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A STOCHASTIC FRONTIER PRODUCTION FUNCTION
WITH FLEXIBLE RISK PROPERTIES⁺

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ABSTRACT

This paper considers a stochastic frontier production function which has additive, heteroscedastic error structure. The model allows for negative or positive marginal production risk of inputs as originally proposed by Just and Pope (1978). An empirical application is presented using data on Central Ethiopian peasant farmers who used no fertiliser in their operations. The null hypothesis of no technical inefficiencies of production among these farmers is accepted. However, the flexible risk model does not represent the data on peasant farmers as well as the traditional stochastic frontier model with multiplicative error structure.

Key Words

Stochastic Frontier Production Function, Production Risks,
Technical Efficiency.

+ Paper presented at the 39th Annual Conference of the Australian Agricultural Economics Society, University of Western Australia, Perth, 14-16 February, 1995. This is a revised version of the paper which was presented at the Georgia Productivity Workshop, University of Georgia, Athens, 21-23 October, 1994.

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1. Introduction

Building models that are consistent with economic theory and reality is the ultimate goal of econometricians. The stochastic frontier production function proposed by Aigner, Lovell and Schmidt (1977) and Meeusen and van den Broeck (1977) is more in line with the definition of a production function than the so-called average production function; and more realistic than the deterministic frontiers pioneered by Farrell (1957) and Aigner and Chu (1968).

However, a significant aspect of production, which has not previously been adequately accounted for in stochastic frontier production models, is production risks. In fact, production risk attracted little attention in the development of conventional production functions (Antle 1983) whereas marketing and price risks have been considered [see Lippman and McCall (1982) and references therein]. Nevertheless, production uncertainty is one of the most important ingredients in the formulation of government policy and the decision making of producers [see Just and Pope (1978), Pope and Kramer (1979), Griffiths and Anderson (1982), Wan, Griffiths and Anderson (1992)].

Incorporating production risk into stochastic frontier models is of particular relevance because the main purpose of frontier production functions is the prediction of technical efficiencies. In essence, technical efficiency measures the degree of utilisation of technologies adopted in the production process. It is commonly accepted that production risks affect the decision making of producers concerning the adoption and utilisation of new technologies. Given the importance of technical changes in production growth and the inevitable existence of the risk effects on economic efficiencies, it can be concluded that risk considerations should be incorporated into stochastic frontier functions in order to realistically account for, and

predict, technical efficiencies.

In this paper, we consider an alternative stochastic frontier model function for cross-sectional data, such that the marginal risks of inputs may be negative or positive. The model incorporates the structure of the stochastic frontier function within the framework of the preferred flexible risk model suggested by Just and Pope (1978). Thus output is specified to be the sum of a deterministic function of inputs and a heteroscedastic error term which depends on a different function of the inputs. The model is a modification of that presented in Wan and Battese (1992). The latter paper did not contain an empirical example. Kumbhakar (1993) recently proposed a production function model with flexible risk properties, but the output values are specified to be a multiplicative function of a function of inputs and an error term of components-of-variance type for panel data. The time and firm effects are, however, specified to be fixed (rather than random) effects.

The model is defined and discussed in Section 2. An empirical application of the model is presented in Section 3. Some basic theoretical results required in the derivation of the likelihood function and the partial derivatives of the logarithm of the likelihood function are presented in the Appendix.

2. Flexible Risk Frontier Model

Consider the stochastic frontier production function for a cross-section of N sample firms

$$Y_i = f(\mathbf{x}_i; \alpha) + g(\mathbf{x}_i; \beta)[V_i - U_i] , \quad i = 1, 2, \dots, N, \quad (1)$$

where Y_i is the production for the i -th firm during the period involved;

x_i is a vector of K explanatory variables for the i -th firm; such that the first element is the base of the natural logarithm, e ;

$$f(x_i; \alpha) \equiv \prod_{k=0}^K x_{ik}^{\alpha_k} \text{ and } g(x_i; \beta) \equiv \prod_{k=0}^K x_{ik}^{\beta_k} \text{ are known functions (here assumed}$$

to be of Cobb-Douglas form) of the explanatory variables, which depend on unknown parameters, $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_K)'$ and $\beta = (\beta_0, \beta_1, \dots, \beta_K)'$, to be estimated;

the V_i s are assumed to be independent and identically distributed standard normal random variables; and

the U_i s are non-negative random variables, associated with the existence of technical inefficiency of the firms in the industry, which are assumed to be independent and identically distributed truncations of the $N(\mu, \sigma^2)$ distribution, independently distributed of the V_i -random errors.¹

The production function (1) is of Cobb-Douglas type for convenience of exposition of the stochastic frontier model involved. It is required that all explanatory variables are parametric functions of inputs and other variables, such that they have positive values. Other functional forms can be used for $f(\cdot)$ and $g(\cdot)$, provided they are non-negative.

The mean and variance of production for the i -th firm, given its level of inputs and technical inefficiency effect, are

¹ The model as originally proposed by Wan and Battese (1992) defined

$$g(x_i; \beta) = \prod_{k=1}^K x_{ik}^{\beta_k} \text{ and } V_i \sim N(0, \sigma_v^2). \text{ The above model is a reparameteri-}$$

sation for which $\sigma_v = e^{\beta_0}$ and the random variables, V_i and U_i , are scaled by dividing through by σ_v . This parameterisation is preferred for estimation.

$$E(Y_1 | \mathbf{x}_1, U_1) = \left(\prod_{k=0}^K x_{1k}^{\alpha_k} \right) - \left(\prod_{k=0}^K x_{1k}^{\beta_k} \right) U_1 \quad (2)$$

and

$$V(Y_1 | \mathbf{x}_1, U_1) = \left(\prod_{k=0}^K x_{1k}^{\beta_k} \right)^2. \quad (3)$$

The marginal production risk associated with the j -th explanatory variable, defined to be the partial derivative of the variance of production (3) with respect to the j -th explanatory variable, is thus

$$\frac{\partial V(Y_1 | \mathbf{x}_1, U_1)}{\partial x_{1j}} = \frac{2\beta_j V(Y_1 | \mathbf{x}_1, U_1)}{x_{1j}}. \quad (4)$$

Clearly, the marginal production risk (4) may be positive or negative, depending on the sign of β_j , which is not necessarily the same as the rate of change of the mean of production with respect to the j -th explanatory variable. This is a more flexible property than is obtained with the traditional production functions with multiplicative errors, for which the marginal production risk is the same sign (generally positive) as the rate of change of the mean of production with respect to a given explanatory variable.

The technical efficiency of the i -th firm, denoted by TE_i , given the values of the explanatory variables, \mathbf{x}_i , is defined by the ratio of the mean of production for the i -th firm, given the realized value of its firm effect, U_i , associated with the inefficiency of production, to the corresponding mean of production if there were no inefficiency of production [cf. Battese and Coelli (1988, p.389)], i.e.,

$$TE_i = \frac{E(Y_i | \mathbf{x}_i, U_i)}{E(Y_i | \mathbf{x}_i, U_i = 0)}.$$

Given the stochastic frontier model (1), it follows from equation (2) that the technical efficiency of the i -th firm is given by

$$TE_i = 1 - U_i \left[\prod_{k=0}^K x_{1k}^{\beta_k - \alpha_k} \right]. \quad (5)$$

Thus the technical efficiency of the i -th firm, given its levels of factor inputs, is not only a function of its firm effect, U_i , but also of the values of the explanatory variables and the parameters of the production frontier, including the risk parameters (the β s).

If the parameters of the stochastic frontier production function were known, then the best predictor for the technical efficiency (5) is the conditional expectation of TE_i , given the realized values of the random variable $E_i \equiv V_i - U_i$. [cf. Jondrow, et al. (1982), Battese and Coelli (1988)],

$$E(TE_i | E_i) = 1 - E(U_i | E_i) \left[\prod_{k=0}^K x_{1k}^{\beta_k - \alpha_k} \right]. \quad (6)$$

It can be shown that, given the assumptions of the model (1), the conditional distribution of U_i , given that the random variable, E_i , has value, e_i , is defined by the positive truncation of the $N(\mu_1^*, \sigma_*^2)$ distribution, where μ_1^* and σ_*^2 are defined by

$$\mu_1^* = \frac{\mu - e_i \sigma^2}{\sigma^2 + 1} \quad (7)$$

$$\sigma_*^2 = \frac{\sigma^2}{\sigma^2 + 1}. \quad (8)$$

The conditional expectation of U_i , given that E_i has value e_i , can be shown to be

$$E(U_1 | E_1 = e_1) = \mu_1^* + \sigma_* \left[\frac{\phi(\mu_1^*/\sigma_*)}{\Phi(\mu_1^*/\sigma_*)} \right] \quad (9)$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ represent the density and distribution functions for the standard normal random variable.

We consider maximum-likelihood estimation of the parameters of the stochastic frontier model (1). The logarithm of the likelihood function for sample observations on the firms involved is presented in the Appendix, together with the first partial derivatives which are used by the Davidon-Fletcher-Powell algorithm to obtain the maximum-likelihood estimates.

Tests of hypotheses for the model can be obtained using the generalized likelihood-ratio statistic and traditional asymptotic methods. There is particular interest in the null hypothesis, $H_0: \sigma = 0$, which implies that the stochastic frontier production function is identical to the preferred Just and Pope (1978) model, in which technical inefficiencies are assumed not to exist.

The null hypothesis, $H_0: \sigma = 0$, is tested by calculating the generalised likelihood ratio statistic, $\lambda = -2\ln[L(H_0)/L(H_1)]$, where $L(H_0)$ is the likelihood function for the Just and Pope (1978) model, $Y_1 = g(x_1; \alpha) + g(x_1; \beta)V_1$, and $L(H_1)$ is the likelihood function for the flexible risk frontier model, defined by equation (1). The null hypothesis, $H_0: \sigma = 0$, is rejected if the value of λ exceeds the $(1-\alpha)100\%$ value for the chi-square distribution with two degrees of freedom, where α is the desired size of the test. The χ_2^2 -distribution is involved because the distribution of U_1 is specified by two parameters, μ and σ^2 . However, if σ is zero, then the technical inefficiency effects are non-existent and hence, given the specification of the frontier model defined by equation (1), the Just and Pope (1978) model applies.

An operational predictor for the technical efficiency, TE_1 , is obtained by substituting estimators for the parameters involved in the expressions of equations (6)-(9). The predictor for the random variable, e_1 , involved in the conditional mean, μ_1^* , defined by equation (7), is

$$\hat{e}_1 = \left[y_1 - \prod_{k=0}^K x_{1k}^{\hat{\alpha}_k} \right] / \prod_{k=0}^K x_{1k}^{\hat{\beta}_k}$$

where y_1 represents the observed value of production for the i -th firm; and the carets above the parameters denote the appropriate maximum-likelihood estimators of the parameters involved.

3. Empirical Application

The flexible risk stochastic frontier model (1) is applied in the analysis of data obtained from a survey of Ethiopian farmers in 1990. The data were analysed in Kidane and Abler (1994) and provided to the senior author by Professors Abler and Kidane. Although the survey involved data from all administrative regions in Ethiopia, we consider only those data for Central Ethiopia.

The output variable for which data were obtained in the survey is the *value of output* for cropping and livestock enterprises for the farmers concerned. The input variables for which observations were obtained for the sample farmers are *equipment*, as measured by the number of implements (hoes, plows, etc.) used in the farming operations; the number of *cattle* owned and the amount of *land* (in hectares) operated. No data were collected on labour inputs of the sample farmers. Data were also collected on the amount of artificial fertilisers used in the cropping enterprises, but about 53 per cent of the sample farmers in Central Ethiopian used no fertiliser. Hence we consider only the data for those farmers who applied no fertiliser

in the sample year. These are considered to be the most traditional group of the farmers involved in the survey. Data for 447 farmers are involved in this data set.

We use these sample data to estimate the flexible risk stochastic frontier, defined by equation (1), for which the output variable is *value of output* divided by 100^2 and the input variables are *equipment, cattle and land* (i.e., $K = 3$), as defined above.

The maximum likelihood estimates for the parameters of the production frontier are obtained by using a set of procedures from the maximum-likelihood module in the GAUSS system.³ The estimates obtained for the flexible risk frontier and the Just and Pope (1978) model are presented in Table 1. Initial estimates for the α - and β -parameters for the maximum-likelihood routine were obtained using the final maximum-likelihood estimates for the Just and Pope (1978) model. Initial estimates for μ and σ were obtained by maximising the likelihood function (conditional on the α - and β -estimates) over a grid of values of μ and σ , where μ ranged from -3σ to 3σ and σ ranged from 0.1 to 2.0 (by steps of 0.1). In general, the search procedure was quite sensitive to the choice of initial values of the parameters.

² Given that *value of output* is the output variable for the stochastic frontier (1), then the random variable, U_1 , arises from all types of inefficiencies of production, including technical inefficiencies. If all farmers faced the same price structure, then U_1 would measure only technical inefficiencies. Hence U_1 is associated with economic inefficiency of production of the farmers involved.

³ The estimation was programmed in GAUSS 3.1.5 Aptech Systems, Inc.

Table 1: Maximum-likelihood Estimates for Parameters of Flexible Risk Frontier Models for Farmers in Central Ethiopia Who Use No Fertiliser¹

Variable	Parameter	Model 1	Model 2	Model 3
Constant	α_0	0.97 (1.86)	0.41 (1.21)	0.38 (0.17)
Equipment	α_1	0.560 (0.092)	0.52 (0.12)	0.520 (0.093)
Cattle	α_2	-0.006 (0.057)	-0.018 (0.076)	-0.02 (0.11)
Land	α_3	0.46 (0.44)	0.65 (0.46)	0.659 (0.075)
Constant	β_0	-0.15 (2.21)	-0.17 (0.23)	-0.17 (0.46)
Equipment	β_1	0.603 (0.093)	0.611 (0.088)	0.61 (0.15)
Cattle	β_2	0.007 (0.099)	0.019 (0.097)	0.02 (0.18)
Land	β_3	0.202 (0.068)	0.199 (0.073)	0.199 (0.094)
	σ^2	0.0042 (515)	0.062 (2.6)	-
	μ	1.4 (6.3)	0	-
Loglikelihood function		-1218.502	-1218.741	-1218.760

¹ Model 1 refers to the general flexible risk frontier model, defined by equation (1). Model 2 refers to the flexible risk model for which the inefficiency effects have half-normal distribution. Model 3 refers to the Just and Pope (1978) model for which the inefficiency effects, U_1 , are absent.

The values of the loglikelihood function for the three models considered in Table 1 are approximately equal, but increase slightly in value from the Just and Pope (1978) model to the frontier model with $\mu = 0$ to the more general frontier model with μ a parameter to be estimated. The estimated standard errors of the maximum-likelihood estimators for the two stochastic frontier models are obtained by the GAUSS system using the first partial derivatives of the logarithm of the likelihood function. These values are generally quite large relative to the corresponding estimates for the Just and Pope (1978) model, whose estimated standard errors are obtained using the matrix of the second partial derivatives. This is particularly the case for estimation of the constant parameters (α_0 and β_0) and the parameters, σ and μ , associated with the inefficiency effects, U_1 .⁴

Given the specifications of the flexible risk frontier model, estimated under the column headed Model 1, in Table 1, tests of hypotheses that simpler distributional assumptions are adequate are presented in Table 2. The first null hypothesis considered in Table 2, $H_0: \mu = 0$, is that the inefficiency effects in the frontier model have half-normal distribution. The generalized likelihood-ratio statistic, λ , has value 0.48, which is not greater than the 95 per cent point for the χ_1^2 -distribution and so the null hypothesis, $H_0: \mu = 0$, would be accepted.

The second null hypothesis considered in Table 2, $H_0: \sigma = 0$, implies that the inefficiency effects are not present in the model, which then

⁴ Efforts to obtain standard errors of the maximum-likelihood estimators based on the second partial derivatives of the logarithm of the likelihood function, for the two frontier models, were not successful because the Hessian was not invertible.

Table 2: Tests of Hypotheses for Parameters of the Inefficiency Effects
in the Flexible Risk Frontier Model

Null hypothesis H_0	Likelihood LLF	Test Statistic λ	χ^2
$\mu = 0$	-1218.741	0.04	3.84
$\sigma = 0$	-1218.760	0.10	5.99

reduces to the Just and Pope (1978) model. The generalised likelihood-ratio statistic, λ , has value 0.52 which is considerably less than the 95 per cent point for the χ^2_2 -distribution. Hence the hypothesis that the inefficiency effects are absent from the stochastic frontier, given the specifications of the general flexible risk frontier model, would also be accepted.⁵

The estimates for the α - and β -parameters of the production functions are very close across the models considered in Table 1. Since the β -parameters are of particular significance in the flexible risk model involved, we note that the estimates associated with the three explanatory variables, equipment, cattle and land, are positive. This implies that increasing the levels of these inputs is estimated to have an increasing effect on the variance of the value of output, i.e., the three variables have positive marginal risks. The marginal risk associated with cattle is not significantly different from zero.

The estimates for the α -parameters in the model are positive for equipment and land, but negative for cattle, although the latter estimate is not significantly different from zero. The α -parameters associated with the

⁵ This result implies that the technical efficiencies of the sample farmers in Central Ethiopia are equal to one.

explanatory variables are not elasticities under the specifications of the flexible risk frontier model, unless the inefficiency effects are zero.⁶

We seek to compare the fit of the proposed flexible risk frontier model with the more traditional stochastic frontier model of Cobb-Douglas type, defined by

$$Y_i = \left(\prod_{k=0}^K x_{ik}^{\alpha_k} \right) e^{V_i - U_i}, \quad i = 1, 2, \dots, N, \quad (10)$$

where the V_i s are independently and identically distributed $N(0, \sigma_v^2)$ -random errors; and the

U_i s are independently and identically distributed, non-negative truncations of the $N(\mu, \sigma^2)$ -distribution.

Maximum-likelihood estimates for this stochastic frontier model are obtained using the program, FRONTIER, Version 2.0, written by Tim Coelli for estimation of a production frontier model for panel data, for which the technical inefficiency effects are an exponential function of time, see Coelli (1992) and Battese and Coelli (1992). The maximum-likelihood estimation of the parameters in the frontier model (10) involving the three explanatory variables, equipment, cattle and land, are listed in Table 3, where $\sigma_s^2 \equiv \sigma_v^2 + \sigma^2$ and $\gamma = \sigma_v^2 / \sigma_s^2$.

Although the estimates for the elasticity for this stochastic model are

⁶ The elasticity of the expected production with respect to the j -th factor input, conditional on the input variables and the inefficiency effect for the i -th farmer is equal to

$$\frac{\partial E(Y_i | x_i, U_i)}{\partial x_{ij}} \frac{x_{ij}}{E(Y_i | x_i, U_i)} = \frac{\alpha_j - \beta_j \left[\prod_{k=0}^K x_{ik}^{\beta_k - \alpha_k} \right] U_i}{1 - \left[\prod_{k=0}^K x_{ik}^{\beta_k - \alpha_k} \right] U_i}.$$

Table 3: Maximum-Likelihood Estimates for Parameters of the
Cobb-Douglas Stochastic Frontier for Farmers in Central Ethiopia
Who Use No Fertiliser

Variable	Parameter	Estimate
Constant	α_0	0.65 (0.18)
Equipment	α_1	0.510 (0.080)
Cattle	α_2	-0.011 (0.048)
Land	α_3	0.735 (0.052)
Variance Parameters		
	σ_s^2	5.5 (3.6)
	γ	0.970 (0.019)
	μ	-10.5 (8.1)
Loglikelihood function		-1129.155

quite similar to the estimates for the corresponding α -parameters for the flexible risk models estimated in Table 1, the logarithm of the likelihood function of this traditional Cobb-Douglas frontier model is somewhat greater than for any of the flexible risk models.⁷ Further, the traditional Cobb-Douglas frontier model differs significantly from the corresponding

⁷ The logarithm of the likelihood function presented in Table 3 is not the value calculated directly by FRONTIER which gives the logarithm of the likelihood function for the *logarithm* of the output values. The sum of the logarithms of the output values is subtracted from the value given by FRONTIER in order to obtain the appropriate logarithm of the likelihood function to compare the two models.

traditional production function, in which the technical inefficiency effects are assumed to be absent (i.e., $U_1 = 0$ for all farmers). This is evident from the estimate for the ratio parameter, $\gamma = \sigma^2 / (\sigma_v^2 + \sigma^2)$, which has value strictly between zero and one if the technical inefficiency effects are present in the stochastic frontier model. The estimated value of γ , 0.970, is highly significant, given its estimated standard error of 0.019.⁸

Although a formal non-nested test procedure for the two stochastic frontier models is not presented, the results obtained suggest that the flexible risk model is not a good fit for the input-output data for traditional farmers in Central Ethiopia.

4. Conclusions

The stochastic frontier production function estimated in this paper has flexible production risks which are desirable for the analysis of data on different production systems. For the empirical application presented in this paper none of the three input variables involved in the frontier had negative marginal risk. However, it appears that the flexible risk model is not an adequate representation of the data involved. However, the model may prove to be better than the traditional non-flexible risk production frontiers.

⁸ Further results on the estimation of traditional Cobb-Douglas and translog stochastic frontier production functions using the data on the sample farmers from Central Ethiopia are presented in Nsanzugwanko (1994).

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Appendix

Given the assumptions on the random variables, V_1 and U_1 , in the stochastic frontier model (1), it follows that the joint density function of V_1 and U_1 is

$$f_{V_1, U_1}(v_1, u_1) = f_{V_1}(v_1) f_{U_1}(u_1) \\ = \frac{\exp[-1/2(v_1^2)]}{\sqrt{2\pi}} \cdot \frac{\exp[-1/2(u_1 - \mu)^2/\sigma^2]}{\sqrt{2\pi} \sigma \phi(\mu/\sigma)} \quad (A.1)$$

The joint density function for $E_1 \equiv V_1 - U_1$ and U_1 can be shown to be

$$f_{E_1, U_1}(e_1, u_1) = \frac{\exp\{-1/2\{e_1^2 + (\mu/\sigma)^2 - (\mu^*/\sigma_*)^2\} + (u_1 - \mu^*)^2/\sigma_*^2\}}{2\pi \sigma \phi(\mu/\sigma)} \quad (A.2)$$

where μ^* and σ_*^2 are defined by equations (7) and (8), respectively.

Further, the density function for E_1 can be shown to be

$$f_{E_1}(e_1) = \frac{\exp\{-1/2[e_1^2 + (\mu/\sigma)^2 - (\mu^*/\sigma_*)^2]\}}{\sqrt{2\pi} (\sigma^2 + 1)^{1/2} [\phi(\mu/\sigma)/\phi(\mu^*/\sigma_*)]} \quad (A.3)$$

It follows readily from equations (A.2) and (A.3) that the conditional density function for U_1 , given the random variable E_1 has value, e_1 , is

$$f_{U_1|E_1=e_1}(u_1) = \frac{\exp[-1/2(u_1 - \mu^*)^2/\sigma_*^2]}{\sqrt{2\pi} \sigma_* \phi(\mu^*/\sigma_*)} \quad , \quad u_1 > 0 \quad (A.4)$$

The density function for the production of the i -th firm is

$$f_{y_1}(y_1) = \frac{\exp\{-1/2\{(y_1 - \prod_{k=1}^K x_{1k}^{\alpha_k})^2 / [(\prod_{k=0}^K x_{1k}^{\beta_k})^2] + (\mu/\sigma)^2 - (\mu_1^*/\sigma_*)^2\}\}}{\sqrt{2\pi} (\sigma^2 + 1)^{1/2} [\Phi(\mu/\sigma) / \Phi(\mu_1^*/\sigma_*)] \left(\prod_{k=1}^K x_{1k}^{\beta_k} \right)} \quad (\text{A.5})$$

$$\text{where } \mu_1^* = \{\mu - \sigma[(y_1 - \prod_{k=0}^K x_{1k}^{\alpha_k}) / (\prod_{k=1}^K x_{1k}^{\beta_k})] / (\sigma^2 + 1)\} \quad (\text{A.6})$$

and σ_*^2 is as defined by equation (8).

From equation (A.5), it is evident that the logarithm of the likelihood function is given by

$$\begin{aligned} L(\theta; y) = & -\frac{N}{2} [\ln(2\pi) + \ln(\sigma^2 + 1)] - N \ln[\Phi(\mu/\sigma)] \\ & + \sum_{i=1}^N \ln[\Phi(\mu_1^*/\sigma_*)] - \sum_{i=1}^N \sum_{k=0}^K \beta_k \ln x_{ik} \\ & - \frac{1}{2} \sum_{i=1}^N \left\{ \left[(y_1 - \prod_{k=0}^K x_{ik}^{\alpha_k}) / (\prod_{k=0}^K x_{ik}^{\beta_k}) \right]^2 \right\} \\ & - \frac{N}{2} (\mu/\sigma)^2 + \frac{1}{2} \sum_{i=1}^N (\mu_1^*/\sigma_*)^2 \end{aligned} \quad (\text{A.7})$$

where $\theta = (\alpha', \beta', \sigma, \mu)'$.

If the technical inefficiency effects, U_1 , are absent from the frontier model (1), then the Just and Pope (1978) model applies, whose logarithm of the likelihood function is given by

$$\begin{aligned} L(\theta^*; y) = & -\frac{N}{2} \ln(2\pi) - \sum_{i=1}^N \sum_{k=0}^K \beta_k \ln x_{ik} \\ & - \frac{1}{2} \sum_{i=1}^N \left\{ \left[(y_1 - \prod_{k=1}^K x_{ik}^{\alpha_k}) / (\prod_{k=0}^K x_{ik}^{\beta_k}) \right]^2 \right\} \end{aligned} \quad (\text{A.8})$$

and $\theta^* = (\alpha', \beta')'$.

The partial derivatives of the logarithm of the likelihood function (A.7) with respect to the parameters, α , β , σ and μ are as follows:

$$\frac{\partial L^*}{\partial \alpha_j} = \sum_{i=1}^N \left\{ z_i^* + \frac{\phi(z_i^*)}{\Phi(z_i^*)} \right\} \frac{\partial z_i^*}{\partial \alpha_j} + \sum_{i=1}^N \left\{ (\ln x_{ij}) \left(\prod_{k=0}^K x_{ik}^{\alpha_k} \right) \cdot \frac{(y_i - \prod_{k=0}^K x_{ik}^{\alpha_k})}{\left(\prod_{k=0}^K x_{ik}^{\beta_k} \right)^2} \right\} \quad (\text{A.9})$$

where $z_i^* = \mu_1^*/\sigma_*$ and $\frac{\partial z_i^*}{\partial \alpha_j} = (\ln x_{ij}) \left(\prod_{k=0}^K x_{ik}^{\alpha_k} \right) / \left\{ \left(\prod_{k=1}^K x_{ik}^{\beta_k} \right) (\sigma^2 + 1)^{1/2} \right\}$,

$j = 0, 1, 2, \dots, K;$

$$\frac{\partial L^*}{\partial \beta_j} = \sum_{i=1}^N \left\{ z_i^* + \frac{\phi(z_i^*)}{\Phi(z_i^*)} \right\} \frac{\partial z_i^*}{\partial \beta_j} + \sum_{i=1}^N \ln(x_{ij}) + \sum_{i=1}^N \left\{ \ln(x_{ij}) (y_i - \prod_{k=0}^K x_{ik}^{\alpha_k})^2 / \left(\prod_{k=0}^K x_{ik}^{\beta_k} \right)^2 \right\} \quad (\text{A.10})$$

where $\frac{\partial z_i^*}{\partial \beta_j} = \ln(x_{ij}) \left\{ (y_i - \prod_{k=0}^K x_{ik}^{\alpha_k}) / \left(\prod_{k=0}^K x_{ik}^{\beta_k} \right) \right\} (\sigma^2 + 1)^{-1/2}$, $j = 0, 1, \dots, K;$

$$\frac{\partial L^*}{\partial \sigma} = -\frac{N\sigma}{\sigma^2 + 1} + \frac{N}{\sigma} \left[z + \frac{\phi(z)}{\Phi(z)} \right] z + \sum_{i=1}^N \left[z_i^* + \frac{\phi(z_i^*)}{\Phi(z_i^*)} \right] \frac{\partial z_i^*}{\partial \sigma}$$

where $z = \mu/\sigma$ and $\frac{\partial z_i^*}{\partial \sigma} = - \left\{ \frac{\mu(1+2\sigma^2) + \sigma^2 \left[(y_i - \prod_{k=0}^K x_{ik}^{\alpha_k}) / \left(\prod_{k=0}^K x_{ik}^{\beta_k} \right) \right]}{\sigma^2 (\sigma^2 + 1)^{3/2}} \right\}$. (A.11)

$$\frac{\partial L^*}{\partial \mu} = -\frac{N}{\sigma} \left[z + \frac{\phi(z)}{\Phi(z)} \right] + \left[\sigma (\sigma^2 + 1)^{1/2} \right]^{-1} \sum_{i=1}^N \left[z_i^* + \frac{\phi(z_i^*)}{\Phi(z_i^*)} \right] \quad (\text{A.12})$$