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Optimal Stocking in a Bioeconomic Model of Grazing with Plant-Herbivore Competition*

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Abstract

Noy-Meirs predator-prey model of the dynamics of a grazing system is extended in this study to include plant-herbivore competition. Such models of plant-herbivore competition although common in the literature on the biology of herbivory, have been little used in range management. In this paper this extended version of Noy-Meirs model is used as the basis for a bioeconomic model of optimal stocking. Initially, a deterministic optimal control model with plant-herbivore competition is developed. This is then extended to the problem of optimal stocking under uncertainty. An optimal stochastic control approach for the case of two state variables is used to determine the optimal stocking rate. The results are general for a wide class of dynamic models.

1 Introduction

A number of optimal stocking models have been developed based on dynamic programming and optimal control methods. Passmore and Brown (1991) developed a discrete-time, stochastic, dynamic programming model of optimal stocking based on rangelands as a renewable resource. Torell, Lyon and Godfrey (1991) use a continuous-time deterministic optimal control problem to analyze the effect of the planning horizon on stocking rate. Standiford and Howitt (1992) develop an optimal stocking model based on discrete-time stochastic optimal control methods. This model is then applied in a mixed land use setting with grazing, hunting and forestry, to determine optimal land use strategies. Virtala (1992) develops a discrete-time deterministic stock-recruitment model for Reindeer. Finally, Perrings (1994) has developed a discrete-time stochastic optimal control model with endogenous range capacity. Such models have in general relied heavily on ideas developed in the fisheries literature, such as linear resource depletion. Within the ecological literature on rangelands this concept has fallen into disrepute¹.

*Thanks are due to Anthony Bloesch, Colin Brown, John Mott, Mal Wegener and Simon Woodward for discussions and comments on the material presented in this paper.

¹See the critique of Noy-Meirs early work by Johnson and Parsons (1982) and the forum edited by Levin (1993).

All of these models have been developed in the context of an economy with private grazing rights to the extent that they ignore institutional features². Furthermore, they have all been based on simple predator-prey models that ignore the dynamics of plant-herbivore competition.

A recent attempt to remedy some of these deficiencies is Swanson (1994). Unfortunately, Swanson's model stops short of interspecific competition by treating the base resource (Land, i.e. pasture biomass) as a parameter or decision variable but not as a state variable. Consequently, rangeland degradation cannot be analyzed within Swanson's model. Nevertheless, his work does point in the right direction.

In this paper an attempt is made to integrate some of the concerns discussed in the ecological literature on range management and the more general biological literature on herbivory within a dynamic bioeconomic model of optimal stocking. A continuous time framework is used for a number of reasons. Firstly, one can argue that real world processes are continuous rather than discrete. Secondly, continuous time problems are easier to handle analytically and modern computer technology no longer presents a barrier to the analytical or numerical solution of continuous time optimal control problems³. A third reason is suggested by Dutta and Radner (1994) who study a class of principle agent contracts called "bankruptcy contracts". If one wished to study pastoral leases from the perspective of agency theory, then it would appear desirable to allow for the bankruptcy of the pastoral enterprise. This would require allowing for "bankruptcy contracts" in the sense of Dutta and Radner. Using discrete time in such a scenario leads to difficulties with regard to overshooting the duration of the contract. This can be avoided by using continuous time⁴. Fourthly, an extension of the model to traditional pastoralism and common property grazing problems is in preparation, modelling such systems requires the use of continuous time models as the use of difference games in natural resource allocation is known to lead to problems concerning the playability of such games⁵.

In the second section an overview of the main issues being debated in range ecology and in the literature on herbivory is presented. In the third section a simple model of grazing technology is derived from a linear weight-gain function typical of the literature. In the fourth section the model of plant-herbivore competition is embedded within a bioeconomic model of optimal stocking in which graziers seek to maximize profit over time. A proposition by Workman and Fowler that the optimal stocking rate is always less than maximum sustainable yield stocking rate is proved. In the fifth section this model is extended to the stochastic case. In the sixth section the results of a sensitivity analysis of key parameters are presented. Finally, conclusions are drawn and directions for further research presented.

²In the case of Perrings and Vartdal's work this is unfortunate as the assumption of private property is in both cases is probably not justified. Both Sahelian and Sami grazing rights have been well documented in anthropological and other literature. For a discussion of grazing rights in Africa see Livingstone (1986) and Behnke (1994). Sami pastoralism from an anthropological perspective is discussed in Ingold (1976) and (1980).

³In this respect MAPLE V.3 has been of some assistance in solving and understanding many of the problems presented in this paper.

⁴Dutta and Radner (1994): p. 487.

⁵Clemhout and Wan (1985): p. 477.

2 Range Ecology and Herbivory

The starting point of any bioeconomic model of grazing would appear to be the predator-prey model of Noy-Meir (1975). Although this model has certain deficiencies with respect to ecological accuracy and detail Noy-Meir's basic conclusions are generally not questioned. The main criticisms of his approach may be divided into three issues that are nevertheless interrelated:

1. Noy-Meir does not take into consideration the coevolutionary nature of the relationship between forage plants and herbivores.
2. Herbivores may in fact "optimize" plant growth either directly or indirectly by their feeding behaviour.
3. Points 1 and 2 may in fact, under certain circumstances, lead to overcompensatory plant growth thus precluding overgrazing.

The range ecological and herbivory literature has followed two separate paths since Noy-Meir's original model. One path has been taken by range ecology with the adoption of the state and transition model. The state and transition model makes use of Markov chains to model changes in key rangeland state variables. The use of Markov chains presupposes numerical data. Nevertheless, the implication is clear, the population dynamics of rangeland ecosystems may be viewed as a Markov process, regardless of whether numerical data are available or not.

The second path has been that taken by pasture and crop modellers. Beginning with the work of Johnson and Parsons (1985) a need was perceived to extend Noy-Meir's original model by introducing senescence into the equation. Johnson and Parsons model is of little use to the economist relying as it does on a great deal of detail concerning plant physiology. Nevertheless, the basic tenor of their argument is important for economists who wish to develop bioeconomic models of stocking decisions. The primary point of Johnson and Parsons paper was that plants are not a passive resource whose biomass is depleted in a linear manner by grazing. Rather grazing has a complex impact on plant physiology.

These issues have largely been ignored in the economics of rangeland management.

It is one of the purposes of this paper to integrate some of these aspects from the ecological literature into a bioeconomic model of grazing.

It should be noted that these issues are controversial and that any a-priori assumptions concerning winners and losers in the plant-herbivore grazing game should, where possible, be avoided.

In recent years considerable controversy has developed within range ecology as to the nature of the interaction between forage plants and herbivores in particular to what extent the impact of grazing (i.e. stocking rate) on forage plants is always detrimental, has been questioned. The traditional approach has been based on ideas of Clementsian succession, where a rangeland is more or less regarded as representing a climax plant community with forage plants being viewed as r-strategists (opportunistic species) and man and his livestock

as something akin to a K-strategist (persistent species). This then leads to a scenario whereby grazing is perceived as being always detrimental and best modelled by a simple predator-prey relationship. Furthermore the assumption of a climax community implies an equilibrium situation in the sense of a steady-state. More recently, this approach has been rejected in favour of models not based on the existence of a climax community, e.g. the State and Transition model and models based on the so-called "herbivore or grazing optimization" hypothesis. This hypothesis states that grazing at moderate stocking rates leads to levels of plant biomass above those that would occur in the absence of grazing. This effect has been extensively documented in grazing trials⁶. Controversy remains however as to how to interpret these results and in particular the likely implications for modelling. The effect is graphed in diagram 1. A simple Noy-Meir type predator-prey model would consider only the downward sloping portion of the curve and ignore the upward sloping portion of the curve which characterises overcompensatory plant growth.

This hypothesis has important implications for optimal stocking. On the one hand it implies non-linearity of the optimal control problem, whereas previous predator-prey models used such as that of Noy-Meir (1975) imply a "bang-bang" control⁷.

⁶See the Forum-Grazing Theory and Rangeland Management, S.A. Levin (1993).

⁷See also Virtala (1992) whose model of optimal stocking of reindeer in Finland, which to some extent is based on Noy-Meir's work, is of the "bang-bang" type. "Bang-bang" problems are characterized by state equations that are linear in the control variable.

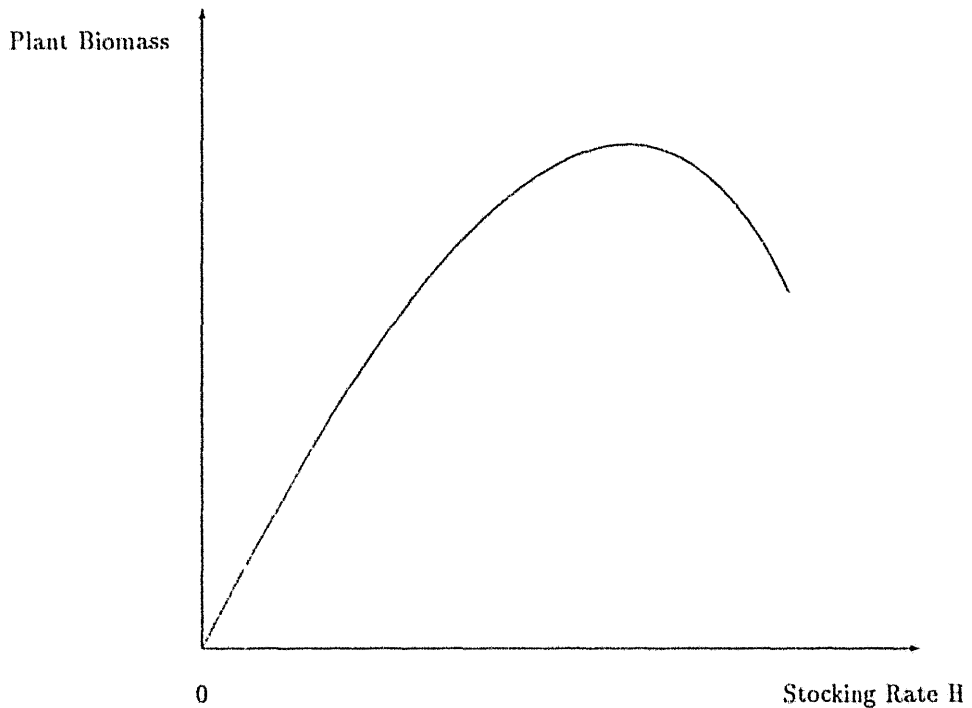


Diagram 1: Herbivore Optimization Curve (adapted from DeAngelis (1992): p. 113. See Also McNaughton (1979): p.696.).

On the other hand it begs the question as to the mechanism by which overcompensatory plant growth may occur. There are three possible explanations for overcompensatory plant growth in circulation in the literature⁸:

1 Direct effects, e.g. nutrient cycling

⁸McNaughton (1979): pp. 693 lists nine possibilities, but all of these imply the use of an interactive model. Thus they fall into the same class as nutrient cycling. I ignore these examples here as many are still controversial. The interested reader is referred to the literature.

2 Plant-plant competition, e.g. reduced self thinning of plants

3 Plant defence mechanisms, e.g. proteinase inhibitors

Some authors support the nutrient cycling hypothesis⁹. This approach would lead one away from the non-interactive model described in this paper to an interactive model in which plant growth would depend directly and in a positive manner on stocking rate. A second hypothesis is the reduced self-thinning hypothesis¹⁰. Basically, this hypothesis explains overcompensatory plant growth via the reduction in competition between plants that is induced by grazing, this in turn leads to increased tillering and so to greater plant biomass. The third candidate to explain the presence of overcompensation is that of grazing induced plant defences. Induced plant defences unlike constitutive plant defences do not require evolutionary time to respond to the impact of grazing. By using a model of interspecific competition between plants and herbivores predator-prey models may be generalized to allow for the possibility of overcompensatory plant growth.

The resistance to herbivore optimization theory within the range management literature has been primarily due to the failure to identify a plausible mechanism by which overcompensatory plant could occur in the presence of grazing. In the more general biological literature on herbivory and in particular the literature on insect herbivory this is not controversial. In this literature the main candidates for such a mechanism are plant defence mechanisms.

In the literature on insect herbivory it is common practice to model the population dynamics of grazing using models of interspecific competition between plants and herbivores. Studies of plant-herbivore competition for mammalian herbivores have been somewhat rarer, but it is thought that the dynamics are analogous to those of insect herbivore systems¹¹. The primary competitive mechanism used by plants to counteract the impact of grazing is that of plant defence mechanisms.

Plant defence mechanisms may be divided by strategy into three types: physical, chemical and behavioural (including informational) and by *telos* into two types: induced and constitutive. Physical defences include thorns and high cellulose concentrations, chemical defences include various plant toxins and proteinase inhibitors, etc., behavioural defences include signalling mechanisms such as infochemicals as well as mimicry and association with known toxic species. The latter leads to the formation of plant defence guilds, i.e. complexes of plant associations that are characterised by the fact that they form a defensive unit consisting of plants of different species.

In this paper these features will not be explicitly treated, I will not distinguish between types of defence mechanisms in the proposed model, but instead will only postulate that such a defence mechanism exists, is a function of total plant biomass and is successful in that it has a detrimental effect upon herbivore fecundity. Such a model is relatively general and would be consistent with a wide range of optimal defence functions.

An indication of the importance of such defence mechanisms is given by Culvenor (1984) who estimated a total cost to the Australian economy of between \$ 70-80 million per annum¹².

⁹Andrew Moore for example has communicated his belief to me that overcompensatory plant growth may be explained by nutrient cycling. Personal Communication, January 1995.

¹⁰See Hiernaux et al. (1994) for a discussion.

¹¹Crawley (1983) for example uses the same techniques to analyze the population dynamics of insects, small herbivores and large herbivores

¹²Culvenor (1984): p. 3.

Note this figure ignores the opportunity cost of weight-gain foregone due to the response of livestock to plant defences. It includes losses due to livestock deaths and the costs of veterinary treatment. What matters to us are the hidden costs of such defences upon which no economic value can be placed.

In general grasses possess relatively poor defences. Crawley (1983) points out that compensatory growth may be associated with the curve of photosynthetic value versus leaf age possessing a maximum and that one would expect chemical defences to be concentrated in those leaves of maximum photosynthetic value¹³. Therefore the issue is not so much the overall level of chemical defences but how a plant, including grasses, concentrates the few chemical defences it may have. Furthermore, the reality is that rangelands consist of a mixture of grasses, woody weeds and other range plants with defensive potential. Such a community of range plants is better characterized by interspecific competition. I will not however analyze the process by which species composition changes, although such models are already beginning to appear in the ecological literature¹⁴.

The following model of the ecosystem is therefore proposed:

$$\dot{V} = G(V) - c(V)H \quad (1)$$

$$\dot{A} = F(V, A) - b(V)H \quad (2)$$

where V is forage plant biomass, $A = HL$ the total number of animals, H stocking rate, $G(V)$ the biomass regeneration function, $F(V, A)$ the herbivore regeneration (reproduction) function and $c(V) > 0$ and $b(V) > 0$ strategy functions assumed to be chosen by herbivores and forage plants respectively. This model represents an extension of Noy-Meir (1975) in that livestock numbers are also treated as a state variable. Note, that such a system is called an indirect feedback control system¹⁵. To see this replace H in equation 1 by $\frac{\dot{A}}{b(V)}$. Clearly, the control H acts on V only indirectly via the time derivative of A .

Some explanation is required of range management terminology. Grazing systems are stock management strategies that are designed to increase yield and avoid land degradation by reducing grazing pressure. Grazing pressure refers to the amount of stress imposed by herbivores on a grassland ecosystem. A common measure of grazing pressure is the stocking rate expressed in animal units per hectare and unit of time. Animal units are not individual animals. Individual animals cannot be directly compared and therefore aggregated. Animal units are calculated based on forage intake. A set of animals with equivalent forage intake is called an animal unit. Such units are aggregable and provide a measure of grazing pressure in a given place and time.

3 Production Functions, Weight-Gain Functions and Stocking Rate

In empirical work done by Van Heerden and Tainton (1989) a negative linear relationship between individual weight-gain and stocking rate was found¹⁶. A linear relationship of the

¹³Crawley (1983): pp. 41-42.

¹⁴See for example Lundberg, Järemo and Nilsson (1994).

¹⁵Lefschetz (1965): p. 18.

¹⁶See also Wheeler and Freer (1986) pp. 176-177.

type discovered by Van Heerden and Tainton conforms to the following form:

$$\bar{Y} = a - bH \quad (3)$$

where \bar{Y} is the average liveweight and $a, b > 0$.

From this one may derive a production function for a fixed area of land (Y_L) (short-run production function) in the following manner⁶:

$$Y_L = kc(V)H - mH^2. \quad (4)$$

This results in the revenue-cost diagram, Diagram 2.

⁶Humphreys (1987): p. 125.

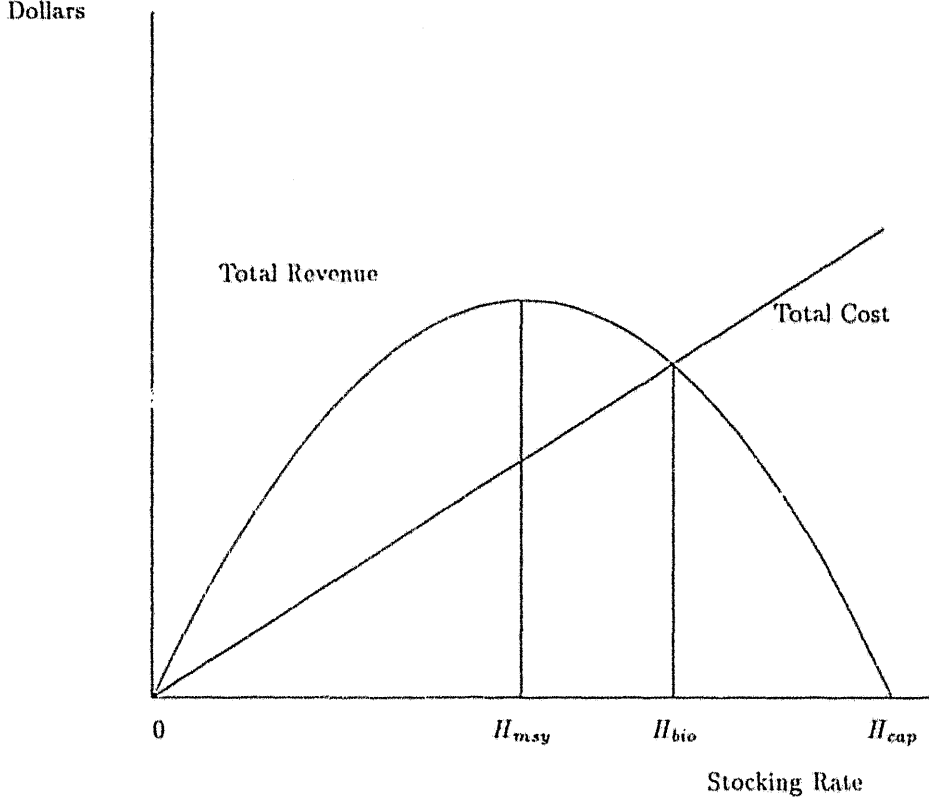


Diagram 2: Revenue-Cost diagram for the optimal stocking problem.

This diagram possesses the typical form of revenue-cost functions used in the literature on renewable resources. This is important because it implies that bionomic equilibria may still exist, whereas the linear weight-gain (production) functions used by Noy-Meir preclude the existence of bionomic equilibria.

Two parameters of the revenue function are particularly important. these are the maximum sustainable yield stocking rate H_{msy} and economic grazing capacity H_{cap} . Note that economic grazing capacity may differ from the ecological grazing capacity. Economic grazing capacity is defined as the least upper bound of the set of zero revenue stocking rates. To calculate H_{msy} differentiate Y_L and set $\frac{dY_L}{dH}$ to zero:

$$\frac{dY_L}{dH} = kc(V) - 2mH = 0 \quad (5)$$

From this one obtains $H_{msy} = kc(V)/2m$. To obtain H_{cap} set $Y_L = 0$ and solve for H . This gives $H_{cap} = kc(V)/m$. Note that $H_{msy} = \frac{1}{2}H_{cap}$.

4 Optimal Stocking in a Continuous-Time Deterministic Model with Sole Ownership

The optimal stocking problem for a grazing enterprise under sole ownership and in continuous time is given by

$$\max_H \int_0^T \{p(kc(V)H - mH^2)L - rA\} e^{-it} dt + R(V, H, T)e^{-iT} \quad (6)$$

given the equations of motion

$$\dot{V} = G(V) - c(V)H \quad (7)$$

$$\dot{A} = F(V, A) - b(V)H \quad (8)$$

and $c(V)$, $b(V)$ and "stocking cost" $r \geq 0$. Note the variables V, H are time dependent. L is a constant parameter. p is the sale price of animal liveweight and i is the discount rate. $R(V, H, T)$ is the value of the grazing enterprise on the expiry of the pastoral lease or on retirement of the grazier¹⁷. thus $R(V, H, T) = (w_1V + w_2H)L$ where w_1 and w_2 are weights which reflect the relative value of available forage and livestock in the final period. The corresponding Hamiltonian is

$$\mathcal{H}(H, \lambda, \mu, t) = \{p(kc(V)H - mH^2)L - rHL\} e^{-it} + \lambda[G(V) - c(V)H] + \mu[F(V, HL) - b(V)H] \quad (9)$$

The maximum principle gives

$$(p(kc(V) - 2mH)L - rL)e^{-it} - \lambda c(V) + \mu \frac{\partial F}{\partial H} - \mu b(V) = 0 \quad (10)$$

and the transversality conditions

$$V, H > 0,$$

then

$$\frac{\partial R}{\partial V} e^{-iT} = \lambda(T)$$

¹⁷Strictly speaking modelling a leasehold enterprise would require the use of a principal-agent framework. It is however not possible to treat principle-agent problems from a deterministic approach. I have elected not to analyse leasehold land tenure using this approach in this paper, as it first requires the development of a stochastic optimal stocking theory. Furthermore, the development of continuous time principal-agent models is still in its infancy. See, however Dutta and Radner (1994) for a discussion of the continuous time principle-agent problem with moral hazard.

and

$$\frac{\partial R}{\partial A} e^{-iT} = \mu(T) \quad (11)$$

Substituting these values into () gives

$$(p(kc(V) - 2mH)L - rL)e^{-it} - w_1 L e^{-iT} c(V) + w_2 e^{-iT} (F_2 L - b(V)) = 0 \quad (12)$$

If one assumes $F(V, HL)$ to be logistic with the form $nHL(1 - \frac{HL}{V})$ then this becomes

$$(p(kc(V) - 2mH)L - rL)e^{-it} - w_1 L e^{-iT} c(V) + w_2 e^{-iT} ((nL(1 - \frac{HL}{V}) - \frac{nHL^2}{V} - b(V))) = 0 \quad (13)$$

From this one obtains

$$H^* = \frac{1}{2} \frac{(e^{-it}(pke(V)L - rL) - e^{-iT}(w_1 Lc(V) + w_2 nL - w_2 b(V)))V}{(L(pne^{-it}V + w_2 e^{-iT}nL))} \quad (14)$$

The impact of the discount rate on the optimal stocking rate is indeterminate due to the exponential nature of the discount factor. Myopia cannot therefore provide a generic explanation for overgrazing. If one were however to assume a steady-state this situation would however change, the discount rate would lead to a reduction in stocking rate and not increase. Similar results have been obtained by Perrings (1994) and Virtala (1992). I will however not assume a steady-state here in order to facilitate a comparison with the stochastic case. A steady-state assumption is incompatible with a stochastic model. Furthermore it is not compatible with more recent thinking in the range management literature such as the "State and Transition" model.

4.1 The Workman-Fowler Proposition

Workman and Fowler (1986) proposed that the optimal stocking rate always lies below the biological optimum (maximum sustainable yield). Although they provided numerical examples they gave no formal proof of this claim. The Workman-Fowler proposition is interesting because it exposes some of the problems involved in applying results from the fisheries literature to pastoral problems.

Proposition 1 (Workman and Fowler (1986)):

$$H^* < H_{msy}.$$

Proof: First note that the revenue function $P(V, H)$ takes on a slope of zero for $H = H_{msy}$. Furthermore as H approaches zero $\frac{\partial P}{\partial H}$ approaches infinity. Given $r > 0$, then by the mean value theorem, there exists a point between $H = 0$ and $H = H_{msy}$, where the total revenue

function and the total cost function are of equal slope. This point is the optimal stocking rate H^* . Therefore the following condition must hold: $0 < H^* < H_{\max}$ and the proposition is shown to be true \square

This result holds even in the presence of discounting. Thus sole ownership in the rangeland setting will not produce overgrazing based on discount effects. This result differs considerably from sole ownership in a fishery¹⁸. Although I shall show later that other factors may contribute to overgrazing in rangelands.

5 Optimal Stocking in a Bioeconomic Model with Stochastic Environmental Fluctuations

The use of deterministic models is subject to considerable criticism and indeed a number of models have attempted to introduce stochastic elements by modelling forage growth as a Markov process¹⁹. Both economic and ecological models that have taken this path have been developed. The difficulties involved in such an approach are considerable, in particular if the derivation of general results is the desired objective. Nevertheless, the development of stochastic models is not only necessary for reasons of realism, but because certain questions can only be analyzed within the context of a stochastic model. An example of such a question is that of the incentive effects of leasehold versus freehold land tenure. This issue is best analyzed within a principle-agent framework. Analyzing optimal stocking within a principal-agent framework will however require the development of a stochastic model as principle agent problems by definition involve decision making under risk.

In this section, the deterministic model developed above is extended by replacing the deterministic state equations by a system of stochastic differential equations.

It has been traditional in stochastic optimal stocking problems to assume particular stochastic processes. Thus, Passmore (1992) and Passmore and Brown (1991) postulate a discrete-time Markov process. Furthermore, this supposition fits in well with the so-called State and Transition model which is currently held in high regard in range ecology and which also views rangeland ecosystem dynamics as a Markov process²⁰.

Perrings (1984) takes a different approach by introducing two state variables: herd size and range carrying capacity. This introduces an "historical" aspect into the model, where the current state of the system depends on all previous states²¹. It is important however to note that this does *not* imply that the underlying stochastic process is not Markovian. Perrings does not specifically assume a particular stochastic process, but the form of the difference equations used would appear to imply that he was considering a Markov process.

In the following a general time-indexed stochastic process is assumed which may or may not be Markovian. Extending the general grazing model used above to the stochastic case gives the following system of stochastic differential equations:

$$\dot{V} = \hat{V}(V, H, t) \quad (15)$$

¹⁸Clark (1990): Ch.2.

¹⁹See Renshaw (1991) for a good discussion of the relationship between deterministic and stochastic models in population biology.

²⁰Westoby, Walker and Noy-Meir (1989a) and (1989b).

²¹This differs from other models, where the current state usually depends only on the previous state.

$$\dot{A} = \hat{A}(V, H, t) \quad (16)$$

Typically, this system of stochastic differential equations would have an additive specification:

$$\dot{V} = V(V, H, t) + g(V)\xi(t) \quad (17)$$

$$\dot{A} = A(V, H, t) + h(H)\zeta(t) \quad (18)$$

where $g(V)$ and $h(H)$ are measures of the intensity of noise and ξ and ζ are noisy processes. The functions $V(V, H, t)$ and $A(V, H, t)$ are the deterministic component of the differential equation.

It should be noted that the State and Transition model may be recovered from the model presented here in the following manner.

Given a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ and interpreting V as a vector of key range condition indicators then the State and Transition model is characterized by the following additional assumptions:

$$1 \quad b(V, t) = 0$$

$$2 \quad \xi(t) \text{ is a continuous-time Markov process}$$

Interestingly, the claims of Westoby, Walker and Noy-Meir that the state and transition model is a disequilibrium model appear somewhat premature. In a stochastic context equilibrium may be interpreted in a number of different ways. The underlying stochastic process may be interpreted as a:

$$1 \quad \text{Strongly Stationary (strong equilibrium concept)}$$

$$2 \quad \text{Weakly Stationary (weak equilibrium concept)}$$

process²².

If one were to totally reject all forms of equilibrium then analysis would become impossible. Such a situation would be characterized as "chaotic" in the sense of non-linear dynamic systems theory. The existence of chaotic behaviour in real biological systems is however contentious to say the least²³.

The stochastic optimal stocking (control) problem may be solved analogously to the deterministic case. The objective functional will however differ slightly as it is necessary to assume that graziers attempt to maximize expected profit:

²²The term ergodic is sometimes used instead of stationary.

²³Renshaw (1991): pp. 4-5.

$$\max_H E \left\{ \int_0^T (p(kc(V)H - mH^2)L - rA)e^{-\alpha t} dt \right\} + R(V, H)e^{-\alpha T} \quad (19)$$

given the equations of motion:

$$\dot{V} = V(V, H, t) + g(V)\xi(t) \quad (20)$$

$$\dot{A} = A(V, H, t) + h(H)\zeta(t) \quad (21)$$

In order to solve this problem a number of extra assumptions are needed. In the following it is assumed that the processes $\xi(t)$ and $\zeta(t)$ are Wiener processes. A Wiener process is a homogenous Markov diffusion process, thus this specification still follows the spirit of the State and Transition model, but introduces the possibility of plant-herbivore competition which the state and transition model ignores. Rewriting the above system of equations as Wiener processes one obtains:

$$\dot{V} = V(V, H, t) + g(V)W(t) \quad (22)$$

$$\dot{A} = A(V, H, t) + h(H)W(t) \quad (23)$$

Note that in our model $V(V, H, t) = G(V) - C(V)H$ and $A(V, H, t) = F(V, H, L) - b(V)H$.

This gives us an optimal stochastic control problem in two state variables, such a problem may be solved by using a "two-state variable" (2SV) analogue of the Hamilton-Jacobi-Bellman (HJB) equation²⁴.

The 2SV HJB equation for this problem is given by:

$$0 = \max \left\{ (p(kc(V)H - mH^2) - rHL)e^{-\alpha t} + J_V[G(V) - c(V)H] + J_A[F(V, H, L) - b(V)H] + \frac{1}{2}J_{VV}\sigma_1(H)^2 + \frac{1}{2}J_{AA}\sigma_2(H)^2 \right\} \quad (24)$$

Evaluating this equation at the maximum gives the following condition:

$$0 = (p(kc(V) - 2mH) - rL)e^{-\alpha t} - J_V c(V) + J_A \left(\frac{\partial F}{\partial H} - b(V) \right) + J_{VV}\sigma_1(H) \frac{\partial \sigma_1}{\partial H} + J_{AA}\sigma_2(H) \frac{\partial \sigma_2}{\partial H} \quad (25)$$

This is a second order partial differential equation which is easily solved given the terminal data (value matching condition)

²⁴For a treatment of the application of "single-state variable" optimal stochastic control problems to natural resource management see Mangel (1985). A discussion of optimal stochastic control with two state variables as applied to renewable resources may be found in Beard (1994).

$$J(V, A, t) = R(V, H)e^{-iT} \quad (26)$$

Furthermore consider the case where $F(V, HL)$ is a logistic function $nHL(1 - \frac{HL}{V})$, then one obtains:

$$0 = (p(kc(V) - 2mH)L - rL)e^{-it} - w_1Le^{-iT}c(V) + w_2e^{-iT}(nL(1 - \frac{HL}{V}) - \frac{nHL^2}{V} - b(V)) \quad (27)$$

This then gives the following optimal stocking rate

$$H^* = \frac{1}{2} \frac{(e^{-it}(pkLc(V) - rL) - w_1Le^{-iT}c(V) + w_2e^{-iT}nL - w_2e^{-iT}b(V))V}{L(e^{-it}pLmV + w_2e^{-iT}nL)} \quad (28)$$

Interestingly, the use of a weighted average to represent the terminal value of the enterprise also leads to a certainty equivalence result. This would not however be the case in the infinite time context. Pastoral enterprises do not however possess infinite time horizons, although such an assumption may well be justified for society as a whole. Further, the assumption that the value of the enterprise is a weighted average of available forage and livestock inventory appears not only plausible but has a close affinity to certain results in the literature on option pricing and valuation. Note that the terminal value of the property can be considered as the option value of the rangeland. It would seem therefore that the certainty equivalence result is a natural result which holds for sole ownership pastoral enterprises with finite time horizons.

Note that this optimal stocking rate varies with regard to available forage between zero and an upper limit dependent upon the digestive ability of the livestock. The existence of an upper limit to the digestive ability of large herbivores was first introduced into the literature by Westoby (1974)²⁵.

The introduction of institutional factors, such as sharecropping, principle-agent relationships, grazing rights characteristic of common property and transhumance may however lead to different results where "uncertainty matters". This has long been recognized by New-Institutional economists such as Douglas North:

The major role of institutions in a society is to reduce uncertainty by establishing a stable (but not necessarily) efficient structure to human interaction²⁶.

The certainty equivalence result obtained here for the sole ownership case is therefore important as a benchmark for comparing the success of various institutional regimes in managing rangeland resources. In addition to sustainability the ability of an institution to manage risk needs to be addressed when making such comparisons.

²⁵The choice of $c(V)$ will determine whether or not such an upper limit exists. A common functional form suggested in the literature is the Michaelis-Menten function used in reaction kinetics. See Murray (1989), Chapter 5.

²⁶North (1990): p. 6.

6 Sensitivity Analysis of Key Parameters

A number of key parameters influence the optimal stocking rate. For example, discount rate i , farm size L and the date t . An analysis of the impact of each of these factors was carried out using asymptotic methods. The results confirm more or less the current state of debate in the literature

The impact of discounting on the optimal stocking rate is, even after repeated application of L'Hopital's rule, indeterminate. The reason for this is that both the numerator and denominator in the optimal stocking rate expression are exponentially decreasing in i . In this context, it is interesting to note that previous optimal stocking studies have found conflicting results with regard to the impact of discounting²⁷. For example, Passmore and Brown (1991) and Passmore (1992) conclude, in line with received opinion in the natural resources literature, that discounting is environmentally detrimental. Perrings (1994) and Virtala (1992) imply the opposite in their conclusions. Furthermore, Perrings (1993) provides a detailed analysis of why the impact does not possess a unique sign²⁸.

The impact of grazing area L (farm size) on the optimal stocking rate H^* is interesting. The limit of H^* as L approaches zero is undefined. If however one takes the limit as L approaches infinity then the optimal stocking rate approaches zero.

$$\lim_{L \rightarrow \infty} H^* = 0 \quad (29)$$

This result is consistent with one of the stylized facts of the range management literature namely, that small farm sizes lead *ceteris paribus* to overgrazing.

Finally, the impact of ageing or the approach of the lease expiry date is of interest. As graziers age or a lease approaches its expiry date, stocking decisions are likely to change. Graziers in different generations are often perceived as making different decisions due to experience or the lack thereof. In modelling the grazing firm with a finite time horizon rather than an infinite time horizon the impact of ageing and /or time remaining to lease expiry can be analyzed.

Consider the limit of H^* as $t \rightarrow T$.

$$\lim_{t \rightarrow T} H^* = \frac{1}{2} \frac{(pLkc(V) + w_2Ln - rL - w_1c(V)L - w_2b(V))V}{L(pmV + w_2nL)} \quad (30)$$

In order to determine whether or not this expression is smaller than the optimal stocking rate for positive gross margins it would be necessary to parameterize the model numerically. Preliminary simulations and intuition suggest that as t approaches T the optimal stocking rate will fall. However, this issue does need further examination and it is still somewhat premature to place a definitive sign on the impact of "date" on stocking rate.

7 Conclusion

In this paper, an extended version of Noy-Meirs original population dynamic model of grazing is analyzed from a bioeconomic perspective. In particular Noy-Meirs model is extended to

²⁷ Passmore and Brown (1991), Passmore (1992), Perrings (1994) and Virtala (1992).

²⁸ Perrings (1993): pp. 89-92.

include features of plant-herbivore competition which appears to give a biologically more realistic account of the interrelationship between range plants and herbivores, than the simple predator-prey model used by Noy-Meir. This extended model was first analysed from a deterministic perspective where it is shown that, given private property rights, overgrazing is impossible even in the presence of discounting. The model is then extended to a stochastic framework, where it is shown that discounting has an unclear impact on stocking rate and that even the presence of uncertainty fails to account for range degradation. The area grazed is the only factor which could possibly account for overgrazing and this result is consistent with that of other studies. The assumption that the dynamics of both range condition and livestock follow a Wiener process requires some ad hoc assumptions to guarantee feasibility of the solution. A more realistic assumption may be to postulate a Poisson process, but as a first approximation a Wiener process does deliver some interesting results. A second avenue of future research may be to examine alternative models of grazing impact, such as an Ivlev model²⁹.

Generic explanations of overgrazing are, with the exception of grazing area (farm size), still out of reach of this model and an extension to other institutional settings appears desirable. In particular an issue that is not addressed here is the "tragedy of the commons" as a theoretical model of overgrazing. The "tragedy of the commons" fails to account for the stylized facts of overgrazing in both developed and developing countries. In addition, one should bear in mind that property rights in Australia's rangelands are hardly "well-defined" in the sense of private property. The model presented here might therefore be extended to include institutional aspects such as grazing rights, land tenure and market institutions, in the hope of developing a theory more in harmony with the stylized facts of land use in the world's rangelands. In comparing such a model with sole ownership one should bear in mind that not only is the overall grazing pressure associated with an institution important but also the capacity of an institution to manage risk.

8 Appendix

A. Linear Weight Gain Functions and Returns to Scale

In this appendix it is shown that both Noy-Meir's linear production function and a Cobb-Douglas function are not compatible with weight-gain stocking rate tradeoffs that have been observed experimentally.

Given the following Cobb-Douglas technology:

$$Y = kc(V)A^\alpha L^{1-\alpha} \quad (31)$$

where A is the total number of animals, L land, $c(V) > 0$ consumption per animal, V plant biomass, k a weight-gain parameter and α the elasticity of substitution.

Taking output per hectare one obtains

$$\frac{Y}{L} = kc(V)A^\alpha L^{-\alpha} = kc(V)\left(\frac{A}{L}\right)^\alpha \quad (32)$$

²⁹ Note an Ivlev model is non-linear in the control variable even in the absence of interspecific competition, for this reason it does not fall within the class of models discussed in this paper.

Defining $Y_L = \frac{Y}{L}$ and $H = \frac{A}{L}$ one obtains the following weight-gain per hectare function dependent upon stocking rate H .

$$Y_L = kc(V)H^\alpha. \quad (33)$$

The individual weight-gain function is then given by

$$\bar{Y} = \frac{Y}{A} = \frac{Y_L}{H} = kc(V)H^{\alpha-1} \quad (34)$$

Differentiating with respect to H gives the slope of the individual weight-gain function.

$$\frac{d\bar{Y}}{dH} = (\alpha - 1)kc(V)H^{\alpha-2} \quad (35)$$

Examination of (45) shows that for $\alpha < 1$ the individual weight gain function has negative slope. Differentiation again one obtains:

$$\frac{d^2\bar{Y}}{dH^2} = (\alpha - 1)(\alpha - 2)kc(V)H^{\alpha-3} = (\alpha^2 - 3\alpha + 2)kc(V)H^{\alpha-3} \quad (36)$$

If $Y(H)$ is linear then

$$\alpha^2 - 3\alpha + 2 = 0 \quad (37)$$

Solving for alpha one obtains $\alpha = 2$ or $\alpha = 1$. If $\alpha = 2$ the technology is not diminishing returns. For a twice differentiable diminishing returns technology $\alpha = 1$, but this implies that the individual weight-gain function has zero slope. Note that substituting $\alpha = 1$ back into (2) gives Noy-Meir's linear production function. Noy-Meir's linear production function which is derived from a Cobb-Douglas function implies therefore that individual animals do not lose condition as a result of increased stocking rate. This result is both counterintuitive and contrary to experimental evidence.

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