MONOTONICITY AND CURVATURE – A BOOTSTRAPPING APPROACH

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Abstract

This research contributes to the ongoing discussion on functional flexibility and theoretical consistency by comparing the empirical performance of two second order flexible functional forms - the Symmetric Generalized McFadden and the Transcendental Logarithmic. It proposes an estimation procedure to enhance the domain of applicability for the Translog by a combination of matrix decomposition, classical non-linear estimation techniques as well as bootstrapping based resampling. The validity of the proposed procedure is exemplified by applying it to a sample of small-scale farmers. The results show that the range of theoretical consistency can be crucially enhanced for the Translog functional form by maintaining its flexibility and statistical significance. Hence, beside its empirical superiority by applying the outlined procedure the Translog can also catch up with respect to the range of functional consistency.

Keywords

Econometric Modeling, Flexible Functional Forms, Theoretical Consistency, Bootstrapping

1 Introduction[1]

As is well known in applied production economics flexible functional forms are considered as superior to model an empirical relationship. According to Diewert (1974) a functional form can be denoted as 'flexible' if its shape is only restricted by theoretical consistency. This implies the absence of unwanted a priori restrictions and is paraphrased by the metaphor of "providing an exhaustive characterization of all (economically) relevant aspects of a technology" (see Fuss et al. 1978). However, for most functional forms there is a fundamental trade-off between flexibility and theoretical consistency as well as the domain of applicability. Following the classical econometric tradition this contribution proposes an estimation procedure to enhance the consistent domain of applicability for a second order flexible functional form by combining matrix decomposition, non-linear estimation techniques as well as bootstrapping based resampling. The validity of the econometric procedure is exemplified by using a curvature constrained estimation of the widely

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applied Transcendental Logarithmic functional form in order to enhance its theoretical consistency by maintaing its superior empirical applicability.

2 The Problem

The functional form of an econometric model as well as the specified probability distribution for the residual are the two major assumptions underlying the empirical investigation of economic hypotheses and are commonly considered as maintained hypotheses of the model. In production economics one basic question to be solved by econometric modeling is the one with respect to an adequate representation of the underlying technology T.

Proposition I: The technology $T = \{(y, x) : x \text{ can produce } y\}$ describes the set of feasible input-output vectors with $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_m)$ respectively. T satisfies the usual properties of a theoretically well-defined production technology. ▪

Proposition II: The technology approximation $T' = \{(y, x, \hat{\beta}) : \hat{\beta}x \text{ can produce } y\}$ approximates the set of feasible input-output vectors with $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_m)$ and $\hat{\beta} = (\beta_1, \beta_2, ..., \beta_n)$ respectively. The approximation T' satisfies the usual properties of a theoretically well-defined production technology and $\hat{\beta}$ satisfies the usual statistical properties of a well-defined estimator. ▪

Economic theory provides no a priori guidance with respect to the functional relationship. Lau's criteria (Lau, 1978, 1986) with respect to the ex ante selection of an algebraic form are valuable for applied modelling but conclude in the magic triangle of functional choice: the researcher should not expect to find a functional form equally satisfying the principles of theoretical consistency, functional flexibility as well as an accurat domain of statistical applicability. The literature on econometric modelling proposes two solutions to this severe problem (Chambers 1988, Lau 1986): (1) to apply functional forms which could be made globally theoretical consistent by corresponding parameter restrictions, here the range of flexibility has to be investigated, or, (2) to opt for functional flexibility and check or impose theoretical consistency for the proximity of an approximation point - usually at the sample mean - only. A globally theoretical consistent as well as flexible functional form can be considered as an adequate representation of the production possibility set. Locally theoretical consistent as well as flexible functional forms can be considered as an i-th order differential approximation of the true production possibilities.

Proposition III: A globally flexible and theoretically consistent constrained technology approximation $T_g' = \left\{ (y, x, \hat{\beta}) : \hat{\beta}x \text{ can produce } y;\ T_g' \left| \frac{dy}{dx_i} > 0 \wedge \mathbf{H}_{T_g'} = nsd \text{ where i = inputs and } \mathbf{H}_{T_g'} \text{ as the global Hessian} \right. \right\}$ globally approximates the set of feasible input-output vectors with $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_m)$ and $\hat{\beta} = (\beta_1, \beta_2, ..., \beta_n)$ respectively. The approximation T_g' globally satisfies the usual properties of a theoretically well-defined production technology. ▪

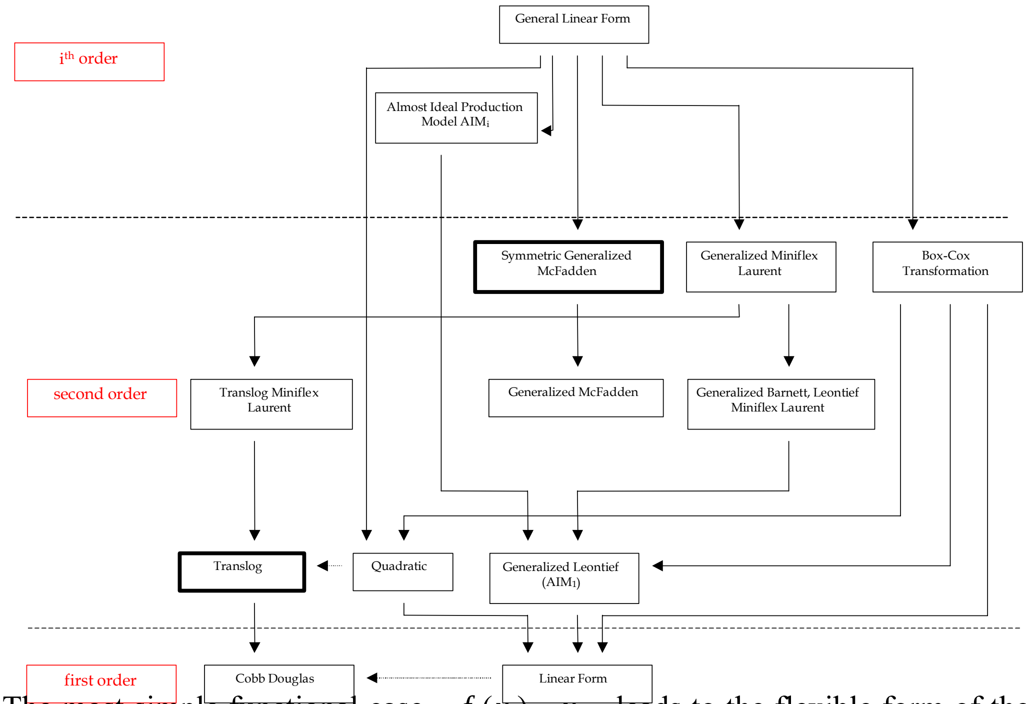
Proposition IV: A locally flexible and theoretically consistent constrained technology approximation

$$T_l' = \left\{ \left(y, x, \hat{\beta}\right) : \hat{\beta}x \text{ can produce } y;\ T_l' \left| \frac{dy}{dx_i} > 0 \wedge \mathbf{H}_k = nsd \right. \text{ for at least observation k} = 1 \text{ where } \mathbf{H}_k \text{ as the local Hessian} \right\}$$

locally approximates the set of feasible input-output vectors with $x = \left(x_1, x_2, ..., x_n\right)$, $y = \left(y_1, y_2, ..., y_m\right)$ and $\hat{\beta} = \left(\beta_1, \beta_2, ..., \beta_n\right)$ respectively. The approximation T_l' locally satisfies the usual properties of a theoretically well-defined production technology and $\hat{\beta}$ satisfies the usual statistical properties of a well-defined estimator. ■

Figure 1 gives a brief overview of the most common flexible functional forms selected with respect to the frequency of empirical usage or the representation of systematic nodes in the development of functional representation.

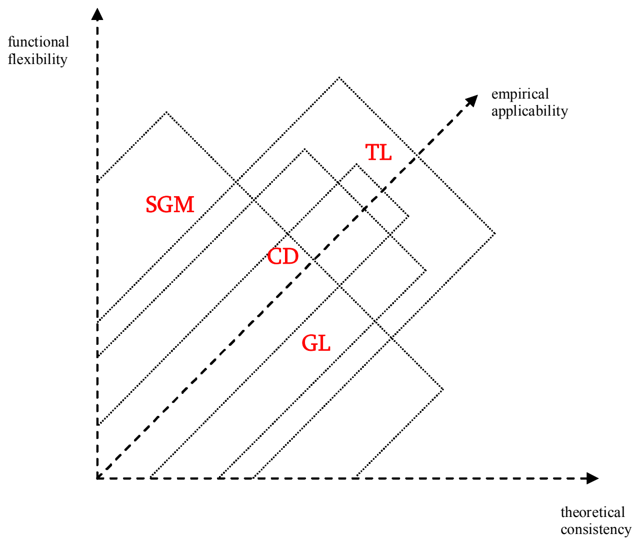
Figure 1: Flexible Functional Forms



(see Morey 1996, Feger 2000)

The most simple functional case - $f_i(x_i) = x_i$ - leads to the flexible form of the Quadratic, whereas the Transcendental Logarithmic (Translog) - $f_i(x_i) = \ln x_i$ - is the historically first invented flexible functional form incorporating the first order case of the Cobb Douglas (CD). Another early invented second order flexible functional form, the Generalized Leontief (GL), is based on $f_{ij}(x_i x_j) = x_i^{1/2} x_j^{1/2}$ with respect to the second order effects. The introduction of the Symmetric Generalized McFadden (SGM) in the mid 80's - following $f_{ijk}(x_i x_j x_k) = \varphi_{ij}\left(x_i x_j / \sum_k v_k x_k \right)$ for the second order effects - marks another milestone in the search for global flexibility. Figure 2 illustrates the different strengths and weaknesses of these functional forms with respect to the magic triangle of functional choice.

Figure 2: Strengths and Weaknesses of Different Functional Forms



Following the Bayesian econometric tradition Terrell (1996) proposes the use of a Gibbs sampler to generate an initial sample from the posterior density for a prior ignoring regularity restrictions. By accept-reject sampling a final sample is then generated which consists only of parameter values adhering to these regularity conditions. Different extensions of this estimation method have been subsequently made (O'Donnel at al. 2003, Griffiths et al. 2000, and Wolff et al. 2006). The following discussion contrasts the SGM as the 'state-of-the-art' with respect to theoretical consistency and the TL as probably the 'best empirical performer' as numerous applied studies show.

3 – Theoretical Consistency: The Symmetric Generalized McFadden

The SGM was introduced by Diewert and Wales in 1987 based on the initial formulation by McFadden (see Diewert/Wales 1987). As the functional form of the Generalized Leontief, the SGM is linearly homogeneous in inputs by construction. Monotonicity - $\left(df_i(x_i)/dx_i\right) > 0$ - can be either imposed locally only, if globally restricted for monotonicity the property of second order flexibility is lost. The crucial feature of the SGM providing the reason for its common distinction as state of the art is the fact that if globally restricted for correct curvature by matrix decomposition the constrained curvature property applies globally. In the case of a production function this means investigating $d^2 f_i(x_i)/dx_i^2$ and $d^2 f_i(x_i x_j)/dx_i dx_j$ to assure that the estimated function is quasi-concave resulting in a negative semi-definite bordered Hessian and consequently alternating determinants of its submatrices D starting with a negative one: $-1^k D_k \geq 0$. However, one has to be aware that in this case the second order flexibility is restricted to only one point (see Feger 2000, Ryan/Mah 1994, Diewert/Wales 1987). A SGM production function can be formulated as follows

$$y = \sum_{i=1}^{n} \beta_i x_i + \frac{1}{2}\left(\sum_{i=1}^{n} \theta_i x_i\right)^{-1} \sum\nolimits_{i=1}^{n} \sum\nolimits_{j=1}^{n} \varphi_{ij}\left(x_i x_j\right) \qquad [1]$$

where as usual x_i and x_j denote inputs, y is the output and β_i, θ_i and φ_{ij} are the parameters to be estimated. By applying either Lau's technique (Lau 1978) based on the Cholesky factorization **H** = **-LBL**' (where **L** is a unit lower triangular matrix and **B** as a diagonal matrix), or the matrix decomposition following Wiley et al. (1973) **H = -ΔΔ'** (where **H** is replaced by the negative product of a lower triangular matrix times its transpose), the bordered Hessian can be constrained to a negative semi-definite matrix assuring quasi-concavity of the estimated production function.

Proposition V: A globally flexible and constrained technology approximation T_g' of the type $T_{sgm}' = \left\{ \left(y, x, \hat{\beta}\right) : \hat{\beta}x \text{ can produce y; } y = \sum_{i=1}^{n} \beta_i x_i + \frac{1}{2}\left(\sum_{i=1}^{n} \theta_i x_i\right)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \varphi_{ij}\left(x_i x_j\right);\ T_{sgm}' \left| \mathbf{H}_{T_g'} = -\mathbf{A} * \mathbf{A}^{\mathrm{T}} \wedge \frac{dy_k}{dx_k} > 0 \text{ for at least k=1} \right. \right\}$ globally approximates the set of feasible input-output vectors with $x = \left(x_1, x_2, ..., x_n\right)$, $y = \left(y_1, y_2, ..., y_m\right)$ and $\hat{\beta} = \left(\beta_1, \beta_2, ..., \beta_n\right)$ respectively. The approximation T_g' globally satisfies the property of a quasi-concave production technology. ▪

To exemplify the described functional properties, the followig SGM production funtion was applied on an arbitrary chosen real world cross-sectional sample of 252 small-scale farmers producing maize by using the inputs s = seed, l = labour and f = fertilizer:

$$y = \beta_s x_s + \beta_l x_l + \beta_f x_f + \frac{1}{2} \frac{\varphi_{ss} x_s^{\ 2} + \varphi_{sl}\left(x_s x_l\right) + \varphi_{sf}\left(x_s x_f\right) + \varphi_{ll} x_l^{\ 2} + \varphi_{lf}\left(x_l x_f\right) + \varphi_{ff} x_f^{\ 2}}{\theta_s x_s + \theta_l x_l + \theta_f x_f} \qquad [2]$$

where the parameter θ_i was set equal to the respective sample mean and each variable has been normalized by its mean (see Diewert/Wales 1987). In a second step the same function was applied in a curvature constrained specification following the technique by Wiley et al.:

$$y = \beta_s x_s + \beta_l x_l + \beta_f x_f +$$

$$\frac{1}{2}\left[\begin{array}{l} (-\Delta_{ss}\Delta_{ss}) x_s^{\ 2} + (-\Delta_{ss}\Delta_{sl})\left(x_s x_l\right) + (-\Delta_{ss}\Delta_{sf})\left(x_s x_f\right) + (-\Delta_{sl}\Delta_{sl} - \Delta_{ll}\Delta_{ll}) x_l^{\ 2} \\ +(-\Delta_{sl}\Delta_{sf} - \Delta_{ll}\Delta_{lf})\left(x_l x_f\right) + (-\Delta_{sf}\Delta_{sf} - \Delta_{lf}\Delta_{lf} - \Delta_{ff}\Delta_{ff}) x_f^{\ 2} \end{array}\right] / \left(\theta_s x_s + \theta_l x_l + \theta_f x_f\right) \quad [3]$$

where again θ_i was set equal to the respective sample mean and each variable has been normalized by its mean. The parameters Δ_{ii} and Δ_{ij} refer to the lower triangular matrix and its transpose respectively with i,j = seed, labour, and fertilizer. Table 1 and 2 summarize the estimation results:

Table 1: Unconstrained SGM

Parameter	Estimate [t-statistics]	Parameter	Estimate [t-statistics]	Parameter	Estimate [t-statistics]
s_{ss}	-0.805 [-3.387]***	s_{sl}	2.972 [34.531]***	s_{sf}	0.997 [3.129]***
s_{ll}	-1.527 [-3.654]***	s_{lf}	0.824 [1.686]*	s_{ff}	0.197 [0.436]
β_s	0.474 [1.182]	β_l	0.315 [1.312]	β_f	-0.187 [-0.817]
adjR^2	0.76	F-value	25.27		
QC (%)	37.31	M (%)	0		

(1) s-seed, l-labour, f-fertilizer; (2) *, **, ***: significance at 10-, 5- or 1%-level; t-values in parentheses; (3) the parameters in the top two rows refer to the Hessian; (4) symmetry - ($s_{ij} = s_{ji}$); (5) QC – quasi-concavity, M – monotonicity.

Table 2: Constrained SGM

Parameter	Estimate [t-statistics]	Parameter	Estimate [t-statistics]	Parameter	Estimate [t-statistics]
s_{ss}	-1.921 [-8.351]***	s_{sl}	1.441 [5.226]***	s_{sf}	0.169 [0.531]
s_{ll}	-1.112 [-2.396]**	s_{lf}	-0.127 [-0.259]	s_{ff}	-0.015 [-0.034]
β_s	1.013 [2.339]**	β_l	0.329 [0.817]	β_f	-0.033 [-0.086]
adjR^2	0.63	F-value	25.05		
QC (%)	100	M (%)	40.36		

(1) s-seed, l-labour, f-fertilizer, (2) *, **, ***: significance at 10-, 5- or 1%-level; t-values in parentheses; (3) the parameters in the top two rows refer to the Hessian; (4) symmetry - ($s_{ij} = s_{ji}$), concavity is imposed globally by constraining S to be nsd by $S = -A*A^T$, monotonicity is imposed at the sample mean; (5) QC – quasi-concavity, M – monotonicity.

The overall model fit of the unconstrained as well as constrained specification seem to be in an acceptable range for cross-sectional data. In the unconstrained specification about 55% of all estimated parameters showed to be significant at least at the 10%-level, in the constrained specification this ratio falls to about 40%. The estimated unconstrained SGM function showed to be quasi-concave for about 37% of all observations but for none of the observations monoton in all inputs. The estimated constrained SGM function showed to be globally quasi-concave as expected and monoton in all inputs for about 40% of all observations. Hence, our exemplary empirical application confirmed our previously made theoretical arguments: the functional form of the symmetric generalized McFadden is highly consistent in its constrained specification but fails to show satisfactorily empirical applicability by a relatively modest statistical significance of the model and the individual parameters estimated.

4 – Empirical Applicability: The Transcendental Logarithmic

The locally flexible functional form following the Generalized Leontief is the Transcendental Logarithmic or Translog (see Christensen et al. 1973). Due to the literature the Translog appears as probably the best investigated second order flexible functional form and surely the one with the most empirical applications as its empirical applicability in terms of statistical significance is outstanding (Feger 2000). A Translog production function can be formulated as follows

$$y = \beta_0 + \sum_{i=1}^{n} \beta_i \ln x_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \ln x_i x_j \qquad [4]$$

where as usual x_i and x_j denote inputs, y is the output and β_i and γ_{ij} are the parameters to be estimated. Locally theoretical consistent as well as flexible functional forms can be considered as an i-th order differential approximation of the true production possibilities. Hence, the popular Translog is considered as a second order differential approximation of the true production possibilities. The theoretical properties of the second order Translog are well known (Lau 1986): it is easily restrictable for global homogeneity as well as homotheticity, correct curvature can be implemented only locally if local flexibility should be preserved, the maintaining of global monotonicity is impossible without losing second order flexibility. Hence, the Translog functional form is fraught with the problem that theoretical consistency can not be imposed globally. Ryan and Wales (2000) argue that a sophisticated choice of the reference point could lead to satisfaction of consistency at most or even all data points in the sample. Jorgenson and Fraumeni (1981) firstly propose the imposition of quasi-concavity through restricting the Hessian to be a negative semidefinite matrix. However, as in the case of the Generalized Leontief, the Hessian of the Translog is not structured in a way that the definiteness property is invariant towards changes in the exogenous variables. Following Jorgenson and Fraumeni (1981) quasi-concavity can be imposed at a reference point (usually at the sample mean) by replacing the bordered Hessian by the negative product of a lower triangular matrix **Δ** times its transpose **Δ'** according to the decomposition proposed by Wiley et al. (1973). Imposing curvature at the sample mean is then attained by setting

$$\gamma_{ij} = -(\Delta\Delta')_{ij} + \beta_i \lambda_{ij} + \beta_i \beta_j \qquad [5]$$

where i, j = 1, …, n, $\lambda_{ij} = 1$ if i = j and 0 otherwise and $(\Delta\Delta')_{ij}$ as the ij-th element of **ΔΔ'** with **Δ** a lower triangular matrix. As our point of approximation is the sample mean all data points are divided by their mean transferring the approximation point to an (n + 1)-dimensional vector of ones. At this point the elements of **H** do not depend on the specific input bundle.

Proposition VI: A locally flexible and constrained technology approximation T_l' of the type $T_{tl}' = \left\{ \left(y, x, \hat{\beta}\right) : \hat{\beta}x \text{ can produce } y;\ y = \beta_0 + \sum_{i=1}^{n} \beta_i \ln x_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij} \ln x_i x_j;\ T_{tl}' \left| \frac{dy}{dx_i} > 0 \wedge \mathbf{H}_k = nsd \text{ for at least observation } k = 1 \right. \right\}$ locally approximates the set of feasible input-output vectors with $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_m)$ and $\hat{\beta} = (\beta_1, \beta_2, ..., \beta_n)$ respectively. The approximation T_l' locally satisfies the usual properties of a theoretically well-defined production technology and $\hat{\beta}$ satisfies the usual statistical properties of a well-defined estimator. ▪

To exemplify the described functional properties, the followig Translog production funtion was applied on the same cross-sectional sample of 252 small-scale farmers producing maize by using the inputs s = seed, l = labour and f = fertilizer:

$$\ln y = \beta_0 + \beta_s \ln x_s + \beta_l \ln x_l + \beta_f \ln x_f + \frac{1}{2}\ln x_s^2 + \frac{1}{2}\ln x_l^2 + \frac{1}{2}\ln x_3^2 + \frac{1}{2}\ln x_s \ln x_l + \frac{1}{2}\ln x_s \ln x_f + \frac{1}{2}\ln x_l \ln x_f \qquad [6]$$

where each variable has been normalized by its mean. In a second step the same function was applied in a curvature constrained specification following the technique illustrated above:

$$\ln y = \beta_0 + \beta_s \ln x_s + \beta_l \ln x_l + \beta_f \ln x_f + \frac{1}{2}\left(-\gamma_{ss}\gamma_{ss} + \beta_s - \beta_s\beta_s\right)\ln x_s^2 + \frac{1}{2}\left(-\gamma_{sl}\gamma_{sl} - \gamma_{ll}\gamma_{ll} + \beta_l - \beta_l\beta_l\right)\ln x_l^2$$
$$+\frac{1}{2}\left(-\gamma_{sf}\gamma_{sf} - \gamma_{lf}\gamma_{lf} - \gamma_{ff}\gamma_{ff} + \beta_f - \beta_f\beta_f\right)\ln x_3^2 + \frac{1}{2}\left(-\gamma_{sl}\gamma_{ss} - \beta_s\beta_l\right)\ln x_s \ln x_l + \frac{1}{2}\left(-\gamma_{sf}\gamma_{ss} - \beta_s\beta_f\right)\ln x_s \ln x_f \qquad [7]$$
$$+\frac{1}{2}\left(-\gamma_{sf}\gamma_{sl} - \gamma_{lf}\gamma_{ll} - \beta_l\beta_f\right)\ln x_l \ln x_f$$

where again each variable has been normalized by its mean. The resulting normalized translog model in [7] is nonlinear in parameters and consequently linear estimation algorithms are ruled out even if the original function is linear in parameters. By this "local" procedure a satisfaction of consistency at most or even all data points in the sample can be reached. The transformation in [5] moves the observations towards the approximation point and thus increases the likelihood of getting theoretically consistent results at least for a range of observations (Ryan/Wales 2000). However, by imposing global consistency on the translog functional form Diewert and Wales (1987) note that the parameter matrix is restricted leading to seriously biased elasticity estimates. Hence, the translog function would lose its flexibility. Table 3 and 4 summarize the estimation results for the Translog:

Table 3: Unconstrained Translog

Parameter	Estimate [t-statistics]	Parameter	Estimate [t-statistics]	Parameter	Estimate [t-statistics]
$ß_0$	-0.784 [-8.837]***	γ_{ss}	0.020 [58.429]***	γ_{sf}	-0.053 [-4.023]***
$ß_s$	0.543 [59.022]***	γ_{ll}	0.957 [1.653]*	γ_{lf}	0.910 [2.318]**
$ß_l$	0.472 [1.619]*	γ_{ff}	0.657 [6.048]***		
$ß_f$	0.238 [1.605]*	γ_{sl}	-0.079 [-2.887]***		
adjR^2	0.93	F-value	59.07		
QC (%)	22.2	M (%)	70.2		

(1) s-seed, l-labour, f-fertilizer; (2) *, **, ***: significance at 10-, 5- or 1%-level; t-values in parentheses;
(3) QC – quasi-concavity, M – monotonicity.

Table 4: Constrained Translog

Parameter	Estimate [t-statistics]	Parameter	Estimate [t-statistics]	Parameter	Estimate [t-statistics]
$ß_0$	-1.217 [-36.249]***	γ_{ss}	0.904 [0.019]	γ_{sf}	0.015 [13.589]***
$ß_s$	1.428 [4.029]***	γ_{ll}	0.007 [0.158]	γ_{lf}	0.003 [0.715]
$ß_l$	0.108 [1.742]*	γ_{ff}	0.014 [98.492]***		
$ß_f$	0.428 [110.929]***	γ_{sl}	0.003 [0.204]		
adjR^2	0.69	F-value	335.58		
QC (%)	86.9	M (%)	86.7		

(1) s-seed, l-labour, f-fertilizer; (2) *, **, ***: significance at 10-, 5- or 1%-level; t-values in parentheses;
(3) QC – quasi-concavity, M – monotonicity.

As expected, the overall model fit of the unconstrained specification is high whereas the fit of the constrained model seems to be in an acceptable range for cross-sectional data. In the unconstrained specification all estimated parameters showed to be significant at least at the 10%-level, in the constrained specification this ratio falls to about 60%. The estimated unconstrained TL function showed to be quasi-concave for only about 22% of all observations but for 70% of the observations monoton in all inputs. The estimated constrained TL function showed to be quasi-concave and monoton in all inputs for about 87% of all observations. Hence, our exemplary empirical application confirmed our previously made theoretical arguments: the unconstrained functional form of the Transcendental Logarithmic is applicable at a high range in its unconstrained specification but fails to show satisfactorily theoretical consistency of the estimated model. By constraining the TL functional form the theoretical consistency of the estimated model can be increased significantly but still fails for more than 10% of all observations. So far, the econometric techniques applied as well as the results with respect to the performance of the functional forms are in line with common practices and expectations. The next section introduces an econometric procedure to enhance the range of theoretical consistency of the Translog functional form by maintaining its superiority with respect to the range of empirical applicability.

5 – Econometric Modeling and Results: Nested Intervals by Resampling

A translog production function model is developed following [6] where the functional form is normalized by the means of the respective variables. After a first estimation the consistency of the estimated production function is tested by checking the first derivatives (monotonicity) as well as the eigenvalues of the Hessian matrix (quasi-concavity). Subsequently correct curvature is imposed locally following Wiley et al. (1973) and Ryan and Wales (1998) and the range of theoretical consistency is again investigated for the estimated function. In a next step bootstrapping techniques are applied to reveal the confidence intervals for the estimated parameters of the function. Based on these bias-corrected statistics, decile intervals for the individual parameter values are defined. A sequence of restricted estimations is then performed for each parameter combination according to these parameter decile intervals and the most appropriate combination(s) of different parameter ranges are determined in terms of the theoretically consistent range of the estimated function. The proposed procedure is exemplified by using again the cross-sectional data set on small-scale farmers.

Step1: estimation of an unconstrained model, and step 2: estimation of a curvature constraint model have been already documented by tables 3 and 4 in the preceeding section. Step 3 involves the application of a simple bootstrapped estimation of the constrained model. Comprehensively described in the literature (Efron 1979 or Efron/Tibshirani 1993) the bootstrapping technique delivers confidence intervals for the individual parameter estimates. If we suppose that $\hat{\psi}_n$ is an estimator of the parameter vector ψ_n including all parameters obtained by estimating [7] based on our original sample of 252 farmers $X = (x_1, ..., x_n)$, then we are able to approximate the statistical properties of $\hat{\psi}_n$ by studying a sample of 1000 bootstrap estimators $\hat{\psi}_n(c)_m, c = 1, ..., C$. These are obtained by resampling our 252 observations – with replacement – from X and recomputing $\hat{\psi}_n$ by using each generated sample. Finally the sampling characteristics of our vector of parameters is obtained from

$$\hat{\Psi} = \left[\hat{\psi}_{(1)m},\ldots,\hat{\psi}_{(100)m}\right] \qquad [8]$$

Table 5 summarizes the bias-corrected bootstrapped confidence intervals for the constrained TL parameters:

Table 5: Bias-Corrected Bootstrapped Confidence Intervals

Parameter	95%-Confidence Interval	Parameter	95%-Confidence Interval
β_0	[-1.218; -0.214]	γll	[-0.061; 0.024]
β_s	[1.223; 1.776]	γff	[0.009; 0.016]
β_l	[0.061; 0.185]	γsl	[0.029; 0.446]
β_f	[0.278; 0.449]	γsf	[-0.013; 0.023]
γss	[0.688; 1.651]	γlf	[-0.009; 0.007]

Table 6 gives the means of the bias-corrected parameter ranges (deciles) based on the bootstrap estimates. Alternatively any other sub-division of the parameter ranges could be applied (e.g. quantilies, quartiles etc.):

Table 6: Means of the Bias-Corrected Nested Parameter Intervals

Decile Parameter	1	2	3	4	5	6	7	8	9	10
β_0	-1.168	-1.067	-0.967	-0.866	-0.766	-0.666	-0.565	-0.465	-0.364	-0.264
β_s	1.251	1.306	1.362	1.417	1.472	1.527	1.583	1.638	1.693	1.749
β_l	0.067	0.079	0.092	0.104	0.117	0.129	0.142	0.154	0.167	0.179
β_f	0.285	0.303	0.320	0.337	0.355	0.372	0.389	0.407	0.424	0.441
γss	0.736	0.832	0.929	1.025	1.121	1.218	1.314	1.410	1.507	1.603
γll	-0.057	-0.048	-0.040	-0.031	-0.023	-0.014	-0.006	0.003	0.011	0.020
γff	0.010	0.010	0.011	0.012	0.012	0.013	0.013	0.014	0.015	0.015
γsl	0.050	0.092	0.133	0.175	0.217	0.259	0.300	0.342	0.384	0.426
γsf	-0.011	-0.008	-0.004	0.000	0.003	0.007	0.011	0.014	0.018	0.022
γlf	-0.009	-0.007	-0.005	-0.004	-0.002	-0.001	0.001	0.003	0.004	0.006

Step 4: using these parameter deciles a sequence of restricted estimations is then performed based on different combinations of parameter ranges. By this procedure the parameter confidence intervals are 'searched' for the crucial values for which the overall functional consistency fails. According to this trial-and-error procedure - comparable to the use of nested intervals – the most appropriate combination(s) of different parameter ranges are determined in terms of the theoretically consistent range of the estimated TL function. By applying this interval procedure on our empirical case study the cross parameter γ_{sf} was detected as most crucial for functional consistency (intervals 1-9). In *step 5* the constrained TL model in [7] is now re-estimated by restricting γ_{sf} to the crucial range following the previously defined intervals. Table A1 summarizes the constrained regression results (see appendix). The nested intervals following parameter search resulted in the parameter region defined by the deciles 1 to 8 for γ_{sf} as the region implying the

highest functional consistency. This econometric procedure could be also applied by using programmed macros in statistical software. Finally in *step 6* the constrained TL model is specified and estimated by restricting the crucial parameter γ_{sf} to the found nested interval, hence, for our example $\gamma_{sf} = [-0.009; 0.016]$. However, analogue to the 2SLS estimation procedure the standard errors for the constrained regressions of the second stage have to be adjusted as the final error term u^*_i is not exactly equal to the variance of the original u_i. This can be simply done by multiplying each standard error of the coefficients estimated in the second stage with the correction factor $\hat{\sigma}_{u*} / \hat{\sigma}_u$ (see e.g. Gujarati, 2003, pp. 773). Table 7 summarizes the final TL model:[2]

Table 7: Constrained Translog by Nested Intervals

Parameter	Estimate [t-statistics]	Parameter	Estimate [t-statistics]	Parameter	Estimate [t-statistics]
$ß_0$	-1.349 [-273.246]***	γ_{ss}	0.005 [0.142]	γ_{sf}	0.016 [10.560]***
$ß_s$	0.493 [30.374]***	γ_{ll}	0.007 [1.192]	γ_{lf}	0.004 [5.389]***
$ß_l$	0.108 [13.129]***	γ_{ff}	0.014 [742.377]***		
$ß_f$	0.428 [836.115]***	γ_{sl}	-0.034 [-1.541]*		
nested parameter restriction: $\gamma_{sf} = [-0.009; 0.016]$					
adjR^2	0.93	F-value	336.01		
QC (%)	92.1	M (%)	98.8		

(1) s-seed, l-labour, f-fertilizer; (2) *, **, ***: significance at 10-, 5- or 1%-level; t-values in parentheses;
(3) QC – quasi-concavity, M – monotonicity; (4) corrected standard errors.

As becomes evident, by this estimation procedure the theoretical consistency of the Transcendental Logarithmic can be crucially enhanced by mainting its statistical superiority and consequently its high range of empirical applicability. Table 8 documents this by comparing the usually constrained as well as the nested interval constrained TL production functions:

Table 8: Constrained TL Comparison

	TL usually constrained	TL nested interval constrained	relative improvement (%)
adj R2	0.69	0.93	34.78
parameter significance (%)	60	70	10
monotonicity	86.7	98.8	13.69
quasi-concavity (%)	86.9	92.1	5.97
regularity	86.9	92.1	5.97

The range of theoretical consistency is enhanced by up to 14% (monotonicity), the overall statistical significance could be even improved by up to 35% for the model. The functional regularity (i.e. monotonicity, diminishing marginal returns and quasi-concavity) increased by up to 6%.

[2] For the chosen example the estimate for the restricted parameter γ_{sf} is on the upper boundary of the defined regular parameter space. Andrews (1999, 2000) discusses different methods to adjust the standard error for the parameter in question with respect to this rather complex case which we do not follow here for the sake of clarity of argumentation.

Corollary: a locally flexible and constrained technology approximation T_l' of the type

$$T_{tl\_nest}' = \left\{ \begin{array}{l} \left(y, x, \hat{\beta}\right): \hat{\beta}x \text{ can produce } y;\; y = \beta_0 + \sum_{i=1}^{n} \beta_i \ln x_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij} \ln x_i x_j;\; T_{tl}' \left| \frac{dy}{dx_i} > 0 \wedge \mathbf{H}_k = nsd \text{ for at least observation k} = 1; \right. \\ \beta \in \left[\beta_1; \beta_2\right] \text{for at least one parameter} \end{array} \right\}$$

locally approximates the set of feasible input-output vectors with $x = \left(x_1, x_2, ..., x_n\right)$, $y = \left(y_1, y_2, ..., y_m\right)$ and $\hat{\beta} = \left(\beta_1, \beta_2, ..., \beta_n\right)$ respectively. The approximation T_l' locally satisfies the usual properties of a theoretically well-defined production technology for a high range of observations and $\hat{\beta}$ satisfies the usual statistical properties of a well-defined estimator. ▪

The estimation results show that the proposed method leads to a significantly enlarged regularity range for the translog functional form. As a global imposition of functional regularity implies the loss of functional flexibility for the translog form, the outlined procedure based on classical econometric methods proved to be an alternative technique to such based on Bayesian econometrics.

7 – Conclusions

This paper proposes a new procedure for the curvature constrained estimation of the widely used Transcendental Logarithmic functional form in order to enhance its theoretical consistency by maintaing its superior empirical applicability. By using an applied example the performance of the TL is compared to the Symmetric Generalized McFadden as the reference for a global curvature consistent functional form. As expected, whereas the TL shows the better empirical performance it scores relatively poor on the functional range of theoretical consistency. The opposite was found for the SGM. By performing a nested interval search on the crucial parameter(s), restricting the latter to a range of values showing the highest range of consistency and estimating the functional form by using the usual Hessian decomposition technique, the theoretical consistency of the TL could be crucially enhanced by maintaining its statistical significance and avoiding a loss of functional flexibility. The empirical results show that the applied estimation procedure – the combination of matrix decomposition, restricted non-linear estimation and nested parameter intervals based on stochastic resampling – can critically contribute to increase the theoretical adherence of a second order flexible model without having to rely on using Bayesian econometric techniques.

8 – References

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9 – Appendix

Table A1: Functional Consistency and Empirical Applicability per Parameter Interval[1]

γ_{sf} - Deciles	1-9	1-8	1-7	1-6	1-5	1-4	1-3	1-2
adjR2	0.92	0.92	0.93	0.93	0.94	0.95	0.95	0.96
M (%)	98.81	98.81	98.81	98.81	98.81	98.81	98.81	98.81
QC (%)	90.08	92.06	92.06	91.67	91.27	91.27	89.68	89.68
R (%)	90.08	92.06	92.06	91.67	91.27	91.27	89.68	89.68

γ_{sf} - Deciles	2-9	3-9	4-9	5-9	6-9	7-9	8-9
adjR2	0.92	0.92	0.92	0.92	0.92	0.92	0.92
M (%)	98.81	98.81	98.81	98.81	98.81	98.81	98.81
QC (%)	90.08	90.08	90.08	90.08	90.08	90.08	90.08
R (%)	90.08	90.08	90.08	90.08	90.08	90.08	90.08

γ_{sf} - Deciles	2-8	3-8	4-8	5-8	6-8	7-8
adjR2	0.92	0.92	0.92	0.92	0.92	0.92
M (%)	98.81	98.81	98.81	98.81	98.81	98.81
QC (%)	92.06	92.06	92.06	92.06	92.06	92.06
R (%)	92.06	92.06	92.06	92.06	92.06	92.06

γ_{sf} - Deciles	2-7	3-7	4-7	5-7	6-7
adjR2	0.93	0.93	0.93	0.93	0.93
M (%)	98.81	98.81	98.81	98.81	98.81
QC (%)	92.06	92.06	92.06	92.06	92.06
R (%)	92.06	92.06	92.06	92.06	92.06

γ_{sf} - Deciles	2-6	3-6	4-6	5-6
adjR2	0.94	0.93	0.93	0.93
M (%)	98.81	98.81	98.81	98.81
QC (%)	91.67	92.06	92.06	92.06
R (%)	91.67	92.06	92.06	92.06

γ_{sf} - Deciles	2-5	3-5	4-5
adjR2	0.94	0.94	0.94
M (%)	98.81	98.81	98.81
QC (%)	91.27	91.27	91.27
R (%)	91.27	91.27	91.27

γ_{sf} - Deciles	2-4	3-4
adjR2	0.95	0.95
M (%)	98.81	98.81
QC (%)	91.27	91.27
R (%)	91.27	91.27

γ_{sf} - Deciles	2-3
adjR2	0.95
M (%)	98.81
QC (%)	89.68
R (%)	89.68

(1) QC – quasi-concavity, M – monotonicity, R – regularity.