the presence of idiosyncratic risk, each firm may have a different present-value vector and, thus, discounts random income streams differently. If markets were complete, these different valuations would be resolved by trading state claims.

Our analysis suggests that an appropriate measure of the cost of the residual idiosyncratic risk is the difference between the actual market value of the firms' production activities and the maximal possible economy-wide value at equilibrium $\bar{\mathbf{q}}(\mathbf{v})$. Symbolically, the cost of idiosyncratic risk (IR) is measured by

$$\begin{aligned} IR &= \sum_{k=1}^{K} c_k^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) - \left[\bar{\mathbf{q}}(\mathbf{v}) \sum_{k=1}^{K} \partial_{\mathbf{q}} c_k^*(\mathbf{w}, \mathbf{q}_k) + \mathbf{w} \sum_{k=1}^{K} \partial_{\mathbf{w}} c_k^*(\mathbf{w}, \mathbf{q}_k)\right] \\ &= \sum_{k=1}^{K} c_k^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) - \sum_{k=1}^{K} c_k^*(\mathbf{w}, \mathbf{q}_k) + \sum_{k=1}^{K} (\mathbf{q}_k - \bar{\mathbf{q}}(\mathbf{v})) \partial_{\mathbf{q}} c_k^*(\mathbf{w}, \mathbf{q}_k). \end{aligned}$$

Several observations. First, by the definition of c^*, $IR \geq 0$. Second, each firm's contribution to IR decomposes into two components. The first, $c_k^*(\mathbf{w}, \bar{\mathbf{q}}(\mathbf{v})) - c_k^*(\mathbf{w}, \mathbf{q}_k)$, is a *distribution effect,* and the second, $(\mathbf{q}_k - \bar{\mathbf{q}}(\mathbf{v})) \partial_{\mathbf{q}} c_k^*(\mathbf{w}, \mathbf{q}_k)$, its entrepreneurial risk. The distribution effect, which measures the difference between the firm's optimal market value and its internal present value, can be either positive or negative for an individual firm as can the entrepreneurial risk. The fact that some firms may have 'too much' present value income and others may have 'too little' is a basic characteristic of Pareto inferior equilibria.

Efficient-set spanning, for example, is sufficient to ensure that the entrepreneurial risk is zero. Notice, however, that even if the entrepreneurial risk is zero for all firms, IR can be strictly positive because of the presence of nonzero distribution effects. Thus, spanning conditions of the type formulated, for example, by Milne (1995) and Magill and Quinzii (1995) are not sufficient to ensure Pareto optimality. The separation conditions of Chambers and Quiggin (2003), which are manifested in the definition of a dual separating entrepreneurial equilibrium, do.

A distinctive feature of this measure of idiosyncratic risk is that it does not require either the existence of an objective probability measure or a common subjective probability (rational expectations) to parse the idiosyncratic risk. This is important analytically because when markets are incomplete, entrepreneurs may choose to bear idiosyncratic risk in production *precisely because of differences in probability judgments* even when $\bar{q}(\mathbf{v}) \in P_k(\mathbf{w})$ for all k. By contrast, the most commonly used approach to idiosyncratic risk measurement rules such occurences out by assumption when it partitions idiosyncratic risk via variances and covariances predicated upon a common probability measure (see, for example, Magill and Quinzii (1995)).

6 Concluding comments

The fact of stochastic production was a primary motivation for the development of many financial markets. Yet, despite important contributions such as those of Milne (1976, 1995), Cochrane (1991, 1996), Magill and Quinzii (1995), Jermann (1998), Tallarini (2000), and real-business cycle models, asset pricing and resource allocation decisions have typically been analyzed separately. And when they were analyzed jointly, it was typically in either purely primal terms or in terms of induced preferences and technologies. Dual methods permit the application of a wide range of powerful analytical tools to the benchmark sole-proprietorship model. In particular, a dual representation of the theory of asset pricing illuminates the firm's resource allocation decisions while linking them clearly and intuitively to its activities in financial markets.

There are a number of directions in which the analysis might be extended. Perhaps the most important is further comparative-static analysis for asset pricing theory, both at the firm and the economy level. The simple example of technological change considered here illustrates some of the possibilities. However, other possibilities including the effect of tax policy appear to flow from a natural extension of these results.

We emphasize, however, that the scope of this paper is limited to the two-period sole proprietorship economy. Particularly in instances where the Fisher separation theorem is known to fail, the derivative-cost function, or its logical extensions may have a relatively small role to play in analyzing the resource allocation and asset pricing decisions of firms.

One might think that it would be similarly difficult to extend the analysis to incorporate the existence of frictions in financial markets within the two-period framework. However, if these frictions can be reasonably treated as convex, as in Prisman (1986) and Ross (1987), a straightforward modification of the basic modelling procedure applies. Similarly, much of the analysis presented here may be applied to the analysis of partnerships and corporations.

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