Nonparametric Analysis of Technology and Productivity under Non-Convexity

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Abstract: This paper investigates the nonparametric analysis of technology under non-convexity. The analysis extends two approaches now commonly used in efficiency and productivity analysis: Data Envelopment Analysis (DEA) where convexity is imposed; and Free Disposal Hull (FDH) models. We argue that, while the FDH model allows for non-convexity, its representation of non-convexity is too extreme. We propose a new nonparametric model that relies on a neighborhood-based technology assessment which allows for less extreme forms of non-convexity. The distinctive feature of our approach is that it allows for non-convexity to arise in any part of the feasible set. We show how it can be implemented empirically by solving simple linear programming problems. And we illustrate the usefulness of the approach in an empirical application to the analysis of technical and scale efficiency on Korean farms.

Keywords: technology, productivity, nonparametric, non-convexity

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1. Introduction

Nonparametric analysis of technology and productivity has been the subject of much interest (e.g., Afriat, 1972; Färe et al., 1994; Varian, 1984). It has provided the basis for Data Envelopment Analysis (DEA) now commonly used in the investigation of productivity and firm efficiency (e.g., Banker, 1984; Banker et al., 1984; Ray, 2004; Cook and Seiford, 2009). DEA has been seen as an attractive approach for three reasons: it allows for a flexible representation of multi-input multi-output technology; it involves solving simple linear programming problems; and it can provide firm-specific estimates of productivity and efficiency. Yet, it has one significant limitation: it assumes that the feasible set is always convex (where diminishing marginal productivity applies everywhere). As such, DEA is not appropriate in the investigation of non-convex technologies. How important are non-convexity issues in the analysis of productivity and firm efficiency? There are situations where non-convexity has significant implications for economics and management. For example, it is an important issue in the analysis of multi-product firms: non-convexity contributes to generating productivity benefits from specialization (e.g., Bogetoft, 1996; Chavas and Kim, 2007). This implies a need to develop empirical methods that can support the analysis of non-convex technology. Such methods are needed to examine empirically when and where non-convexity may arise.

The objective of this paper is to propose a refined nonparametric method for the analysis of technology under non-convexity. Note that non-parametric representations of technology under non-convexity are not new. Relaxing convexity assumptions in DEA has been explored by Deprins et al. (1984), Petersen (1990), Bogetoft (1996), Kerstens and Eeckaut (1999), Bogetoft et al. (2000), Briec et al. (2004), Podinovski (2005), Leleu (2006, 2009), De Witte and Marques

(2011) and others. The most common approach is the "free disposal hull" (FDH) representation investigated by Deprins et al. (1984) and Kerstens and Eeckaut (1999). But while the FDH model allows for non-convexity, we argue that its representation is too extreme: it tends to find evidence of non-convexity "too often". Note that other approaches have also been used to relax the convexity assumption in nonparametric analyses. They include Petersen (1990), Bogetoft (1996), Agrell et al. (2005) and Podinovski (2005). Petersen (1990) and Bogetoft (1996) have proposed to restrict convexity only to the input space or the output space. Agrell et al. (2005) have considered technology represented by unions of pairs of convex input and output sets. And Podinovski (2005) has put forward an approach where convexity is evaluated individually for each input or output.

In this paper, we propose a new nonparametric model that relies on a neighborhood-based assessment of technology. Our approach allows for non-convexity to arise in any part of the feasible set. As such, it extends previous nonparametric analyses of non-convex technology. Our model nests as (restrictive) special cases both the DEA model and the FDH model. We show how it can be implemented empirically by solving simple linear programming problems. As such, our new nonparametric approach extends the related literature both theoretically and empirically. Its usefulness is illustrated in an application to the analysis of technical and scale efficiency on Korean farms.

The new model and its neighborhood-based assessment of technology is presented in section 2. Its use in the evaluation of non-convex technologies is discussed in section 3. Using a directional distance function, section 4 presents productivity analysis under non-convexity and proposes a new measure to evaluate the extent of non-convexity. Section 5 examines the evaluation of returns to scale and scale efficiency under non-convexity. In section 6, we show how our approach can be implemented easily by solving simple optimization problems. The usefulness of the method is illustrated in an application presented in section 7. Finally, section 8 concludes.

2. The Model

Consider the observation of production activities on a set of N firms in an industry. Each firm produces m netputs $z \in R^m$ and faces a production technology represented by the feasible set $T \subset R^m$. We use the netput notation where inputs are negative and outputs are positive. Let $z_i \equiv (z_{1i,} \ldots, z_{mi}) \in R^m$ be the netput vector produced by the i-th firm, where z_{ji} is the j-th netput used/produced by the i-th firm, and $z_i \in T$ means that netputs z_i are feasible, $i \in \mathcal{N} \equiv \{1, \ldots, N\}$. The technology T may exhibit different scale properties. It is said to exhibit

$$\left\{\begin{array}{c}\text{non-decreasing returns to scale (NDRS)}\\ \text{constant returns to scale (CRS)}\\ \text{non-increasing returns to scale (NIRS)}\end{array}\right\} \text{ if } T \left\{\begin{array}{c}\supset\\ =\\ \subset\end{array}\right\} \delta\, T \text{ for any scalar } \delta > 1.$$

And the technology is said to exhibit variable returns to scale (VRS) if no *a priori* restriction is imposed on returns to scale. Throughout the paper, we assume that the technology T satisfies free disposal, where free disposal means that $T = T - R^m_+$.

First, consider the case where T is convex.[1] Then, under free disposal, a nonparametric representation of the technology is given by

$$T_v = \{z\colon z \le \Sigma_{i\in \mathcal{N}}\ \lambda_i\, z_i;\ \lambda_i \in R_+,\ i \in \mathcal{N};\ \Sigma_{i\in \mathcal{N}}\ \lambda_i = 1\}. \tag{1}$$

T_v in (1) is the smallest convex set containing all data points $\{z_i\colon i \in \mathcal{N}\}$ under free disposal and variable returns to scale (VRS) (e.g., Afriat, 1972; Varian, 1984). It is the representation commonly used in Data Envelopment Analysis (DEA) (e.g., Banker, 1984; Banker et al., 1984; Ray, 2004; Cook and Seiford, 2009).

Alternative representations have been proposed depending on the scale properties of the technology. Following Färe et al. (1994) and Banker et al. (2004), they are

$$T_s = \{z\colon z \le \Sigma_{i\in \mathcal{N}}\ \lambda_i\, z_i;\ \lambda_i \in R_+,\ i \in \mathcal{N},\ \Sigma_{i\in \mathcal{N}}\ \lambda_i \in S_s\}, \tag{2}$$

where $s \in \{v, c, ni, nd\}$, with $S_v = 1$ under variable returns to scale (VRS), $S_c = [0, \infty]$ under constant returns to scale (CRS), $S_{ni} = [0, 1]$ under non-increasing returns to scale (NIRS), and $S_{nd} = [1, \infty]$ under non-decreasing returns to scale (NDRS). Indeed, when $S_v = 1$, T_v in (2) reduces to equation (1) under variable returns to scale (VRS). Alternatively, when $S_c = [0, \infty]$, T_c in (2) provides a representation of a convex technology under constant returns to scale (CRS). T_c is the smallest convex cone containing all data points $\{z_i: i \in N\}$. When $S_{ni} = [0, 1]$, T_{ni} in (2) provides a representation of a convex technology under non-increasing returns to scale (NIRS). Finally, when $S_{nd} = [1, \infty]$, T_{nd} in (2) represents a convex technology under non-decreasing returns to scale (NDRS). Since $S_v \subset S_{ni} \subset S_c$ and $S_v \subset S_{nd} \subset S_c$, it follows from (2) that $T_v \subset T_{ni} \subset T_c$ and $T_v \subset T_{nd} \subset T_c$. Also, $S_c = S_{ni} \cup S_{nd}$ implies that $T_c = T_{ni} \cup T_{nd}$. Note that the sets T_v, T_{ni}, T_{nd} and T_c are all convex.

Next, we want to introduce non-convexity in the analysis. For that purpose, consider the following nonparametric representation of technology

$$T_{FDHv} = \{z: z \leq \Sigma_{i \in N}\ \lambda_i\ z_i;\ \lambda_i \in \{0, 1\},\ i \in N;\ \Sigma_{i \in N}\ \lambda_i = 1\}, \tag{3}$$

where FDH stands for “free disposal hull” (Deprins et al., 1984; Kerstens and Eeckaut, 1999). Under free disposal, T_{FDHv} is the smallest set containing all data points $\{z_i: i \in N\}$ under variable returns to scale (VRS). It provides a non-convex representation of the technology under VRS.

Alternative non-convex representations have been proposed depending on the scale properties of the technology. Following Kerstens and Eeckaut (1999), they include

$$T_{FDHs} = \{z: z \leq \Sigma_{i \in N} \lambda_i\ z_i;\ \lambda_i \in \{0, \delta\},\ i \in N;\ \Sigma_{i \in N}\ \lambda_i = \delta;\ \delta \in S_s\}. \tag{4}$$

where $s \in \{v, c, ni, nd\}$, and the S_s's are as defined above. When $S_v = 1$, T_{FDHv} in (4) reduces to equation (3) under variable returns to scale (VRS). Alternatively, when $S_c = [0, \infty]$, T_{FDHc} in (4) provides a representation of a FDH technology under constant returns to scale (CRS). T_{FDHc} is the smallest cone containing all data points $\{z_i: i \in N\}$. When $S_{ni} = [0, 1]$, T_{FDHni} in (4) provides a representation of a FDH technology under non-increasing returns to scale (NIRS). Finally, when $S_{nd} = [1, \infty]$, T_{FGHnd} in (4) represents a FDH technology under non-decreasing returns to scale (NDRS). Since $S_v \subset S_{ni} \subset S_c$ and $S_v \subset S_{nd} \subset S_c$, it follows from (4) that $T_{FDHv} \subset T_{FDHni} \subset T_{FDHc}$ and $T_{FDHv} \subset T_{FDHnd} \subset T_{FDHc}$. Also, $S_c = S_{ni} \cup S_{nd}$ implies that $T_{FDHc} = T_{FDHni} \cup T_{FDHnd}$. Note that each of the sets T_v, T_{ni}, T_{nd} and T_c is in general non-convex. Finally, note that the λ's are restricted to take discrete values in (4) but not in (2). It follows that $T_{FDHs} \subset T_s$, i.e. that T_{FDHs} is a subset of T_s, for $s \in \{v, c, ni, nd\}$.

The sets T_v, T_c and T_{FDHv} are illustrated in Figure 1. Figure 1 shows that these sets satisfy $T_{FDHv} \subset T_v \subset T_c$. Note that the sets T_v and T_c are convex, but that the set T_{FDHv} is in non-convex. This indicates that DEA is clearly inappropriate in the analysis of non-convexity. Indeed, since T_v is always convex, DEA offers no prospect to uncover any evidence of non-convexity and produces biased estimates of technical efficiency under a non-convex technology. In contrast, FDH can provide a basis to represent a non-convex technology. Yet, it has a rather undesirable characteristic: it has a tendency to find non-convexity at many places. This can be seen in Figure 1, where the frontier technology is given by the line ABDHJ under T_v and by ABCDEFGHJ under T_{FDHv}. While the frontier line ABDHJ is concave, the frontier line ABCDEFGHJ is not. The two lines coincide only along the segments AB and HJ, where marginal products are either zero or infinite under T_v. At all other points, the two lines differ. It means that, under FDH, the frontier technology would basically exhibit non-convexity at all points where marginal products are positive and bounded under T_v. Yet, we are usually interested in situations where marginal products are positive and bounded. The fact that FDH would always reveal non-convexity in these situations seems undesirable. In other words, while T_{FDHv} can provide a representation of non-convexity, it may reveal it "too often". This indicates a need to develop alternative representations of technology that can capture non-convexity in a more useful and credible way. Below, we explore alternative formulations that allow for flexible representations of the technology T under non-convexity.

Define a neighborhood of $z \equiv (z_1, \ldots, z_m) \in R^m$ as $B_r(z, \sigma) = \{z': D_p(z, z') \leq r: z' \in R^m\} \subset R^m$, where $r > 0$ and $D_p(z, z') \equiv \sum_{j=1}^{m} [(|z_j - z_j'|/\sigma_j)^p]^{1/p}$ is a weighted Minkowski distance between z and z', with weights $\sigma = (\sigma_1, \ldots, \sigma_m) \in R^m_{++}$ and based on a p-norm $1 \leq p < \infty$.[2] Let $I(z, r) = \{i: z_i \in B_r(z, \sigma), i \in N\} \subset N$, where $I(z, r)$ is the set of firms in N that are located in the neighborhood $B_r(z, \sigma)$ of z.[3] Define a <u>local</u> representation of the technology T in the neighborhood of point z as:

$$T_{rv}(z) = \{z: z \leq \Sigma_{i \in I(z,r)} \lambda_i z_i; \lambda_i \in R_+, i \in I(z, r); \Sigma_{i \in I(z,r)} \lambda_i = 1\}. \quad (5)$$

Equation (5) corresponds to equations (1) except that it applies locally using information limited to points in the neighborhood $B_r(z, \sigma)$ of z under variable returns to scale (VRS). Using (2), alternative local representations of the technology can be obtained depending on its scale properties. They are

$$T_{rs}(z) = \{z: z \leq \Sigma_{i \in I(z,r)} \lambda_i z_i; \lambda_i \in R_+, i \in I(z, r); \Sigma_{i \in I(z,r)} \lambda_i \in S_s\}. \quad (6)$$

where $s \in \{v, c, ni, nd\}$, and the S_s's are as defined above. When $S_v = 1$, $T_{rv}(z)$ in (6) reduces to equation (5) under variable returns to scale (VRS). Alternatively, when $S_c = [0, \infty]$, $T_{rc}(z)$ in (6) provides a local representation of the technology under constant returns to scale (CRS). When $S_{ni} = [0, 1]$, $T_{rni}(z)$ in (6) is a local representation of the technology under non-increasing returns to scale (NIRS). Finally, when $S_{nd} = [1, \infty]$, $T_{rnd}(z)$ in (6) gives a local representation of the technology under non-decreasing returns to scale (NDRS). Since $S_v \subset S_{ni} \subset S_c$ and $S_v \subset S_{nd} \subset S_c$, it follows from (6) that $T_{rv}(z) \subset T_{rni}(z) \subset T_{rc}(z)$ and $T_{rv}(z) \subset T_{rnd}(z) \subset T_{rc}(z)$. Also, $S_c = S_{ni} \cup S_{nd}$ implies that $T_{rc}(z)= T_{rni}(z) \cup T_{rnd}(z)$. Finally, note that, for a given z, the sets $T_{rv}(z)$, $T_{rni}(z)$, $T_{rnd}(z)$ and $T_{rc}(z)$ are all convex.

Definition 1: Consider the following neighborhood-based representation of the technology T:

$$T_{rs}^* = \cup_{i \in N}\ T_{rs}(z_i), \tag{7}$$

for $s \in \{v, c, ni, nd\}$.

Equation (7) defines the set T_{rs}^* as the union of the sets $T_{rs}(z_i)$, $i \in N$. In the neighborhood of point z_i , the set $T_{rs}(z_i)$ is convex and provides a local representation of the technology T under free disposal and returns to scale characterized by $s \in \{v, c, ni, nd\}$. Since the union of convex sets is not necessarily convex, it follows that T_{rs}^* defined in (7) is not necessarily convex for each $s \in \{v, c, ni, nd\}$.

Equation (7) is our proposed neighborhood-based representation of technology. It allows for non-convexity to arise in any part of the feasible set. It differs from the approaches proposed by Petersen (1990), Bogetoft (1996), Agrell et al. (2005) or Podinovski (2005), who explored departures from non-convexity based on inputs and/or outputs. As showed below, T_{rs}^* has three useful characteristics: 1/ it provides a flexible representation of non-convexity; 2/ it nests as (restrictive) special cases both the DEA model and the FDH model; and 3/ it is easy to implement empirically.

3. Evaluating Non-Convexity

Our evaluation of non-convexity of the technology relies on the properties of the representations T_s and T_{rs}^*. The following properties will prove useful. .

Lemma 1: For $s \in \{v, c, ni, nd\}$, the set T_{rs}^* satisfies

$$\lim_{r\to\infty} T_{rs}^* = T_s. \tag{8}$$

Proof: Note that $\lim_{r\to\infty} I(z, r) = N$ for any finite $z \in R^m$. Using equations (2), (6) and (7), it follows that $T_s = \lim_{r\to\infty} T_{rs}(z_i) = \lim_{r\to\infty} T_{rs}^*$ for any $i \in N$ and $s \in \{v, c, ni, nd\}$.

Lemma 2: For $s \in \{v, c, ni, nd\}$, the set T_{rs}^* satisfies

$$\lim_{r\to 0} T_{rs}^* = T_{FDHs}. \qquad (9)$$

Proof: Note that $\lim_{r\to 0} B_r(z_i, \sigma) = \{z_i\}$ and $\lim_{r\to 0} I(z_i, r) = \{i\}$ for any $i \in N$. Using equation (6), we have $\lim_{r\to 0} T_{rs}(z_i) = \{z: z \leq \gamma z_i, \gamma \in S_s\}$. Equation (7) can be alternatively written as $T_{rs}^* = \{\Sigma_{i\in N} \alpha_i T_{rs}(z_i): \alpha_i \in \{0, 1\}, i \in N; \Sigma_{i\in N} \alpha_i = 1\}$. Letting $\eta_i = \alpha_i \gamma$, this implies that $\lim_{r\to 0} T_{rs}^* = \{z: z \leq \Sigma_{i\in N} \eta_i z_i; \eta_i \in \{0, \gamma\}, i \in N; \Sigma_{i\in N} \eta_i = \gamma, \gamma \in S_s\}$. Using equation (4), this gives (9).

Given $s \in \{v, c, ni, nd\}$, equations (8) and (9) show that T_{rs}^* includes two important special cases. From equation (8), the set T_{rs}^* reduces to the set T_s when $r \to \infty$, i.e. when the neighborhood $B_r(z, \sigma)$ of any z becomes "very large". And from equation (9), the set T_{rs}^* reduces to the set T_{FDHs} when $r \to 0$, i.e. when the neighborhood $B_r(z_i, \sigma)$ become "very small" for any $i \in N$.

Proposition 1: For $s \in \{v, c, ni, nd\}$, the sets satisfy

$$T_{FDHs} \subset T_{rs}^* \subset T_{r's}^* \subset T_s, \text{ for any } r' > r > 0. \qquad (10)$$

Proof: Note that $\lim_{r\to 0} B_r(z_i, \sigma) \subset B_r(z_i, \sigma) \subset B_{r'}(z_i, \sigma) \subset \lim_{r\to\infty} B_r(z_i, \sigma)$ for any $r' > r > 0$. Thus, for any $r' > r > 0$, $\lim_{r\to 0} I(z_i, r) \subset I(z_i, r) \subset I(z_i, r') \subset \lim_{r\to\infty} I(z_i, r) = N$. Then, equation (6) implies that $\lim_{r\to 0} T_{rs}(z_i) \subset T_{rs}(z_i) \subset T_{r's}(z_i) \subset \lim_{r\to\infty} T_{rs}(z_i)$ for any $r' > r > 0$ and any $i \in N$. Using equations (7), (8) and (9), this proves (10).

Proposition 1 states that T_{FDHs} is in general a subset of T_s: $T_{FDHs} \subset T_s$, for $s \in \{v, c, ni, nd\}$. It also establishes that the set T_{rs}^*, our neighborhood-based representation of technology, is bounded between T_{FDHs} and T_s, with T_{FDHs} as lower bound and T_s as upper bound. Noting that the set T_s is convex, and the set T_{FDHs} is in general non-convex, it means that T_{rs}^* provides a generic way of introducing non-convexity in production analysis. The sets T_v, T_{FDHv} and T_{rv}^* are illustrated in Figure 2 under VRS. Figure 2 shows that these sets satisfy $T_{FDHv} \subset T_{rv}^* \subset T_v$. Note that the set T_v is convex, but that the sets T_{rv}^* and T_{FDHv} are non-convex. These representations apply under alternative scale properties: under VRS when s $\in$ v (with $S_v = 1$), under CRS when s = c (with $S_c = [0, \infty]$), under NIRS when s = ni (with $S_{ni} = [0, 1]$), as well as under NDRS when s = ni (with $S_{nd} = [1, \infty]$). Finally, equation (10) states that the set T_{rs}^* becomes larger when r increases, i.e. when the neighborhoods used to evaluate T_{rs}^* become larger. As further discussed below, this provides some flexibility in the empirical analysis of non-convexity issues.

4. Productivity under Non-Convexity

Let $g \in R_m^+$ be a reference bundle satisfying $g \neq 0$. Following Chambers et al. (1996), consider the directional distance function[4]

$$D(z, T) = \sup_\beta \{\beta: (z + \beta\, g) \in T\} \text{ if there is a scalar } \beta \text{ satisfying } (z + \beta\, g) \in T\}, \tag{11}$$

$$= -\infty \text{ otherwise.}$$

The directional distance function is the distance between point z and the upper bound of the technology T, measured in number of units of the reference bundle g. It provides a general measure of productivity. In general, D(z, T) = 0 means that point z is on the frontier of the technology T. Alternatively, D(z) > 0 implies that z is technically inefficient (as it is below the frontier),[5] while D(z, T) < 0 identifies z as being infeasible (as it is located above the frontier). Luenberger (1995) and Chambers et al. (1996) provide a detailed analysis of the properties of D(z, T). First, by definition in (11), $z \in T$ implies that $D(z, T) \geq 0$ (since $\beta = 0$ would then be feasible in (11)), meaning that $T \subset \{z: D(z, T) \geq 0\}$. Second, $D(z, T) \geq 0$ in (11) implies that $(z + D(z, T)\, g)) \in T$. When the technology T exhibiting free disposal, it follows that $D(z, T) \geq 0$ implies that z

$\in$ T, meaning that T $\supset$ {z: D(z, T) $\geq$ 0}. Combining these two properties, we obtain the following result: under free disposal, T = {z: D(z, T) $\geq$ 0} and D(z, T) provides a complete representation of the technology T. Importantly, besides being convenient, this result is general: it allows for an arbitrary multi-input multi-output technology; and it applies with or without convexity.

Using (10) and (11), we obtain the following key result.

Proposition 2: For any point $z \in R^m$ where $D(z, T_s) > -\infty$, the directional distance function satisfies

$$D(z, T_{FDHs}) \leq D(z, T_{rs}^*) \leq D(z, T_{r's}^*) \leq D(z, T_s), \text{ for any } r' > r > 0, \tag{12}$$

for s $\in$ {v, c, ni, nd}.

Proposition 2 shows that $D(z, T_{rs}^*)$ is bounded between $D(z, T_{FDHs})$ and $D(z, T_s)$, with $D(z, T_{FDHs})$ as lower bound and $D(z, T_s)$ as upper bound. When s = v, equation (12) implies that DEA (relying on T_v) is more likely to find evidence of technical inefficiency than FDH. This is illustrated in Figure 1, which shows that the production frontier tends to be higher under DEA compared to FDH. With s $\in$ {v, c, ni, nd}, equation (12) shows that this result applies under alternative characterizations of returns to scale. It also shows that $D(z, T_{rs}^*)$ tends to increase with r, where T_{rs}^* is our neighborhood-based representation of technology given in (7). Finally, as discussed next, Proposition 2 provides a basis to evaluate the role of non-convexity in productivity analysis.

Definition 2: At point z, define the following measure of non-convexity

$$C_{rs}(z) \equiv D(z, T_s) - D(z, T_{rs}^*), \tag{13}$$

for s $\in$ {v, c, ni, nd}.

Proposition 3: At point z where $D(z, T_v) > -\infty$,

$$\lim_{r\to 0} C_{rs}(z) \geq C_{rs}(z) \geq C_{r's}(z) \geq \lim_{r\to\infty} C_{rs}(z) = 0, \text{ for any } r' > r > 0, \tag{14}$$

for s $\in$ {v, c, ni, nd}.

Proof: The inequalities in (14) are obtained from combining (12) and (13), and using equations (8) and (9).

Proposition 3 applies under alternative characterizations of returns to scale: under VRS (when s = v), CRS (when s = c), NIRS (when s = ni), as well as NDRS (when s = nd). Equation (13) defines $C_{rs}(z)$ as a measure of non-convexity, evaluated in number of units of the reference bundle g. From equation (14), this measure is always non-negative: $C_{rs}(z) \geq 0$. Equation (14) states that $\lim_{r\to\infty} C_{rs}(z) = 0$. This is intuitive: DEA assumes convexity and does not provide any opportunity to uncover the presence of non-convexity. It means that the search for non-convexity must rely on the case where $r < \infty$. Then, for a given $r < \infty$, finding $C_{rs}(z) > 0$ at some point z implies that the set T_{rs}^* is non-convex. In addition, (14) states that $\lim_{r\to 0} C_{rs}(z)$ is an upper bound measure for $C_{rs}(z)$. This reflects the fact that, under free disposal, FDH offers the greatest prospects to uncover non-convexity. Finally, equation (14) shows that $C_{rs}(z)$ tends to decrease with r, indicating that the opportunity to uncover non-convexity declines with the size of the neighborhoods used to evaluate T_{rs}^*. The effects of r on the evaluation of non-convexity are further discussed below.

5. Evaluating Returns to Scale

Since our analysis applies under alternative scale characterization, it can also be used to investigate returns to scale. While evaluating scale efficiency is well known under convexity (e.g., Färe et al. 1994; Banker et al. 2004), this section explores how this can be done under non-convexity.

Proposition 4: The sets satisfy

$$T_{rv}^* \subset T_{rni}^* \subset T_{rc}^*, \qquad (15a)$$

$$T_{rv}^* \subset T_{rnd}^* \subset T_{rc}^*. \tag{15b}$$

Proof: We have seen that $T_{rv}(z) \subset T_{rni}(z) \subset T_{rc}(z)$ and $T_{rv}(z) \subset T_{rnd}(z) \subset T_{rc}(z)$. Then, (15a) and (15b) follow from (7).

Definition 3: At point z, define the following measure of scale efficiency

$$SE_{rs}(z) \equiv D(z, T_{rc}^*) - D(z, T_{rs}^*), \tag{16}$$

for $s \in \{v, c, ni, nd\}$.

Proposition 5: At point z where $D(z, T_v) > -\infty$, the scale efficiency measures $SE_{rs}(z)$ satisfy

$$SE_{rv}(z) \geq SE_{rni}(z) \geq 0, \tag{17a}$$

$$SE_{rv}(z) \geq SE_{rni}(z) \geq 0. \tag{17b}$$

Proof: Equations (11), (15a) and (15b) imply that $D(z, T_{rc}^*) \geq D(z, T_{rni}^*) \geq D(z, T_{rv}^*)$, and $D(z, T_{rc}^*) \geq D(z, T_{rnd}^*) \geq D(z, T_{rv}^*)$. Using (16), this gives (17a) and (17b).

Equation (16) defines $SE_{rs}(z)$ as a measure of departure from constant returns to scale (CRS), evaluated in number of units of the reference bundle g. From equations (17a)-(17b), evaluated under VRS (with s = v), the measure is always non-negative: $SE_{rv}(z) \geq 0$. This is intuitive: it follows from the fact that the set T_{rc}^* is always at least as large as T_{rv}^*, as stated in (15a)-(15b). In addition, (17a) states that, under NIRS (with s = ni), $SE_{rni}(z)$ is also non-negative but has $SE_{rv}(z)$ as an upper bound. This follows from the fact that the set T_{rni}^* is always at least as large as T_{rv}^* but never larger than T_{rc}^*, as stated in (15a). And (17b) establishes a similar result under NDRS (with s = nd): $SE_{rnd}(z)$ is non-negative but has $SE_{rv}(z)$ as an upper bound. This shows how $SE_{rs}(z)$ in equation (16) provides a basis to measure scale efficiency under non-convexity. Indeed, finding $SE_{rs}(z) > 0$ at point z implies that the set T_{rs}^* exhibits a departure from CRS and that point z is scale inefficient. The effects of r on the evaluation of scale efficiency will be evaluated below.

6. Empirical Assessment

Consider a data set involving observations of m netputs chosen by N firms: $\{z_i = (z_{1i}, \ldots, z_{mi}): i \in N\}$, where z_{ji} is the j-th netput used by the i-th firm. As suggested in propositions 2-5, we want to find some convenient way to solve for the directional distance function D(z, T) under alternative representations of the technology T.

6.1. Empirical evaluation of directional distance functions

This section examines empirical applications using the data $\{z_i = (z_{1i}, \ldots, z_{mi}): i \in N\}$. First consider the optimization problem (11) under T_s in (2), where $s \in \{v, c, ni, nd\}$, $S_v = 1$, $S_c = [0, \infty]$, $S_{ni} = [0, 1]$ and $S_{nd} = [1, \infty 1]$. For each $s \in \{v, c, ni, nd\}$ and assuming that a solution exists, this gives the standard linear programming (LP) problems $D(z, T_s) = \max_\beta \{\beta: z + \beta\, g \leq \Sigma_{i \in N}\, \lambda_i\, z_i; \lambda_i \in R_+, i \in N, \Sigma_{i \in N}\, \lambda_i \in S_s\}$. In all these cases, convexity is imposed. Second, consider the optimization problem (11) under T_{FDHa} in (4) for $s \in \{v, c, ni, nd\}$. Assuming that a solution exists, this gives $D(z, T_{FDHs}) = \max_\beta \{\beta: z + \beta\, g \leq \Sigma_{i \in N} \lambda_i\, z_i; \lambda_i \in \{0, \delta\}, i \in N; \Sigma_{i \in N}\, \lambda_i = \delta; \delta \in S_s\}$, which is a mixed integer linear programming (MILP) problem for s = v (where $S_v = 1$), but a mixed integer nonlinear programming (MINLP) problem for $s \in \{c, ni, nd\}$.[6]

Below, we explore how to solve (14) under T_{rv}^*, the neighborhood-based representation of technology given in (7). For $s \in \{v, n, ni, nd\}$, note that equation (7) can be alternatively written as

$$T_{rs}^* = \{\Sigma_{j \in N} \alpha_j\, T_{rs}(z_j); \alpha_j \in \{0, 1\}, j \in N; \Sigma_{j \in N}\, \alpha_j = 1\}, \tag{18}$$

for $s \in \{v, c, ni, nd\}$. Let λ_{ij} be the weight λ_i associated with $z = z_j$ in (7). Letting $\eta_{ij} = \alpha_j\, \lambda_{ij}$, it follows from (6), (11) and (18) that

$$D(z, T_{rs}^*) = \text{Max}_{\beta,\lambda,\eta,\alpha} \{\beta: (z + \beta g) \leq \Sigma_{j\in N} \Sigma_{i\in I(z_j,r)} \eta_{ij} z_i: \eta_{ij} = \alpha_j \lambda_{ij}, \lambda_{ij} \in R_+,$$

$$\Sigma_{i\in I(z_j,r)} \lambda_{ij} \in S_s, \alpha_j \in \{0, 1\}, \Sigma_{j\in N} \alpha_j = 1, i \in I(z_j, r), j \in N \} \text{ if a solution exists,} \quad (19)$$

$$= -\infty \text{ otherwise,}$$

for s $\in$ {v, c, ni, nd}. Equation (19) is a mixed integer nonlinear programming (MINLP) problem. Solving it numerically can provide a way to assess the directional distance functions $D(z, T_{ra}^*)$ for s $\in$ {v, c, ni, nd}.

Yet, dealing with non-linear constraints in (19) can be empirically challenging. In this context, alternative formulations that avoid non-linear constraints are of interest. One such formulation is the following optimization problem

$$D^+(z, T_{rs}^*) = \text{Max}_{\beta,\eta,\alpha} \{\beta: (z + \beta g) \leq \Sigma_{j\in N} \Sigma_{i\in I(z_j,r)} \eta_{ij} z_i: \eta_{ij} \in R_+, \Sigma_{i\in I(z_j,r)} \eta_{ij} \in \alpha_j S_s,$$

$$\alpha_j \in \{0, 1\}, \Sigma_{j\in N} \alpha_j = 1, i \in I(z_j, r), j \in N \} \text{ if a solution exists,} \quad (20)$$

$$= -\infty \text{ otherwise.}$$

for s $\in$ {v, c, ni, nd}. Equation (20) is a mixed integer linear programming (MILP) problem. Because it does not include the nonlinear restrictions $\eta_{ij} = \alpha_j \lambda_{ij}$, solving (20) is simpler than solving (19). But the absence of the restrictions $\eta_{ij} = \alpha_j \lambda_{ij}$ in (20) implies that $D^+(z, T_{rs}^*)$ is in general an upper bound to $D(z, T_{rs}^*)$: $D^+(z, T_{rs}^*) \geq D(z, T_{rs}^*)$. When would the two objective functions coincide? They would coincide (with $D^+(z, T_{rs}^*) = D(z, T_{rs}^*)$) when the solution to (20), (η^*, α^*), satisfies $\eta_{ij}^* = 0$ for all i when $\alpha_j^* = 0$, $j \in N$. Otherwise, they would differ, and $D^+(z, T_{rs}^*)$ would be strictly larger than $D(z, T_{rs}^*)$: $D^+(z, T_{rs}^*) > D(z, T_{rs}^*)$. In this later case, solving the simpler problem (20) would provide upward biased estimates of $D(z, T_{rs}^*)$.

6.2. Linear programming formulation

Given the potential empirical difficulties in solving the nonlinear optimization problem (19), we now explore a simpler way to evaluate $D(z, T_{rs}^*)$ in (19). From (7), note that T_{rs}^* is defined from $T_{rs}(z_j)$, $j \in N$. This suggests obtaining $D(z, T_{rs}^*)$ using the following two-step approach.

In step one, solve (11) under $T_{rs}(z')$ in (6). For $s \in \{v, c, ni, nd\}$, this corresponds to the (primal) linear programming (LP) problem

$$D(z, T_{rs}(z')) = \mathrm{Max}_{\beta,\lambda}\ \{\beta: (z + \beta\ g) \leq \Sigma_{i \in I(z',r)}\ \lambda_i\ z_i\ ;\ \lambda_i \in R_+,\ i \in I(z', r);\ \Sigma_{i \in I(z',r)}\ \lambda_i \in S_s\},\quad (21)$$

if a solution exists,

$= -\infty$ otherwise,

or its dual LP formulation

$$D(z, T_{rs}(z')) = \mathrm{Min}_{u,v}\ \{v - z^T u: z_j^T u \leq v,\ j \in I(z', r);\ g^T u = 1;\ u \in R_+^m;\ v \in V_s\},\qquad (21')$$

if a solution exists,

$= -\infty$ otherwise,

where u and v are the Lagrange multipliers associated with the constraints $[(z + \beta\ g) \leq \Sigma_{i \in I(z',r)}\ \lambda_i\ z_i]$ and $[\Sigma_{i \in I(z',r)}\ \lambda_i \in S_s]$ in (21), with $V_v = [-\infty, \infty]$, $V_c = 0$, $V_{ni} = [0, \infty]$ and $V_{di} = [-\infty, 0]$.

Then, in step two, assuming that $D(z, T_{rs}(z_i)) > -\infty$ for some $i \in I$, and using (18), $D(z, T_{rs}^*)$ can be obtained as

$$D(z, T_{rs}^*) = \mathrm{Max}_i\ \{D(z, T_{rs}(z_i)): i \in N\}.\qquad (22)$$

In this two-step approach, step one involves solving linear programming (LP) problems in (21) or (21'). And step 2 stated in (22) is a simple maximization problem. This shows how (21)-(22) can be used to obtain $D(z, T_{rs}^*)$ by solving simple linear programming problems. This provides a convenient way to solve (11) under T_{rs}^*, our neighborhood-based representation of technology given in (7).

6.3. Defining the neighborhood $B_r(z, \sigma)$

As discussed in section 2, our analysis relies on the definition of a neighborhood $B_r(z, \sigma) = \{z': D_p(z, z') \leq r: z' \in R^m\} \subset R^m$, where $D_p(z, z')$ is a weighted Minkowski distance with $1 \leq p < \infty$. Below, it will be convenient to rely on a weighted Chebyshev distance defined as $\lim_{p\to\infty} D_p(z, z') = \text{Max}_j \{|z_j - z_j'|/\sigma_j: j = 1, \ldots, m\}$. In this context, $B_r(z, \sigma)$ can be written as $B_r(z, \sigma) = \{z': -r\,\sigma_j \leq z_j - z_j' \leq r\,\sigma_j;\ j = 1, \ldots, m;\ z' \in R^m\}$ and $I(z, r)$ can be written as $I(z, r) = \{i: -r\,\sigma_j \leq z_j - z_{jj}' \leq r\,\sigma_j;\ j = 1, \ldots, m;\ i \in N\}$.

We can choose this neighborhood in at least two ways. Sometimes, we may have *a priori* information about the regions where non-convexity is likely to arise. Assume that one of these regions is region A(z) around point z. In general we want to choose the neighborhood of $B_r(z, \sigma)$ to be no larger than A(z). Indeed, choosing $B_r(z, \sigma) \supset A(z)$ may just "hide" the non-convexity in A(z) within the larger region $B_r(z, \sigma)$. This generates the following rules:

Rule R1: Around point z, choose a neighborhood $B_r(z, \sigma)$ that is no larger than the region A(z) where non-convexity is suspected: $B_r(z, \sigma) \subset A(z)$.

Rule R1 assumes that we do have *a priori* information about the presence of non-convexity. What if we do not have such information? Then we need to find other ways to identify the neighborhood $B_r(z, \sigma)$. In this context, we can use the data to help choose these neighborhoods. To see that, let $M_j \equiv [\text{Max}_{i\in N} \{z_{ji}\} - \text{Min}_{i\in N} \{z_{ji}\}]$ be the range of observations for z_j, j = 1, …, m. For the j-th netput, consider partitioning the line $[\text{Min}_{i\in N} \{z_{ji}\}, \text{Max}_{i\in N} \{z_{ji}\}]$ into k intervals, j = 1, …, m, where k is an integer satisfying $1 \leq k \leq N$. One way is to choose these intervals to be equally spaced.[7] Then, for the j-th netput, the width of an interval is M_j/k. Given

$B_r(z, \sigma) = \{z': -r\, \sigma_j \leq z_j - z_j' \leq r\, \sigma_j;\ j = 1, \ldots, m;\ z' \in R^m\}$, associate these intervals with a neighborhood of point z by letting $r\, \sigma_j = M_j/k$, k being a positive integer, $j = 1, \ldots, m$. For a given k, it follows that the neighborhood of z can be written as $B_r(z, \cdot) = \{z': -M_j/k \leq z_j - z_j' \leq M_j/k;\ j = 1, \ldots, m;\ z' \in R^m\}$. When z ad z' are points within the range of the data, then choosing k = 1 implies that $B_r(z, \cdot)$ is a "large neighborhood" of z which includes all data points. And choosing k > 1 means that we partition the range of each netput into k equally-spaced intervals, the neighborhood $B_r(z, \cdot)$ of z becoming smaller as k becomes larger.

Next, we propose the following rule to guide us in the choice of neighborhoods.

<u>Rule R2</u>: Around point z, choose a neighborhood $B_r(z, \sigma)$ that includes more than one data point.

R2 has important implications. First, it implies that point z cannot be outside the range of the data. That is intuitive: in any analysis, we should always try to avoid extrapolating beyond the data. Second, Rule 2 requires that there are sufficient data points to support the analysis. It hints that the number of observations N should be "large enough" to provide credible evidence on non-convexity in the neighborhood of point z. Third, R2 rules out FDH. Indeed, from equation (9) in Lemma 2, FDH is obtained when $r \to 0$, implying that the neighborhood of any point z_j would include just the point z_j. This would be inconsistent with R2. As discussed in section 2, the FDH approach seems undesirable as it can find evidence of non-convexity "too often". Intuitively, R2 stresses the importance of having a minimal number of observations (more than 1) to evaluate the characteristics of technology in any neighborhood within the data. As such, R2 can help improve the credibility of finding evidence that a technology is non-convex. Fourth, Rule 2 puts some upper bound on the number of intervals k discussed above. Indeed, increasing k would also reduce the number of observations in each interval. Again, to be credible, evidence of non-convexity in the neighborhood of point z should rely on a sufficient number of data points. Overall, Rule R2 implies that the number of observations N should be "large enough" while the number of intervals should "not be too large". As such, it provides useful guidance to support productivity analysis under non-convexity.

7. Empirical Illustration

To illustrate the usefulness of our proposed approach, we apply it to a data set on production activities from a sample of Korean farm households.

7.1. Data

The data were collected in 2007 in a Farm Household Economy Survey conducted by the Korean National Statistical Office. Our analysis focuses on a sample of farms classified as paddy rice farms located in the Jeon-Nam province, a rice-producing province in the southern part of Korea. Being in the same region, all farms face similar agro-climatic conditions. The sample includes 122 rice farms. It provides data on ten outputs: rice, vegetable, soybean, fruit, potato, barley, miscellaneous, specialty, livestock, and others; and four inputs: labor, size of paddy land, size of upland, and other inputs. Labor input is measured in hours, and land inputs are measured in hectares (ha). Other netputs are measured in value, assuming that all farmers face the same prices.

Descriptive statistics on the variables used in our analysis are presented in table 1. The average revenue from rice production is 15,398.81 (measured in 1,000 won[8]), accounting for 62.7% of total farm revenue. The second largest source of revenue is vegetable production: 3,608.15 (measured in 1,000 won), accounting for 14.7% of total farm revenue. The average size of a farm is 1.31 ha (including both paddy land and upland).

7.2. Results

Our analysis uses data on production activities from our sample of 122 Korean farms. It covers 14 netputs: 10 outputs treated as positive, and 4 inputs treated as negative. For the i-th farm, the netputs are $z_i = (z_{ji}: j = 1, \ldots, 14)$, $i \in N \equiv \{1, 2, \ldots, 122\}$.

The estimation of the directional distance function in (11), (19) or (21)-(22) produces a nonparametric estimate of the distance between point z and the boundary of the feasible set, as measured by the number of units of the reference bundle g. When z is the netput vector for the i-th farm, then the distance function $D(z_i, T) \geq 0$ provides a measure of technical inefficiency for the i-th farm, with $D(z_i, T) > 0$ when the i-th farm is technically inefficient. The reference bundle $g = (g_1, \ldots, g_{14})$ is chosen as follows. We let $g_j = 0$ when j is an input, and g_j = sample mean for the j-th output when j is an output. Thus, our reference bundle $g = (g_1, \ldots, g_{14})$ is the typical bundle associated with the outputs of an average farm. This choice leads to a simple interpretation of our directional distance estimates. For example, for a given T, finding that $D(z_i, T) = 0.2$ would mean that the i-th farm is technically inefficient: it could move the production frontier and increase its outputs by a maximum of 20 percent of the average outputs in our sample by becoming technically efficient. Note that this interpretation remains valid under alternative characterizations of the technology T.

We evaluate the directional distance function $D(z_j, T)$ in (11) for each farm under alternative representations of the technology. First, we start with DEA analysis and solve for $D(z_j, T)$ under technologies T_v under VRS and T_c under CRS (as given in equations (1) and (2)). Second, using T_{FDHv} in (3), we obtain FDH measures $D(z_j, T_{FDHv})$ under VRS technology by solving the corresponding mixed integer programming problems. The results are reported in the Appendix for each farm. Since our neighborhood-based representation of technology allows for non-convexity to arise in any part of the feasible set, it can provide a basis to evaluate productivity and non-convexity for different firm types. We investigate this issue for three categories of farms: small farms, medium farms, and large farms.[9] The results are summarized in Table 2 Table 2 presents the average technical inefficiency estimates $D(z_j, T)$ for each group of farms under alternative representation of the technology. It shows that DEA finds evidence of technical inefficiency across all farm sizes. The mean value of $D(z_j, T_v)$ is 0.063 for small farms, 0.159 for medium farms, and 0.119 for large farms. Table 2 also reports that FDH finds that all farms are technically efficient, with $D(z_j, T_{FDHv}) = 0$ for all $j = 1, \ldots, 122$. Note that this is consistent with Proposition 2, which showed that DEA (relying on T_v) is more likely to find evidence of technical inefficiency than FDH (as the production frontier tends to be higher under DEA compared to FDH). But in this case, allowing for non-convexity under FDH eliminates all evidence of technical inefficiency. This has two implications. First, there can be a large difference between the DEA measure of technical inefficiency $D(z_j, T_v)$, and its FDH counterpart $D(z_j, T_{FDHv})$. Second, this difference is due entirely to relaxing the convexity assumption. One must wonder whether this difference is "credible". As discussed in section 2, this raises the question: Does the FDH approach finds non-convexity "too often"? We believe that it does (as further discussed below).

Next, using the neighborhood-based representation of technology T_{rs}^* in (7) or (18), we obtain estimates of the directional distance $D(z_j, T_{rs}^*)$ by solving the linear programming problems in (21)-(22). Assuming equally spaced intervals, we let $r\ \sigma_j = M_j/k$, where T_{rv}^* is defined using $B_r(z, \cdot) = \{z': -M_j/k \leq z_j - z_j' \leq M_j/k;\ j = 1, \ldots, m;\ z' \in R^m\}$ as neighborhood of z, k denoting the number of intervals within the data range. The analysis is repeated for alternative numbers of intervals k: k = 1, 2, 4, 6, 8, 10, 12. The distances $D(z_j, T_{rs}^*)$ are estimated under VRS (with s = v) for each farm. The results are reported in the Appendix for each farm. Summary measures are presented in Table 2 for our three farm sizes: small farms, medium farms, and large farms. The results are consistent with Proposition 2. First, as expected, $D(z, T_{rv}^*)$ is bounded between D(z,

T_{FDHv}) and D(z, T_v), with D(z, T_{FDHv}) as lower bound and D(z, T_v) as upper bound. Second, D(z, T_{rv}^*) tends to increase with the size of the neighborhood r, or equivalently decrease with the number of intervals k (given r $\sigma_j = M_j/k$). Third, Table 2 shows that our estimates D(z, T_{rv}^*) nest DEA estimates and FDH estimates as special cases. Indeed, D(z, T_{rv}^*) becomes equal to D(z, T_v) when neighborhoods become "large" (in our case, when k = 1), and it becomes equal to D(z, T_{FDHv}) when neighborhoods become "small" (in our case, when k = 12). Yet, neither case seems realistic. Indeed, choosing k = 1 imposes a convex technology and prevents any possibility of uncovering evidence of non-convexity. Alternatively, choosing k = 12 likely finds non-convexity "too often". As noted above, FDH does not satisfy our "Rule 2". In this case, 12 intervals are "too many" as there are not enough points in each neighborhood to obtain a reliable estimate of marginal productivity around each data point. And this has adverse effects on the ability to find evidence of technical inefficiency. Indeed, in this case FDH or k = 12 fails to find any evidence of technical inefficiency. These results help document why FDH does not provide a reasonable approach in the analysis of non-convexity.

One advantage of our approach is that it allows us to choose neighborhoods that satisfy our Rules R1 and R2. These rules seek a balance between finding evidence of technical inefficiency versus finding evidence of non-convexity. In our application, we believe that choosing k = 4 is a good choice: it is between k = 1 (corresponding to DEA) and k = 12 (corresponding to FDH). It identifies neighborhoods that are "not too large" to allow us to uncover evidence of non-convexity, and "not too small" to generate a more reliable estimate of the production technology around any data point. Indeed, with 122 data points, choosing k = 4 means that there are on average about 30 data points per interval, points that provide sample information used to evaluate our neighborhood-based representation of technology. Interestingly, when k = 4, we still find evidence of technical inefficiency. Indeed, Table 2 reports mean estimates of technical inefficiency of 0.025

for small farms (with 62.2% of small farms being technically efficient), 0.035 for medium farms (with 75.5% of medium farms being technically efficient), and 0.003 for large farms (with 100.0% of large farms being technically efficient). In addition, Table 2 reports estimates of the non-convexity measure $C_{rv}(z)$ given in equation (13). When k = 4, the mean estimates of $C_{rv}(z)$ are 0.039 for small farms, 0.123 for medium farms, and 0.116 for large farms. For example, it means that, for medium farms, the effects of non-convexity amount to a 12.3 percent change in average outputs. These estimates indicate that the technology facing Korean farmers exhibit significant non-convexity. They also show that the extent of non-convexity is larger on medium and large farms (compared to small farms). As analyzed by Chavas and Kim (2007), non-convexity contributes to increasing the productivity benefits of specialization. This would indicate that large farms have stronger incentives to specialize than smaller farms. To our knowledge, this is the first evidence that non-convexity appears to vary with firm size.

Finally, we evaluate returns to scale under non-convexity. Using (16), we use our neighborhood-based representation T_{rv}^* under VRS to evaluate scale efficiency $SE_{rv}(z)$. The results are summarized in in Table 3 for our three farm sizes. Recall that $SE_{rv}(z) = 0$ when point z is scale efficient, and $SE_{rv}(z) > 0$ implies a departure from CRS and measures the magnitude of scale inefficiency. The evidence against CRS is in general modest. Under DEA (obtained when r is large and k =1), the average SE is 0.026 for small farms, 0.024 for medium farms, and 0.13 for large farms. Alternatively, under FDH (obtained when r is large and k = 12), all farms are found to be scale efficient (with all SE = 0). Using our neighborhood-based representation of technology with k = 4, the average SE is 0.02 for small farms, 0.041 on medium farms, and 0.030 on large farms.

These results have several implications. First, Korean farms exhibit a high level of scale efficiency. This is consistent with the dominant small-scale rice farming system commonly found in Korea. Second, introducing non-convexity affects the estimate of scale effects. Table 3 shows that the relationship between SE and k is not always monotonic. For example, in the case of medium farms, the average SE first rises then declines with k. This indicates that there is no general relationship between non-convexity and returns to scale. Yet, our results indicate that non-convexity matters in the analysis of scale effects. Indeed, table 3 shows that neglecting non-convexity (by using DEA) would generate upward-biased estimates of SE, while relying on FDH would likely generate downward-biased estimates of SE. Finally, table 3 indicates that these biases vary with farm size. In particular, the estimate of SE is found to be more sensitive to the choice of k for large farms. This is likely due to the fact that non-convexity effects are more important on large farms. This stresses the need to account for non-convexity in the evaluation of returns to scale. This also illustrates the usefulness of our approach in understanding and evaluating the technical and scale efficiency of firms under non-convexity.

8. Concluding Remarks

This paper has presented a new nonparametric approach to the analysis of technology and productivity under non-convexity. Our approach relies on a neighborhood-based representation of technology. We investigate the general properties of our model and its use in the evaluation of technology and productivity under non-convexity. Our approach nests two well-known approaches as special cases: Data Envelopment Analysis (DEA), and Free Disposal Hull (FDH) models. Yet either of these two approaches is overly restrictive: DEA because it does not allow for any non-convexity; and FDH because it allows for "too much" non-convexity. We argue that our new nonparametric model allows for non-convexity in a more flexible way. Its neighborhood-based representation of technology allows for non-convexity to arise in any part of the feasible set. In this context, we propose a measure capturing the extent of non-convexity. We also use our approach to evaluate scale efficiency under non-convexity. We show how our approach can be applied by solving simple optimization problems. Finally, we illustrate its usefulness through an empirical application to Korean farms.

Note that our proposed approach could be extended in number of directions. First, exploring linkages with stochastic frontier analysis (e.g., Kumbhakar et al. 2007; Simar and Zelenyuk, 2011) is a good topic for further investigation. Second, the economics and management implications of non-convexity need to be examined in more details. For example, evaluating the productivity effects of firm specialization is a good topic for further research. Finally, empirical applications to different industries are needed to uncover evidence of situations where non-convexity may be important.

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Table 1. Descriptive statistics

Variable	Obs.	Sample mean	Std. deviation	Min.	Max.
rice revenue (in 1,000 won)	122	15398.81	20251.10	892.04	133825.21
vegetable revenue (in 1,000 won)	122	3608.15	4470.39	0	24964.58
soybean revenue (in 1,000 won)	122	448.82	689.75	0	4471.78
fruit revenue (in 1,000 won)	122	255.16	663.12	0	5272.20
potato revenue (in 1,000 won)	122	592.49	3444.29	0	37230.10
barley revenue (in 1,000 won)	122	1536.48	4212.75	0	26533.03
miscellaneous revenue (in 1,000	122	19.02	52.03	0	402.40
specialty revenue (in 1,000 won)	122	579.57	1510.00	0	9897.81
other revenue (in 1,000 won)	122	92.33	445.88	0	4292.03
livestock revenue (in 1,000 won)	122	2014.37	4325.63	0	21604.84
production costs (in 1,000 won)	122	13863.11	16470.88	868.75	115432.24
family labor (hours)	122	641.25	469.48	71.50	3112.10
paddy land (in ha)	122	1.07	1.36	0	9.71
upland (in ha)	122	0.24	0.30	0	1.61

Note: Note that 1,000 won (the Korean currency) is approximately equivalent to 0.89 US dollar.

Table 2. Average Technical Inefficiency D(z, T) and Non-Convexity C(z) under Alternative Representations of the Technology, by Farm Size.

Farm Size[a]		Small farm		Medium farm		Large farm	
Technology T		technical inefficiency D(z, T)	non-convexity $C_{rv}(z)$	technical inefficiency D(z, T)	non-convexity $C_{rv}(z)$	technical inefficiency D(z, T)	non-convexity $C_{rv}(z)$
T_v (DEA)		0.063 (51.4)[c]		0.159 (49.0)		0.119 (61.1)	
T_{FDHv} (FDH)		0.000 (100.0)		0.000 (100.0)		0.000 (100.0)	
T_{rv}^* (Neighborhood-based representation of technology)	k=1[b]	0.063 (51.4)	0.000	0.159 (49.0)	0.000	0.119 (61.1)	0.000
	k=2	0.038 (62.2)	0.025	0.082 (65.3)	0.077	0.017 (86.1)	0.103
	k=4	0.025 (62.2)	0.039	0.035 (75.5)	0.123	0.003 (100)	0.116
	k=6	0.013 (64.9)	0.050	0.009 (89.8)	0.150	0.000 (100)	0.119
	k=8	0.011 (70.3)	0.052	0.001 (95.9)	0.158	0.000 (100)	0.119
	k=10	0.000 (94.6)	0.063	0.000 (100)	0.159	0.000 (100)	0.119
	k=12	0.000 (100)	0.063	0.000 (100)	0.159	0.000 (100)	0.119

Notes: a/ Farm size is identified by the size of total land. Small farms are defined as farms being in the 0 to 30 percentile of the sample distribution of farm size, medium farms are between the 30 percentile and 70 percentile, and large farms are in the 70 to 100 percentile.

b/ Assuming equally spaced intervals, we let $r\,\sigma_j = M_j/k$, where T_{rv}^* is defined using $B_r(z, \cdot) = \{z': - M_j/k \leq z_j - z_j' \leq M_j/k;\ j = 1, \ldots, m;\ z' \in R^m\}$ as neighborhood of z, and k denotes the number of intervals within the data range.

c/ Next to the average technical inefficiency in each group, the number in parentheses is the percentage of technically efficient farms within the group.

Table 3. Scale Efficiency $SE_{rs}(z)$ under Alternative Representations of the Technology, by Farm Size.

Farm Size[a]		Small farm		Medium farm		Large farm	
Technology T		scale efficiency $SEr_s(z, T)$	% of scale-efficient farms	scale efficiency $SE_{rs}(z, T)$	% of scale-efficient farms	scale efficiency $SE_{rs}(z, T)$	% of scale-efficient farms
T_v (DEA)		0.026	35.1	0.024	49.0	0.130	50.0
T_{FDHv} (FDH)		0.000	81.1	0.000	95.9	0.000	100.0
T_{rv}^* (Neighborhood -based representation of technology)	k=1[b]	0.026	35.1	0.024	49.0	0.130	50.0
	k=2	0.034	43.2	0.050	57.1	0.156	66.7
	k=4	0.020	45.9	0.041	63.3	0.030	77.8
	k=6	0.014	54.1	0.033	69.4	0.011	88.9
	k=8	0.013	56.8	0.022	69.4	0.004	94.4
	k=10	0.000	70.3	0.000	79.6	0.000	97.2
	k=12	0.000	81.1	0.000	95.9	0.000	100.0

Notes: a/ Farm size is identified by the size of total land. Small farms are defined as farms being in the 0 to 30 percentile of the sample distribution of farm size, medium farms are between the 30 percentile and 70 percentile, and large farms are in the 70 to 100 percentile.

b/ Assuming equally spaced intervals, we let $r\,\sigma_j = M_j/k$, where T_{rv}^* is defined using $B_r(z, \cdot) = \{z': -M_j/k \le z_j - z_j' \le M_j/k;\ j = 1, \ldots, m;\ z' \in R^m\}$ as neighborhood of z, and k denotes the number of intervals within the data range.

Figure 1- Representations of Technology under T_v, T_c and T_{FDHv}.

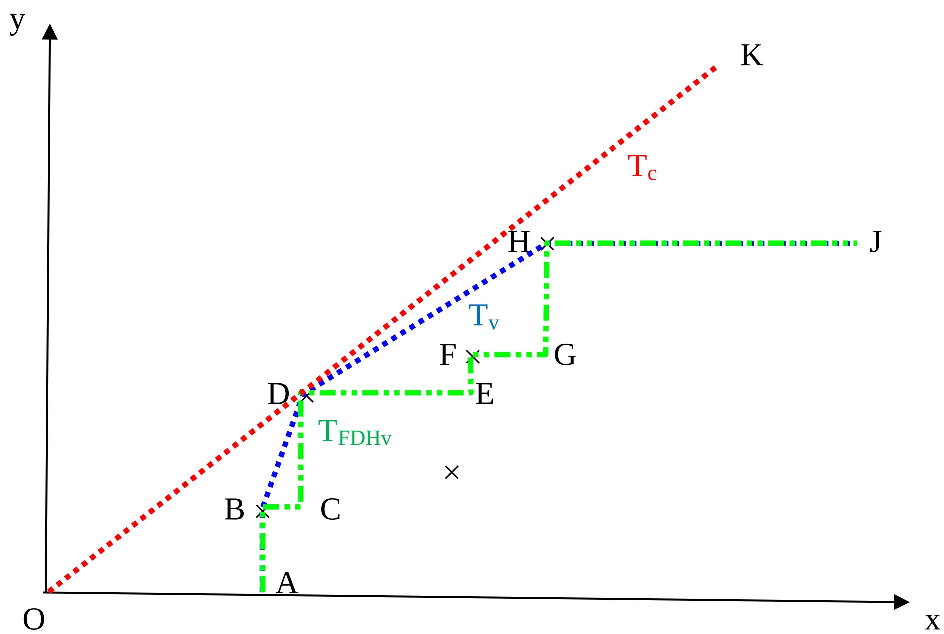
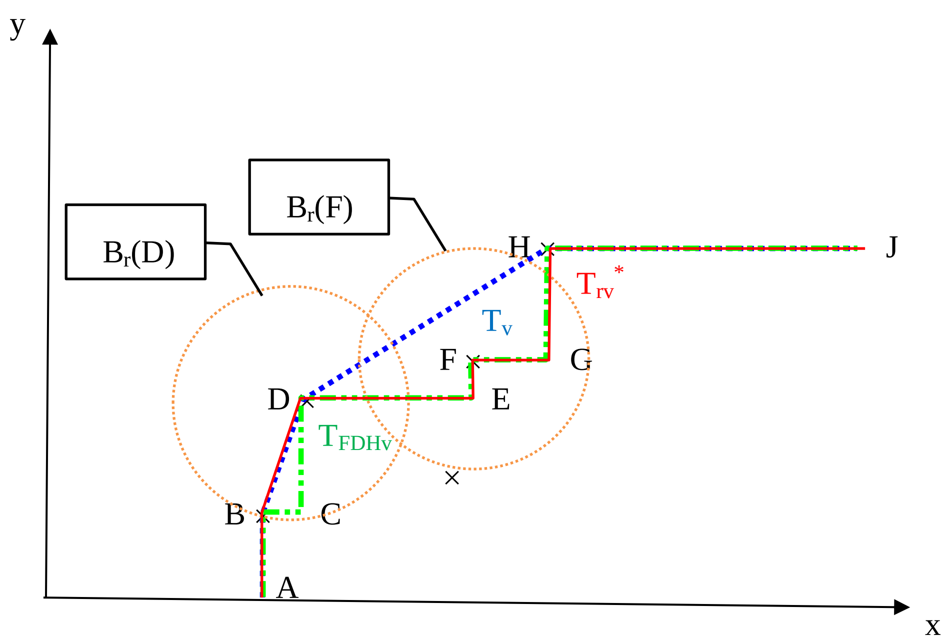


Figure 2- Representations of Technology under T_v, T_{FDHv} and T_{rv}^*.



Appendix 1. Technical Inefficiency D(z, T) for Each Farm under T_{FDHv}, T_v and T_{rv}^*.

	$D(z,T_v)$	$D(z,T_{FDH})$	$D(z, T_{rv}^*)$ (Neighborhood-based representation of technology)						
Farm	(DEA)	(FDH)	k=1[a]	k=2	k=4	k=6	k=8	k=10	k=12
1	0.05807	0	0.05807	0.0479	0.03516	0.03384	0.01293	0	0
2	0.10923	0	0.10923	0	0	0	0	0	0
3	0.18831	0	0.18831	0.07778	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0
7	0	0	0	`0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0
9	0.31028	0	0.31028	0.22411	0.19119	0	0	0	0
10	0.05524	0	0.05524	0.04318	0.02931	0.02182	0.0169	0	0
11	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0
14	0.26196	0	0.26196	0.20173	0.0349	0	0	0	0
15	0.51059	0	0.51059	0.39757	0.17102	0	0	0	0
16	0	0	0	0	0	0	0	0	0
17	0.02883	0	0.02883	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0
20	0.21193	0	0.21193	0.16466	0.12403	0.09727	0.09617	0	0
21	0.18871	0	0.18871	0.12015	0.09719	0.03407	0	0	0
22	0.07299	0	0.07299	0.05625	0.05354	0.04957	0.03771	0	0
23	0.39656	0	0.39656	0.28693	0.19527	0.00464	0	0	0
24	0.18342	0	0.18342	0.11953	0.06646	0.05183	0.04381	0	0
25	0.53594	0	0.53594	0.23268	0.10263	0	0	0	0
26	0	0	0	0	0	0	0	0	0
27	0.02422	0	0.02422	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0
32	0.08687	0	0.08687	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0
35	0.44221	0	0.44221	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0
37	0.12865	0	0.12865	0.05661	0.03722	0	0	0	0
38	0	0	0	0	0	0	0	0	0
39	0.00554	0	0.00554	0	0	0	0	0	0
40	0	0	0	0	0	0	0	0	0
41	0.20641	0	0.20641	0.16212	0	0	0	0	0
42	0	0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0	0	0
44	0.57311	0	0.57311	0.36557	0.20672	0.16115	0	0	0
45	0	0	0	0	0	0	0	0	0
46	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0
48	0.57423	0	0.57423	0.44153	0	0	0	0	0
49	0.0673	0	0.0673	0.05051	0.04933	0.01082	0.00971	0.00002	0
50	0.31944	0	0.31944	0.25168	0.21994	0.16052	0.0001	0	0
51	0.06644	0	0.06644	0	0	0	0	0	0
52	0	0	0	0	0	0	0	0	0
53	0.12894	0	0.12894	0.08894	0.05457	0.04748	0.04748	0	0
54	0.16182	0	0.16182	0.13281	0.11595	0.06936	0.03247	0	0
55	0	0	0	0	0	0	0	0	0
56	0	0	0	0	0	0	0	0	0
57	0.22042	0	0.22042	0.14666	0.11822	0.07754	0.07754	0.00008	0
58	0.06441	0	0.06441	0	0	0	0	0	0
59	0	0	0	0	0	0	0	0	0
60	0.20663	0	0.20663	0.02283	0	0	0	0	0

61	0.42666	0	0.42666	0.13009	0	0	0	0	0
62	0.27122	0	0.27122	0	0	0	0	0	0
63	0.42611	0	0.42611	0.0579	0	0	0	0	0
64	0.11448	0	0.11448	0	0	0	0	0	0
65	0.10595	0	0.10595	0	0	0	0	0	0
66	0.1534	0	0.1534	0.10582	0.02841	0.02265	0.02012	0	0
67	0	0	0	0	0	0	0	0	0
68	0	0	0	0	0	0	0	0	0
69	0.06306	0	0.06306	0.03721	0.03029	0.02786	0.00003	0	0
70	0	0	0	0	0	0	0	0	0
71	0.28092	0	0.28092	0.17012	0.14187	0	0	0	0
72	0	0	0	0	0	0	0	0	0
73	0.34399	0	0.34399	0.1714	0.12672	0	0	0	0
74	0	0	0	0	0	0	0	0	0
75	1.01598	0	1.01598	0.50718	0.14919	0	0	0	0
76	0	0	0	0	0	0	0	0	0
77	0	0	0	0	0	0	0	0	0
78	0	0	0	0	0	0	0	0	0
79	0.3037	0	0.3037	0	0	0	0	0	0
80	0	0	0	0	0	0	0	0	0
81	0	0	0	0	0	0	0	0	0
82	0	0	0	0	0	0	0	0	0
83	0	0	0	0	0	0	0	0	0
84	0	0	0	0	0	0	0	0	0
85	0	0	0	0	0	0	0	0	0
86	0.06372	0	0.06372	0	0	0	0	0	0
87	0	0	0	0	0	0	0	0	0
88	0.41894	0	0.41894	0	0	0	0	0	0
89	0.02432	0	0.02432	0	0	0	0	0	0
90	0	0	0	0	0	0	0	0	0
91	0.05367	0	0.05367	0.05293	0.03424	0.02802	0.00002	0	0
92	0.53044	0	0.53044	0.37717	0	0	0	0	0
93	0	0	0	0	0	0	0	0	0
94	0.38548	0	0.38548	0.09954	0.01607	0	0	0	0
95	0	0	0	0	0	0	0	0	0
96	0.05229	0	0.05229	0	0	0	0	0	0
97	0.05395	0	0.05395	0.02783	0.01066	0.009	0	0	0
98	0.74293	0	0.74293	0	0	0	0	0	0
99	0	0	0	0	0	0	0	0	0
100	0.13083	0	0.13083	0.00503	0	0	0	0	0
101	0.30122	0	0.30122	0	0	0	0	0	0
102	0	0	0	0	0	0	0	0	0
103	0	0	0	0	0	0	0	0	0
104	0	0	0	0	0	0	0	0	0
105	0.43983	0	0.43983	0	0	0	0	0	0
106	0.20599	0	0.20599	0	0	0	0	0	0
107	0.00137	0	0.00137	0	0	0	0	0	0
108	0.69812	0	0.69812	0.31127	0.17893	0	0	0	0
109	0	0	0	0	0	0	0	0	0
110	0	0	0	0	0	0	0	0	0
111	0	0	0	0	0	0	0	0	0
112	0.43043	0	0.43043	0.30188	0.15191	0.00995	0	0	0
113	0	0	0	0	0	0	0	0	0
114	0	0	0	0	0	0	0	0	0
115	0	0	0	0	0	0	0	0	0
116	0	0	0	0	0	0	0	0	0
117	0	0	0	0	0	0	0	0	0
118	0	0	0	0	0	0	0	0	0
119	0	0	0	0	0	0	0	0	0
120	0	0	0	0	0	0	0	0	0
121	0	0	0	0	0	0	0	0	0
122	0	0	0	0	0	0	0	0	0

Note: a/ Assuming equally spaced intervals, we let $r\,\sigma_j = M_j/k$, where T_{rv}^* is defined using $B_r(z, \cdot) = \{z': -M_j/k \le z_j - z_j' \le M_j/k;\ j = 1, \ldots, m;\ z' \in R^m\}$ as neighborhood of z, and k denotes the number of intervals within the data range.

Footnotes

[1] The technology T is convex if, for any z and z' $\in$ T, then $(\theta z + (1-\theta) z') \in T$ for any scalar $\theta \in [0, 1]$.

[2] For example, when p = 2, this corresponds to the Euclidean distance: $D_2(z, z') \equiv \sum_{j=1}^{m} [(|z_j - z_j'|/\sigma_j)^2]^{1/2}$.

And when $p \to \infty$, this corresponds to the Chebyshev distance: $\lim_{p\to\infty} D_p(z, z') = \text{Max}_j \{|z_j - z_j'|/\sigma_j: j = 1, \ldots, m\}$.

[3] The choice and evaluation of the neighborhood $B_r(z, \sigma)$ will be further discussed in section 4.2 below.

[4] The directional distance function D(z, T) in (11) is the negative of Luenberger's shortage function (see Luenberger, 1995).

[5] Note that D(z, T) includes as special cases many measures of technical inefficiency that have appeared in the literature. Relationships with Shephard's distance functions (Shephard, 1953) or Farrell's measure of technical efficiency (Farrell, 1957) are discussed in Chambers et al. (1996) and Färe and Grosskopf (2000).

[6] Since dealing with non-linear constraints can be empirically challenging, note that alternative formulations have been proposed avoiding non-linear constraints in productivity analysis (e.g., Podinosvki, 2004; Leleu,, 2006; Soleimani-Damneh and Reshadi, 2007; De Witte and Marques, 2011).

[7] An alternative way to choose the intervals would be to rely on the empirical distribution of netputs. In this context, one option would be to choose the intervals such that, for each netput, each interval includes the same number of sample observations.

[8] Note that 1,000 won (the Korean currency) = 0.89 US dollars.

[9] Farm size is measured by the total amount of land (in ha). Small farms are defined as farms being in the 0 to 30 percentile of the sample distribution of farm size, medium farms are between the 30 percentile and 70 percentile, and large farms are in the 70 to 100 percentile. The average farm size of small, medium and large farms are 0.574 ha, 1.624 ha, and 5.965 ha, respectively.