Expected Utility Theory: Rest in Peace?

by

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Abstract

From all reports, expected utility theory is dead. The reports are greatly exaggerated. This study makes two modifications which revive expected utility theory. Rather than directly modelling risk preferences by a von Neumann-Morgenstern utility function of wealth, risk preferences and the expected utility of wealth are derived from consumption and investment decisions over time. Rather than using future wealth as the reference point for evaluating risk preferences, current wealth is used instead. The revived theory is both normative and descriptive. It specifies how rational people ought to make decisions under risk and explains the major empirical findings about how people actually make decisions. For example, the Allais Paradox and its variations, preference reversals and framing effects all result from rational decisions by uniformly risk-averse people. Moreover, apparently risk-seeking behaviour can result from risk-averse people with low rates of time preference taking risks to save for the future. The revived theory also shows why eliciting certainty equivalents cannot measure peoples' risk preferences but leads to new procedures for measuring both time and risk preferences. (JEL D81, D91)

Expected Utility Theory: Rest in Peace?

As a decision theory, expected utility theory has distinguished ancestors and has had a long and fruitful life. Now it has been overcome by too many paradoxes and its eulogy is being written in the economics and psychology literature.

The history of decision theories is a history of theoretical explanations followed by paradoxes followed by new explanations (Machina, 1987, Camerer). In the 17th century, Pascal, de Fermat and others proposed the expected value of a lottery as a decision theory. Nicholas Bernoulli provided a counter-example, known as the St. Petersburg Paradox, in which people would never be willing to pay the expected value. Daniel Bernoulli hypothesized what we now call a von-Neumann-Morgenstern utility function to resolve the paradox. In this theory, people evaluate a gamble, not by its expected value, but by their expected utility from the wealth which the gamble provides. Expected utility theory was born. During its lifetime, it has greatly increased our understanding of decisions under risk and has been widely applied to many problems in economics and psychology

Unfortunately, one of the axioms of expected utility theory requires linearity in probabilities. Allais provided the first example in which peoples' choices violate the linearity assumption. To resolve the Allais Paradox and its variations, several researchers (Chew, Fishburn, 1983, Quiggin, Machina, 1982) formulated new decision theories, collectively called non-expected utility models, which are nonlinear in probabilities. At about the same time as the Allais Paradox was resolved, Lichtenstein and Slovic discovered preference reversals. People appear to reverse their preferences by assigning a higher certainty equivalent to one risky alternative and choosing another. Several theories, collectively called nontransitive choice models, have been developed to explain preference reversals (Fishburn, 1991), but perhaps the most prominent is expected regret theory (Bell, Fishburn, 1982, Loomes and Sugden). Also at about this time, Kahneman and Tversky discovered framing effects. People appear to be risk-averse if a problem is framed so that the outcome is a gain and to be risk-seeking if the same outcome is a loss. Prospect theory has been developed to explain different risk preferences for gains and losses (Tversky and Kahneman).

People who violate only the linearity assumption still seem to be rational and the non-expected utility models are worthy successors to expected utility theory. However, people who reverse their preferences or are susceptible to framing effects appear to be irrational. Psychologists are unconcerned and posit expected regret theory and prospect theory as decriptive of how people actually choose rather than normative theories of how people ought to choose. Economists are extremely concerned. The presumption of rational behavior underlies both expected utility and non-expected utility theories, yet even non-expected utility theories are only partially consistent with preference reversals and cannot begin to explain framing effects (Tversky, Slovic and Kahneman, Safra, Segal and Spivak). Studies are now determining which of the new theories predicts better (Harless and Camerer, Hey and Orme) Expected utility theory and its descendents may be the last of their line.

Yet there is hope. Consider a further anomaly which none of the decision theories can explain. Farmers in many lesser-developed countries appear to be very risk averse but talk of living for the present and not worrying about the future Bryan has travelled the world working with farm households and has formulated the following conundrum.

> *Bryan's Conundrum:* Risk preferences are confounded with time preferences. How can you distinguish a person who is afraid to get out of bed in the morning from a person who lays around in bed all day, enjoying the present rather than working and saving for the future?

Farmers are not the only ones who confound time and risk preferences. So do many economists and psychologists. In the United States, for example, most academics invest in TIAA-CREF. The TIAA account provides a guaranteed annuity upon retirement. The CREF account is a high-risk stock market account with potentially high returns. Academic economists and psychologists can allocate their retirement savings to either of the accounts and revise their allocation periodically. In a leaflet accompanying the prospectus, the managers of TIAA-CREF offer the following advise for choosing an allocation:

> "Most experts agree that you should not take too much risk with your pension accumulation. On the other hand, if you don't take *enough* risk, you might not build sufficient assets for a comfortable retirement. So you need to find a risk-reward balance that's comfortable and appropriate for you." (Italics in original.)

Economists and psychologists routinely choose greater risks for greater savings, yet they formulate theories of decisions under risk which are devoid of a savings motive.

The purpose of this article is to revise expected utility theory to incorporate a savings motive and resolve Bryan's Conundrum. Serendipitously, the revised theory resolves the other paradoxes as well. Therefore, it is both normative, describing how people ought to make decisions, and descriptive, explaining more of the empirical evidence than other decision theories.

The revised expected utility theory is based upon a consumption and investment model first constructed by Merton to analyze financial decisions and later generalized by Hertzler to analyze household consumption and production decisions. The theory differs from other decision theories in two fundamental ways. First, there is no utility function for wealth. Instead, the expected utility of wealth is an indirect function derived from time preferences and the utility of consumption. Second, people do not use future wealth as a reference point. Instead, they use current wealth as the reference point for evaluating their risk preferences, forming expectations about the future and making decisions. This preserves three important properties of optimal decisions over time, namely the Markov, martingale and nonanticipating properties. These properties ensure that time is modelled as asymmetric, moving only forward.

I. A Dynamic Decision Theory

As a benchmark, consider a consumer on a fixed salary with no risky investments. A consumer may behave as if they are maximizing the present value of utility over time subject to a budget constraint for the change in wealth.

$$(1)\qquad V(W,s) = \max_{q} \int_{s}^{\infty} e^{-\rho t} \beta (q-\gamma)^{\alpha}\, dt;$$

subject to:

$$dW = (iW - pq + Y)dt.$$

Indirect utility of wealth, V, at initial time s, depends upon an endowment of wealth, W. It results from choosing consumption, q, in each time period, t, to maximize direct utility, $\beta(q-\gamma)^{\alpha}$, discounted at the rate of time preference, ρ. Direct utility is the so-called Stone-Geary function in which consumption is measured against a subsistence level, γ, and α and β are parameters. Wealth increases with investment income at the risk-free rate, i, decreases with expenditures on consumption at price p, and increases with risk-free salary income, Y. For convenience, the time horizon is infinite. This assumption can be relaxed to include a finite time horizon and a utility function for bequests to future generations without substantially changing the results to follow.

Maximizing utility over the entire time horizon is equivalent to maximizing the Hamilton-Jacobi-Bellman equation in each time period.

(2) $$0 = V_t + \max_q \left\{ e^{-\rho t}\beta(q-\gamma)^\alpha + V_W(iW - pq + Y) \right\}.$$

Consumption is chosen to maximize the discounted direct utility of consumption plus the marginal indirect utility of wealth, V_W, multiplied by the change in wealth. The maximized result and the marginal indirect utility of time, V_t, sum to zero.

By assuming the Stone-Geary function for direct utility, the Hamilton-Jacobi-Bellman equation can be integrated into a closed-form solution for indirect utility.

(3) $$V(W,t) = \frac{(1-\alpha)e^{-\rho t}\beta}{p^\alpha}(\rho - \alpha i)^{\alpha-1}\left(\frac{iW - p\gamma + Y}{(1-\alpha)i}\right)^\alpha.$$

Proof is in the Appendix. Indirect utility is a Stone-Geary function of disposable income, $iW - p\gamma + Y$, augmented by a shift term containing the real rate of time preference, $\rho - \alpha i$. The advantage of a closed-form solution is the exact measurement of welfare effects from income changes.

Consumption above subsistence also has a closed form defined by a linear expenditure equation.

(4) $$p(q-\gamma) = (\rho - \alpha i)\left(\frac{iW - p\gamma + Y}{(1-\alpha)i}\right).$$

A fixed proportion of disposable income is expended in each time period with the trade-off between present and future consumption determined by the real rate of time preference.

Next consider a consumer whose income is risky. In addition to choosing consumption, this consumer must choose among risky investments.

(5) $$U(W,s) = \max_{q,R} E\left\{ \int_s^\infty e^{-\rho t}\beta(q-\gamma)^\alpha \, dt \right\};$$

subject to:

$$dW = (iW - pq + (r-i)R)dt + \sigma R dZ.$$

Expected utility, U, is maximized by forming expectations of the future, E, and choosing consumption, q, and investments, R. Risky investments generate income at the rate r and must pay an opportunity rate i because wealth is diverted from risk-free investments. Although risky investments are expected to return the real rate $r - i$, expectations will be in error by $\sigma R dZ$, where σ is a standard deviation and dZ is a Weiner process (Gard pp. 24-25). Squaring the error gives the variance of changes in wealth, $\sigma^2 R^2 dt$.

The Hamilton-Jacobi-Bellman equation now includes a variance term.

$$(6) \qquad 0 = U_t + \max_{q,R}\left\{e^{-\rho t}\beta(q-\gamma)^\alpha + U_W(iW - pq + (r-i)R) + \tfrac{1}{2}U_{WW}\sigma^2 R^2\right\}.$$

Consumption and risky investments are chosen to maximize the discounted utility of consumption plus the marginal utility of wealth, U_W, multiplied by the expected change in wealth, plus one-half the second derivative, U_{WW}, multiplied by the variance of changes in wealth. Marginal utility of wealth is positive and, if the consumer is risk-averse, the second derivative is negative. As before, the maximized result and the marginal utility of time, U_t, sum to zero.

Expected utility also has a closed form solution

$$(7) \qquad U(W,t) = \frac{(1-\alpha)e^{-\rho t}\beta}{p^\alpha}\left(\rho - \alpha\left(i + \tfrac{1}{2}\frac{(r-i)^2}{(1-\alpha)\sigma^2}\right)\right)^{\alpha-1}\left(\frac{iW - p\gamma}{(1-\alpha)i}\right)^\alpha.$$

Proof is in the Appendix. Disposable income, $iW - p\gamma$, no longer includes salary income, but the real rate of time preference, $\rho - \alpha i - \tfrac{1}{2}(\alpha/(1-\alpha))((r-i)^2/\sigma^2)$, now subtracts a risk-adjusted rate of return on investment. As part of this risk-adjusted rate, the expected real rate of return, (r - i), is divided by its coefficient of variation, $\sigma^2 / (r - i)$.

Expected utility in equation (7) can be discussed in two parts. The part containing disposable income is a member of the hyperbolic absolute risk-aversion family of functions and can describe any sort of risk preferences, including all feasible combinations of increasing, constant or decreasing absolute risk aversion and increasing, constant or decreasing relative risk aversion (Merton). The coefficient of absolute risk-aversion measures the degree of curvature of expected utility.

$$(8) \qquad -U_{WW} / U_W = \frac{(1-\alpha)i}{iW - p\gamma}.$$

For a risk-averse consumer with positive disposable income, α is less than one and the coefficient of absolute risk-aversion is positive. The coefficient is a function of observed current wealth. At the beginning of each time period, a consumer takes stock of their wealth, evaluates their risk preferences, forms expectations about the future and makes a decision. In this way, the Markov, martingale and nonanticipating properties (Gard, pp. 25-26, 49, and 41) are preserved and time is asymmetric, moving only forward. In contrast, other decision theories use future wealth as the reference point. For example, the Arrow-Pratt coefficient of risk aversion may be calculated from a von Neumann-Morgenstern utility function which depends upon future wealth. This leads to the following circular logic. Risk preferences cannot be evaluated until future wealth is calculated; future wealth must be calculated from current wealth by adding the gains or losses from the decisions being made; decisions cannot be made until risk preferences are evaluated. Other decision theories resolve this circular logic by assuming risk preferences are evaluated and all decisions are made simultaneously in the future. However in a dynamic model, decisions cannot be made in the future about what to do today and the reference point must be current wealth.

The other part of expected utility in equation (7) contains the real rate of time preference. This part shifts expected utility depending upon a consumer's impatience for current versus future consumption and the riskiness of investments. Traditional expected utility theory shifts expected utility for the riskiness of investments by multiplying the utility of wealth by probabilities. Rank-dependent expected utility theory and other non-expected utility theories shift expected utility by a nonlinear weighting of probabilities (Quiggin). However, these decision theories ascribe all behavior to risk preferences and do not consider time preferences.

As in the previous case without risk, the linear expenditure equation for consumption depends upon disposable income and the real rate of time preference.

$$(9) \qquad p(q-\gamma)=\left(\rho-\alpha\left(i+\tfrac{1}{2}\frac{(r-i)^2}{(1-\alpha)\sigma^2}\right)\right)\left(\frac{iW-p\gamma}{(1-\alpha)i}\right).$$

In this case, however, expenditure can be interpreted as the real rate of time preference divided by the coefficient of absolute risk-aversion.

The demand for risky investments depends upon disposable income, the expected real rate of return and the variance for the rate of return.

$$(10) \qquad R=\frac{(r-i)}{\sigma^2}\left(\frac{iW-p\gamma}{(1-\alpha)i}\right).$$

Alternatively, demand can be interpreted as the inverse of the coefficient of variation divided by the coefficient of absolute risk-aversion. If investments are quite risky, the coefficient of variation will be large and demand will be small. Conversely, if investments are less risky, the coefficient of variation will be small and demand will be large. Investments need not be positive. If forward contracts or futures markets are available, a risk-averse person who expects a negative real rate of return will go "short" and make a negative investment by selling more than they own on the promise of buying in the future.

Most methods for measuring risk preferences elicit people's certainty equivalents. A certainty equivalent is the risk-free income for which indirect utility without risk equals expected utility with risk.

$$(11) \qquad C = \left(\left(1 - \tfrac{1}{2} \cdot \frac{\alpha}{\rho - \alpha i} \cdot \frac{(r-i)^2}{(1-\alpha)\sigma^2} \right)^{\frac{\alpha - 1}{\alpha}} - 1 \right) (iW - p\gamma).$$

Substituting certainty equivalent C to replace risk-free income Y equates indirect utility in equation (3) to expected utility in equation (7). A certainty equivalent depends upon time and risk preferences. Therefore, an elicitation method based solely upon certainty equivalents will confound the two.

Studies to elicit risk preferences usually describe risky alternatives as lotteries having various probabilities of gains or losses. The model here describes risky investments by a stochastic process having an expected return and a variance.

$$(12) \qquad dR = (r - i)R\,dt + \sigma R\,dZ.$$

This stochastic process can be converted to probabilities by specifying its transition density and integrating to find transition probabilities. Its tr ion density is log-normal.

$$(13) \qquad p(t, R_t, \tau, R_\tau) = \left(2\pi\sigma^2(\tau - t)\right)^{-\frac{1}{2}} e^{-\left(\ln R_\tau - \left(\ln R_t + (r-i)(\tau - t) + \frac{1}{2}\sigma^2(\tau - t)\right)\right)^2 / 2\sigma^2(\tau - t)}.$$

Proof is in the Appendix. Transition density, p (not to be confused with the price of consumption) depends upon the current time, t, the current demand for risky investments, R_t, a future time, τ, and the future outcome for risky investments, R_τ. Given current investments, the probability that the future outcome will be less than some constant, say R_c, is the area under the transition density from

-∞ to R_l. The probability that it will be greater than R_l but less than R_u is the area from R_l to R_u, and the probability that it will be greater than R_u is the area from R_u to ∞. These three transition probabilities will be called P_1, P_2 and P_3 for low, medium and high outcomes.

(14) $$P_1 = \int_{-\infty}^{R_l} p(t, R_t, \tau, x)dx;$$

$$P_2 = \int_{R_l}^{R_u} p(t, R_t, \tau, x)dx;$$

$$P_3 = \int_{R_u}^{\infty} p(t, R_t, \tau, x)dx.$$

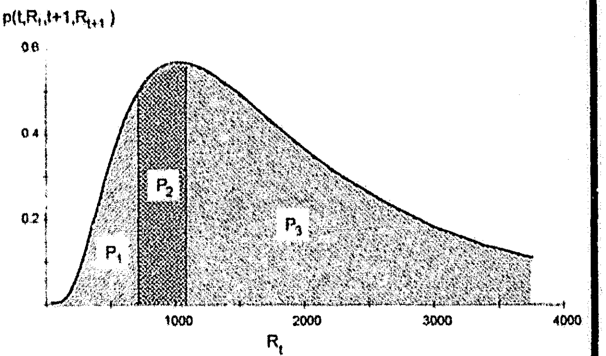

Figure 1. Probabilities 1, 2 and 3 for Low, Medium and High Outcomes.

Transition probabilities are illustrated in Figure 1 with future time τ equal to $t + 1$, the bound R_l equal to 95% of the original investments, R_t, and the bound R_u equal to 145% of the original investments. Event 1 is an outcome on the x-axis of less than 713, Event 3 is an outcome of more than 1088 and Event 2 is an outcome between 713 and 1088. Probabilities for these events are the areas P_1, P_2 and P_3 under the transition density for the log-normal distribution. As areas, they depend upon the mean and variance. A large P_3, for example, results from either a high expected rate of return, which shifts the transition density to the right, or from a large variance, which flattens the peak and thickens the tails of the density.

Using these probabilities, implications of the model can be related to empirical findings in the literature. The model is implemented in Microsoft Excel using the built-in function for the cumulative normal probability and a copy is available from the author. Baseline assumptions for parameters are listed

Table 1. Baseline Parameters for Simulation.

t	0
α	0.6
β	1
γ	100
ρ	0.05
i	0.04
r	0.1
σ^2	0.5
p	1
W	5000
Y	50
R_l	$0.95R$
R_u	$1.45R$
τ	$t + 1$

in Table 1. In particular, people are moderately risk-averse and investments are very risky.

II. Gamblers as Savers

People can be risk-averse but have a strong savings motive and appear to be seeking risks. Binswanger (1980) found that farmers in India have similar degrees of risk aversion but nominate a wide range of certainty equivalents. The explanation is shown in Figure 2 for three people having the same baseline parameters from Table 1 except for different rates of time preference. First, certainty equivalents, C, are calculated from equation (11) and expected returns from risky investments, $(r - i)R$, are calculated as the real rate of return multiplied by the quantity invested from equation (10). Then, ratios of certainty equivalents to expected returns are plotted versus returns from risky investments. Traditionally, if the ratio is greater than one and the certainty-equivalent exceeds the expected return, a person is labelled as risk seeking. If the ratio equals one, a person is labelled as risk-neutral, and if the ratio is less than one, as risk-averse. In Figure 2, the person with a lower rate of time preference ($\rho = 0.0316$) has a ratio of 2.8 but is just as risk averse as the person with an intermediate rate ($\rho = 0.0366$) and a ratio of 1 and as risk-averse as the person with a higher rate ($\rho = 0.0466$) and a ratio of 0.4. Although returns from risky investments are unaffected, certainty equivalents are extremely sensitive to rates of time preference.

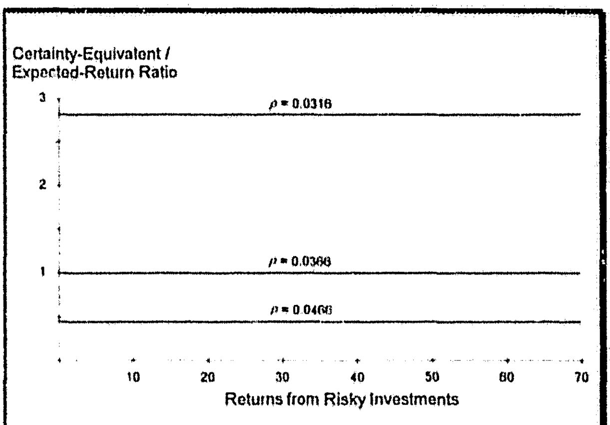

Figure 2. Ratios of Certainty Equivalents to Expected Returns for Three Rates of Time Preference.

Plotting utility in Figure 3 verifies that all three people have the

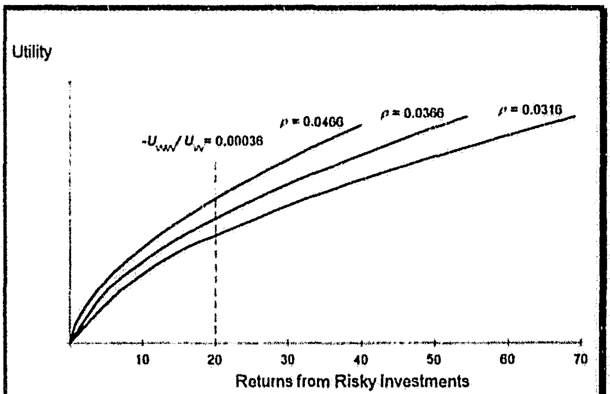

Figure 3. Utility Functions Shifted by Rates of Time Preference.

same degree of risk aversion. Utility is shifted by each person's rate of time preference but, at a given level of returns, has the same degree of curvature. For example, at returns of $20, all three people have a coefficient of absolute risk-aversion, $-U_{ww} / U_w$, equal to 0.00036.

Certainty equivalents can be explained graphically by comparing expected utility with utility. Figure 4 plots utility and expected utility for the person who has a higher rate of time preference. The result is apparently consistent with traditional theory in which expected utility is below a concave utility function and the certainty equivalent is less than the expected returns from risky investments. For example, risky returns of $20 give the same utility as a certainty equivalent of $8, for a ratio of 0.4.

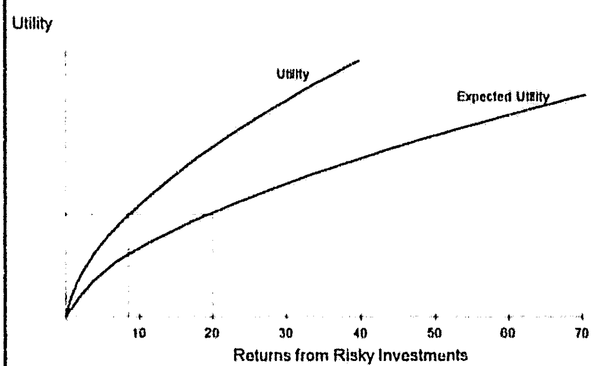

Figure 4. Utility, Expected Utility and Certainty Equivalents for a Higher Rate of Time Preference.

Although not shown graphically, utility coincides with expected utility for the person who has an intermediate rate of time preference. The certainty equivalent always equals expected returns, giving a ratio of 1, even though utility is concave. Figure 5 plots utility and expected utility for the person who has a lower rate of time preference. This result has no counterpart in other decision theories. A lower rate of time preference shifts expected utility above utility and the certainty equivalent exceeds expected returns. In this example, a risky return of $20 has a certainty equivalent of $56, for a ratio of 2.8.

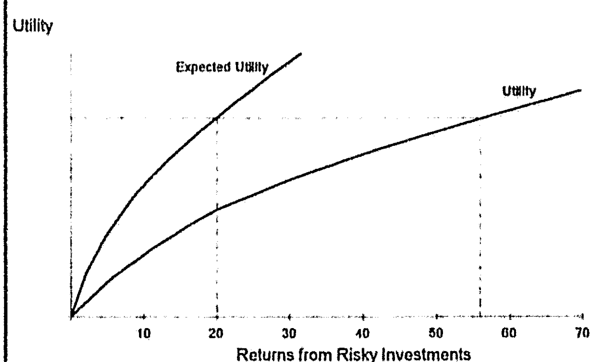

Figure 5. Utility, Expected Utility and Certainty Equivalents for a Lower Rate of Time Preference.

A person may be extremely averse to risk but have a low enough rate of time preference to shift expected utility until the certainty equivalent equals or exceeds expected returns. A desire to save can easily dominate aversion to risk. As a consequence, studies that elicit certainty equivalents confound time and risk preferences and people labelled as risk neutral or risk seeking may simply wish to save for the future.

III. From a Gambler to a Miser

People appear to change their risk preferences as probabilities change. Kachelmeier and Shehata paid high monetary incentives to students in China to accurately elicit their certainty equivalents. Ratios of certainty equivalents to expected returns ranged from around 2 to 4 at low probabilities of winning a lottery but fell to around 1 or less at high probabilities of winning. The results were replicated for low monetary incentives paid to students in Canada and the United States. Non-expected utility theories can explain these results if the same person is interpreted as being a gambler at low probabilities and a miser at high probabilities. An alternative explanation comes from expected utility in equation (7) and the certainty equivalent in equation (11). Both can be nonlinear in the real rate of return, in the variance and, as a consequence, in probabilities, even though people are uniformly averse to risks.

Figure 6 illustrates. For the same risk-averse people as before, ratios of their certainty equivalents to expected returns are graphed versus the probability of winning. The probability of winning is the chance that any money put at risk will remain the same or increase. It is calculated as the transition probability P_3 in equation (14) with the bound R_u set equal to the risky investment, R_r. For a positive real rate of return, the probability of winning will start above 0.5 and increase to 1 as the rate of return increases. In Figure 6,

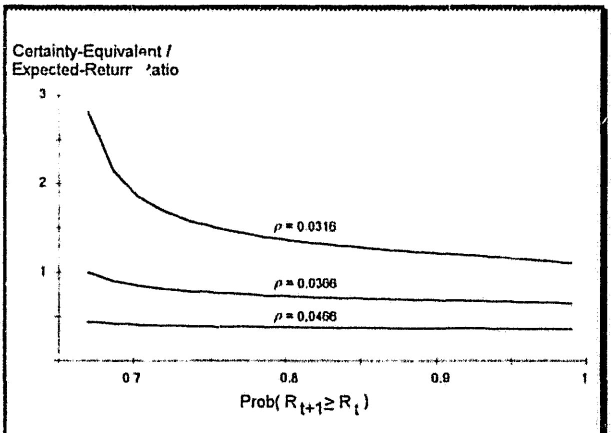

Figure 6. Ratios of Certainty Equivalents to Expected Returns for Increasing Probabilities Winning.

the ratio of certainty equivalents to expected returns falls from 2.8 to just above 1 for the person with a lower rate of time preference. In this case, the ratio is highly nonlinear in probabilities. The ratio starts at 1 and declines gradually for the person with a medium rate of time preference, and starts at 0.4 and stays about constant for the person with the higher rate of time preference. In these cases, the ratios are almost linear. Nonlinearity of the ratios, then, is evidence of a lower rate of time preference, linearity is evidence of a medium to higher rate but neither provides evidence about risk preferences.

Binswanger (1980), followed by Kachelmeier and Shehata found that people appeared to be less risk averse or more risk-seeking when the payoffs from a lottery were lower. Binswanger (1981) attributed the findings to increasing relative risk aversion. For risk aversion to change as the payoff of the lottery changes, people must use future wealth as their reference point in evaluating their risk preferences. This is consistent with other decision theories but inconsistent with dynamically optimal decisions under risk. Kachelmeier and Shehata attributed the apparent differences in risk preferences to the enjoyment of gambling as entertainment which enters directly into peoples' utility. At low payoffs, the marginal utility of gambling becomes more noticeable. Certainly, people do enjoy gambling (Conlisk). The question is whether the marginal utility of gambling is necessary to explain apparently greater risk-seeking for low payoffs.

Figure 7 shows the effect of low payoffs on a person with a low rate of time preference. To model a low payoff, the real rate of return and the standard deviation for the stochastic process of equation (12) are multiplied by a constant, k, which is less than one: $dR = k(r - i)Rdt + k\sigma RdZ$. If a person is restricted to the same investment as with the high payoff, their certainty equivalent in equation (11) is unaffected but expected returns, $k(r - i)R$, will be smaller by the proportion k. In Figure 7, the ratio of certainty equivalents to expected returns is higher at all probabilities if investment is restricted. However, a person would rather choose a larger investment, larger by the proportion $1 / k$, and expected returns, $(r - i)R$, will

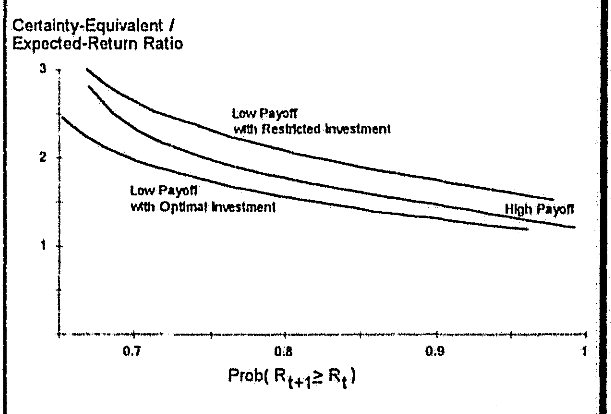

Figure 7. Ratios of Certainty Equivalents to Expected Returns for High and Low Payoffs.

be the same as with a higher payoff. Due to the lower real rate of return, however, the probability of winning is lower and the ratio of certainty equivalents to expected returns in Figure 7 is lower with optimal investment.

Even though time and risk preferences are constant, the ratios of certainty equivalents to expected returns will vary with the payoffs to a lottery. How the ratios will vary depends upon the rules of the lottery. Binswanger (1980) and Kachelmeier and Shehata restricted people's choices and found higher ratios.

IV. Allaying the Allais Paradox

According to traditional theory, people's choices should be linear in probabilities. Figure 8 graphs these indifference curves over probabilities for three possible events. Event 1 has the lowest payoff and a probability shown along the x-axis. Event 3 has the highest payoff and a probability along the y-axis. Event 2 has an intermediate payoff and a probability which equals one minus the sum of Probabilities 1 and 3. Along the hypotenuse of the triangle, for example, Probabilities 1 and 3 sum to one and Probability 2 equals zero. At the origin, Probabilities 1 and 3 equal zero and Probability 2 equals one. Elsewhere, Probability 2 is between zero and one.

In Figure 8, the indifference curves are parallel lines. Moving from right to left, Probability 1 for the low payoff event decreases and expected utility increases. Moving from bottom to top, Probability 3 for the high payoff event increases and expected utility also increases. Expected utility is greatest at the upper left-hand corner of the triangle where Probability 3 equals one and the other probabilities are zero. Inscribed within the probability triangle is a parallelogram with corners A_1, A_2, B_1 and B_2. At point A_1, Event 1 has a probability of 0.05, Event 2 has a probability of 0.95 and Event 3 has a probability of 0. At point A_2, Events 1, 2 and 3 have probabilities 0.1, 0.6 and 0.3.

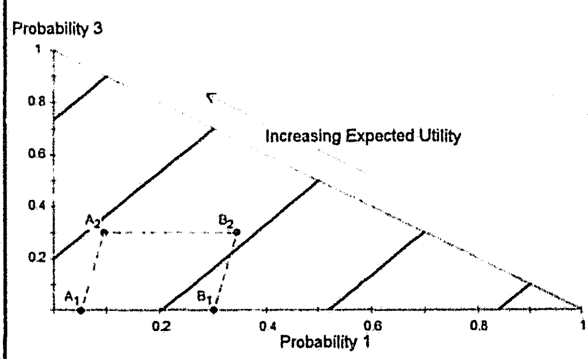

Figure 8. Indifference Curves for Traditional Expected Utility Theory.

Compared to A_1 and A_2, points B_1 and B_2 have larger probabilities for Event 1, smaller probabilities for Event 2 and the same probabilities for Event 3. For the indifference curves in Figure 8, people will prefer point A_2 to A_1 and point B_2 to B_1 along the upper edge of the parallelogram. When asked to choose, however, people tend to prefer points A_1 and B_2 at opposite corners of the parallelogram. This is the Allais Paradox. Variations of the paradox are the common consequence effect, the common ratio effect and the utility evaluation effect.

Machina (1987) offers an explanation for the Allais Paradox in which indifference curves are not parallel, but rather "fan out." In Figure 9, indifference curves are still straight lines but have a shallower slope in the lower right-hand corner of the probability triangle and a steeper slope in the upper left-hand corner. For these indifference curves, people will prefer events at opposite corners of the parallelogram, points A_1 and B_2.

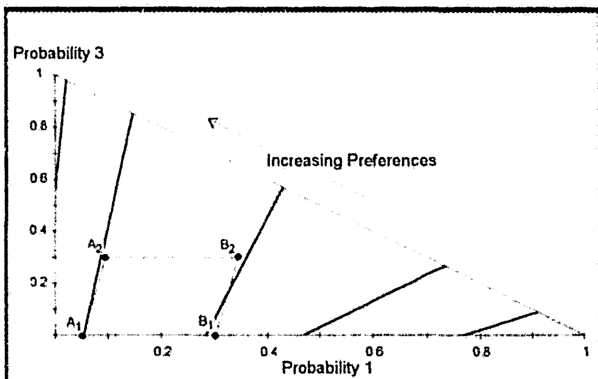

Figure 9. Indifference Curves for Machina's Hypothesis of "Fanning Out".

Empirical evidence suggests that indifference curves are not straight lines. Camerer and Ho graphed the indifference curves for a selection of non-expected utility theories and for prospect theory. Those for prospect theory are very similar to the indifference curves for expected utility in equation (7) of this study. Figure 10 illustrates. Expected utility increases by moving up and to the left and is greatest where Probability 3 approaches 0.8 and Probability 1 approaches zero. People do not prefer to be at the upper left-hand corner of the probability triangle because Event 3 has a high expected return

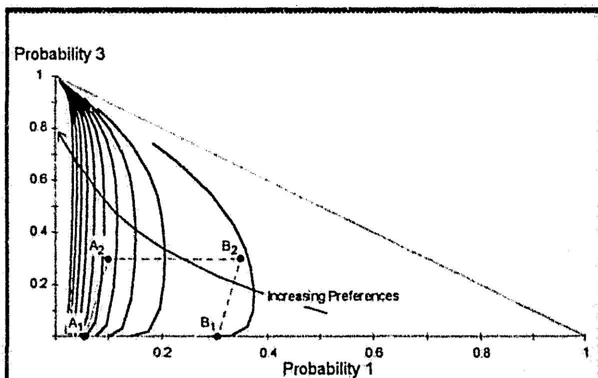

Figure 10. Indifference Curves for Expected Utility in Equation (7).

and a high variance and Event 2 has an intermediate expected return but a low variance. Risk-averse people prefer a combination of the two. The lower right-hand corner is avoided because Event 1 has a low return and a high variance and risk-averse people will never prefer it. Moreover, the indifference curve nearest the center of the probability triangle demarcates desireable risks on the left from undesireable risks on the right. To the left, expected utility exceeds the utility a person could achieve if they took no risks and the certainty equivalent is positive. People are willing to pay for the opportunity to invest. To the right, expected utility is less than the utility a person could have by avoiding risks entirely and the certainty equivalent is negative. People would be willing to pay to avoid these undesireable risks and will invest only if forced to do so.

If asked to nominate certainty equivalents for the desireable risks A_1 and A_2 and for the desireable risks B_1 and B_2, people will tend to nominate higher certainty equivalents for A_1 and B_2 on opposite corners of the parallelogram, in accordance with the Allais Paradox. However, careful distinction must be made between certainty equivalents on one hand and the actual choice among risky alternatives on the other. This topic is discussed in the next section.

V. Reversing Preference Reversals

People may place a higher value on one alternative, yet choose another. Lichtenstein and Slovic presented people with lotteries called the P-bet and the \$-bet. The P-bet had a high probability of an intermediate payoff and the \$-bet had an intermediate probability of a high payoff. When elicited for their certainty equivalents, many people put a higher value on the \$-bet. When asked to choose between bets, they chose the P-bet. By valuing the \$-bet higher but choosing the P-bet it appears they reversed their preferences.

In a dynamic model with consumption and risky investments, people can nominate a higher certainty equivalent for one investment yet choose another without reversing their preferences. A certainty equivalent in equation (11) is affected by choices for both consumption and risky investments and depends upon both time and risk preferences but demand for risky investments in equation (10) depends only upon risk preferences. Figure 11 plots iso-demand curves within the probability triangle. Along an iso-demand curve different combinations of Events 1, 2 and 3 have the same coefficient of variation and result in the same level of investment. Nearest to the center of the probability triangle is the iso-demand curve for zero investment at the demarcation between desireable and undesireable risks. Moving to the left, investment increases and the iso-demand curves become almost linear. Notice that the iso-demand curves are also consistent with the Allais Paradox as discussed in the previous section.

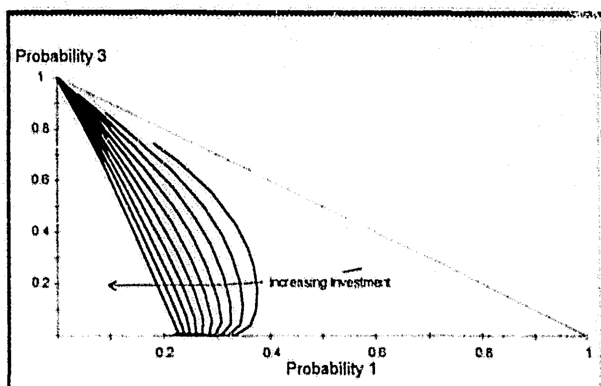

Figure 11. Iso-demand Curves for Risky Investments over Combinations of Events.

Now suppose the P-bet and the $-bet have the following forms.

$$\text{P-bet:}\begin{cases}0.20 \text{ chance of Event } 1\\ 0.65 \text{ chance of Event } 2\\ 0.15 \text{ chance of Event } 3\end{cases} \qquad \text{\$-bet}\begin{cases}0.15 \text{ chance of Event } 1\\ 0.35 \text{ chance of Event } 2\\ 0.50 \text{ chance of Event } 3\end{cases}$$

The P-bet has a large chance of an intermediate payoff and the $-bet has a reasonable chance of a large payoff. In Figure 12, the P-bet is at the intersection of a low indifference curve and a high iso-demand curve. The $-bet is at the intersection of a higher indifference curve but a lower iso-demand curve. A person may be willing to pay a larger certainty equivalent for the opportunity of investing in the $-bet but choose the P-bet by risking more of their wealth on it. Lichtenstein and Slovic also constructed $-bets which had a greater demand for investment than the P-bet. Such a bet is any point to the left of the iso-demand curve and above the indifference curve which intersect at the P-bet. In summary, eliciting a person's certainty equivalent does not simultaneously elicit their choice among risky alternatives because there is no correspondence between the two. A certainty

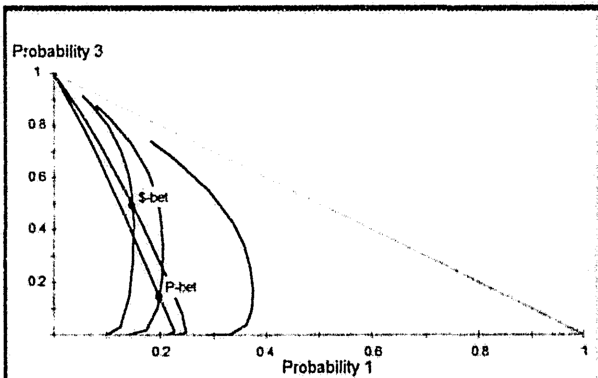

Figure 12. Higher Certainty Equivalent for the $-bet and Greater Investment in the P-bet.

equivalent is an income measure similar to the compensating variation of consumer theory but choices are governed by a demand equation.

VI. Constrained by the Frame

People appear to be risk-averse when expecting a gain and risk-seeking when expecting a loss. Kahneman and Tversky demonstrated this by giving people two decision problems with identical outcomes.

1. In addition to whatever you own, you have been given $1,000. You are now asked to choose between a $^1/_2$: $^1/_2$ chance of a gain of $1,000 or $0 or a sure gain of $500.

2. In addition to whatever you own, you have been given $2,000. You are now asked to choose between a $^1/_2$. $^1/_2$ chance of a loss of $1,000 or $0 or a sure loss of $500.

From the reference point of final wealth, the two decision problems are symmetric. A person who chooses the sure gain in problem 1 and the sure loss in problem 2 will have a final wealth of $1,500. A person who gambles in both problems will have a final wealth of either $1,000 or $2,000. Although the problems have identical outcomes, most people choose the sure gain in the first problem and the gamble in the second. Apparently identical problems lead to different decisions depending upon how they are framed. Tversky and Kahneman postulate that people are risk-averse for gains and risk-seeking for losses and have advanced prospect theory to predict peoples' behavior.

In a dynamic model of decisions under risk, the reference point is not final wealth, but current wealth. Several empirical studies have observed the stability of risk preferences over risky alternatives (Markowitz, Machina, 1982) and are consistent with current wealth as the reference point. People observe their wealth, evaluate their risk preferences, form expectations about the future and make their decisions. From this vantage, decision problems 1 and 2 are not symmetric. Problem 2 contains an element of compulsion which problem 1 does not. To elaborate, suppose people are given a third alternative in each problem. Once the reference point of initial wealth is established, they may choose the gamble or the sure bet, as before, but they may also choose neither. If they choose neither, they simply retain their initial wealth. In problem 1, everyone would choose either the gamble or the sure bet and expect to be better off than at their initial wealth. In problem 2, everyone would choose neither because both the gamble and the sure bet would make them worse off than at their initial wealth. The framing of problem 2 constrains people to take losses which they would otherwise avoid.

From the reference point of initial wealth, problems 1 and 2 are completely different. A decision problem for losses which is symmetric to problem 1 for gains would be as follows:

3 In addition to whatever you own, you have been given \$1,000. You are now asked to choose between a $^1/_2 : ^1/_2$ chance of a loss of \$1,000 or \$0 or a sure gain of \$500. You may hedge the gamble if you wish by going "short" and choosing a negative gamble to convert the loss into a gain.

In a world with risk markets, people can invest positive or negative amounts. If a risk-averse person expects a gain, they will go "long" and make a positive investment. If they expect a loss, they will go "short" and make a negative investment by selling more than they own on the promise of buying in the future. A negative gamble in problem 3 turns the chance of a loss into a positive expected return on investment.

The demand for investment in equation (10) places no restrictions on R, the amount of wealth put at risk. If a risk-averse person expects the real rate of return, $(r - i)$, to be positive, they will invest a positive amount. Otherwise, they will invest a negative amount. Either way, expected return on investment, $(r - i)R$, is positive and the investment is a desireable risk. A person must be constrained to take an undesireable risk on the right-hand side of the probability triangle. The constraint could take many forms but, for illustration, assume a person is constrained to invest an amount which is equal in magnitude but opposite in sign to the amount they would freely choose. This particular constraint allows a closed-from solution in which the coefficient of $+^1/_2$ in equations (7), (9) and (11) is replaced by $-1^1/_2$. Figure 13 shows the indifference curves in the right-hand side of the probability triangle in the region of undesireable risks. As before, expected utility increases from right to left. Figure 14 shows the iso-demand curves. Investment increases from left to right in the direction

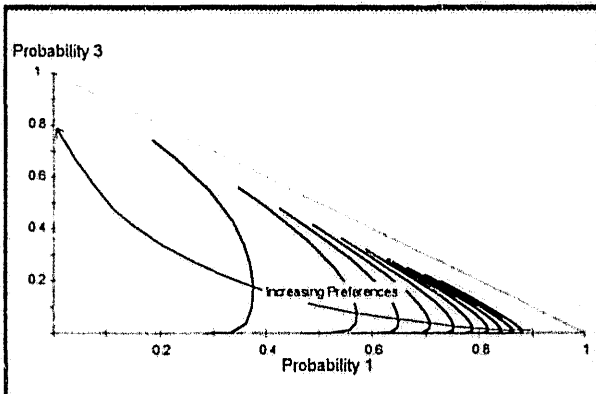

Figure 13. Indifference Curves in the Region of Undesireable Risks.

opposite to preferences. Camerer and Ho measured what they thought were indifference curves but were actually iso-demand curves in the right-hand side of the probability triangle. They found the curves to be concave, similar to those in Figure 14. These iso-demand curves are very different from those for desireable risks and someone who is forced to take a loss will not behave symmetrically.

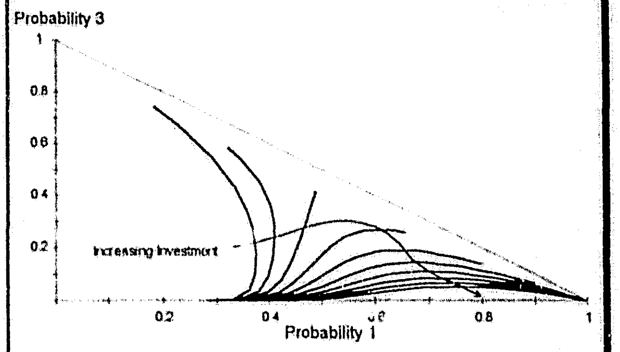

Figure 14. Iso-demand curves in the Region of Undesireable Risks.

Figure 15 illustrates the unconstrained and constrained decision problems. It shows three indifference curves and two iso-demand curves. The iso-demand curves have points C and D on them. As before, the indifference curve near the center of the probability triangle demarcates desireable investments on the left from undesireable investments on the right. A positive certainty equivalent will increase utility from that of the indifference curve near the center to that of the indifference curve on the left. A negative certainty equivalent, which is equal in magnitude, will decrease utility to that of the indifference curve on the right. The iso-demand curve on the left is for a desireable investment, either positive or negative. The iso-demand curve on the right is for a forced investment equal in magnitude but opposite in sign.

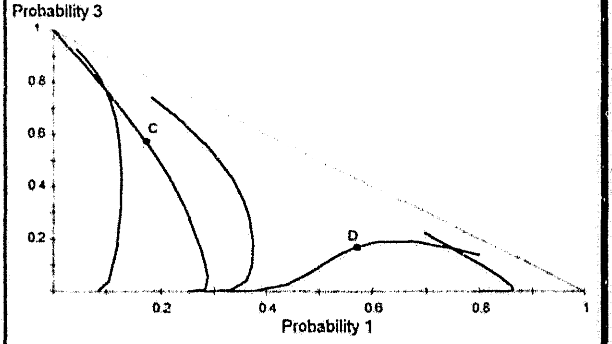

Figure 15. Equal and Opposite Unconstrained and Constrained Investments.

Analogous to problems 1 and 2, the two decision problems for Figure 15 are as follows:

I. Given your current wealth, choose between i) a 17% chance of Event 1, a 26% chance of Event 2 and a 57% chance of Event 3 at point C; ii) a certainty equivalent to reach the indifference curve shown furthest to the left in the triangle; or iii) no change at the indifference curve nearest the center of the probability triangle.

II. Given your current wealth, choose between i) a 57% chance of Event 1, a 26% chance of Event 2 and a 17% chance of Event 3 at point D; ii) a certainty equivalent to reach the indifference curve shown furthest to the right in the triangle; or iii) no change at the indifference curve nearest the center of the probability triangle.

In decision problem I, a risk-averse person may choose a certainty equivalent to reach the highest indifference curve rather than the risky investment at point C. In decision problem II, a risk-averse person may choose no change along the indifference curve nearest the center. If this option is eliminated, they will be forced to choose the risky investment at point D. People need not flip their risk preferences for their decisions to be asymmetric. They may be uniformly averse to risk but make asymmetric decisions because, from the vantage of current wealth, the decision problems are asymmetric.

VII. Resolving Bryan's Conundrum

Eliciting a person's certainty equivalent cannot disentangle their time and risk preferences but eliciting both a certainty equivalent and the demand for investment can. First, a person can be presented with an investment opportunity which has either a positive or negative real rate of return, $(r - i)$, and a variance, σ^2. If it will make the presentation easier, equation (14) can be applied to convert the rate and variance to events with probabilities. Then the person can be asked how much they wish to invest, R. Substituting equation (8) into equation (10) and rearranging allows the calculation of their coefficient of absolute risk-aversion.

$$(15) \qquad -U_{WW} \,/\, U_W = R\frac{\sigma^2}{(r-i)}.$$

A risk-averse person with a positive coefficient will invest a positive or negative amount depending whether the real rate of return is expected to be positive or negative. Either way, a risk-averse person expects to make a gain. Interestingly, a risk-seeking person invests the opposite way and expects to make a loss. Defining behavior in this way suggests that most people are risk-averse and few are truly risk-seeking. Even gamblers may be at the casino for entertainment (Conlisk) and might appreciate better odds. In their study, Kachelmeier and Shehata explained students' behavior as risk-seeking but provide anecdotal evidence that the students were actually risk averse.

Students all chose to participate because they expected to gain. Risk-seeking students would have participated even if they expected to lose.

For the next step in the elicitation, a person could be asked for their disposable income, $iW - p\gamma$. Using information on investments and disposable income, the coefficient α can be calculated after rearranging equation (10).

$$(16) \qquad \alpha = 1 - \frac{(r-i)}{\sigma^2}\left(\frac{iW - p\gamma}{iR}\right).$$

Coefficient α will be less than one for risk-aversion. Finally, a certainty equivalent could be elicitied and the rate of time preference, ρ, calculated after rearranging equation (11).

$$(17) \qquad \rho = \alpha\left(i + \tfrac{1}{2}\frac{(r-i)^2}{(1-\alpha)\sigma^2}\left(1 - \left(\frac{C}{iW - p\gamma} + 1\right)^{\frac{\alpha}{\alpha-1}}\right)^{-1}\right).$$

Alternatively, a person's time and risk preferences could be ascertained from their actual consumption and investments. From information about rates of return, variances of the rates, disposable income and investments, α could be calculated using equation (16). With further information about expenditures on consumption above subsistence, ρ could be calculated after rearranging equation (9).

$$(18) \qquad \rho = p(q-\gamma)\left(\frac{iW - p\gamma}{(1-\alpha)i}\right)^{-1} + \alpha\left(i + \tfrac{1}{2}\frac{(r-i)^2}{(1-\alpha)\sigma^2}\right).$$

Time and risk preferences can be disentangled either by eliciting peoples' choices in experiments or by observing their actual behavior. Either method requires a considerable amount of information but is otherwise straightforward to implement.

VIII. Conclusions

The decision theory proposed in this study is an expected utility theory and, as a consequence, is normative. It's underlying assumptions differ from those of other decision theories in two ways. First, people are assumed to have time and consumption preferences. Their expected utility of wealth and, hence, their risk preferences are derived indirectly from these. Second, the reference point is not final wealth, but current wealth. People observe their current wealth, evaluate their risk preferences, form expectations about the future and make their decisions. They make two decisions, one for consumption and another for risky investments. Although risky investments depend only upon risk preferences, consumption depends upon both time and risk preferences. As a result, expected utility of wealth and the corresponding certainty equivalent also depend upon time and risk preferences.

The expected utility theory in this study is also descriptive because it is consistent with the empirical evidence about people's decisions under risk. Their choices are consistent with the Allais Paradox because their indifference curves and iso-demand curves are nonlinear in probabilities. People do not reverse their preferences when they assign a higher certainty equivalent to a $-bet but then choose the P-bet because there is correspondence between certainty equivalents and choices. A certainty equivalent is an income measure and choices are governed by a demand equation. People do not flip from being risk-averse for gains to being risk-seeking for losses. Although gains and losses may appear symmetric from a reference point of future wealth, from the vantage of current wealth they are always asymmetric. Expected gains are desireable risks; expected losses are undesireable. Risk-averse people may be constrained by the framing of a decision problem to take losses they would otherwise avoid. Finally, further studies could use the theory of this study to design experiments which resolve Bryan's Conundrum and disentangle time preferences from risk preferences.

The bad news is that many empirical studies have elicited certainty equivalents and have incorrectly attributed the results to risk preferences. The good news is that other empirical studies (Binswanger, 1980, Cicchetti and Dubin, Dalal and Arshanapalli) have determined the demand for risky investments and have correctly measured risk preferences. Other good news is that applied decision analysis using such techniques as portfolio theory and mean-variance analysis can continue as before, with the caveat that people may be more averse to risk than previously thought. Further investigations of certainty equivalents and demands for risky investments may reveal that people are significantly averse to risk but take risks anyway to save for the future. The final good news is that the eulogies for expected utility theory are premature.

APPENDIX: PROOFS

Closed-form solutions

For the expected utility function in equation (7), partially differentiate once with respect to time and twice with respect to wealth.

$$U_t = -\rho U.$$

$$U_W = \frac{\alpha e^{-\rho t}\beta}{p^{\alpha}}\left(\rho - \alpha\left(i + \tfrac{1}{2}\frac{(r-i)^2}{(1-\alpha)\sigma^2}\right)\right)^{\alpha-1}\left(\frac{iW - p\gamma}{(1-\alpha)i}\right)^{\alpha-1} = \frac{\alpha i}{iW - p\gamma}U;$$

$$U_{WW} = \frac{\alpha(\alpha-1)i^2}{(iW - p\gamma)^2}U$$

Find the optimality conditions for consumption and risky investments from the Hamilton-Jacobi-Bellman equation in (6).

$$\alpha e^{-\rho t}\beta(q-\gamma)^{\alpha-1} - U_W p = 0;$$

$$U_W(r-i) + U_{WW}\sigma^2 R = 0$$

Solve for q - γ and R and substitute the partial derivatives U_W and U_{WW} to get the demand for consumption above subsistence and the demand for risky investments.

$$(q-\gamma) = \left(\frac{e^{\rho t}U_W p}{\alpha\beta}\right)^{\frac{1}{\alpha-1}} = \frac{1}{p}\left(\rho - \alpha\left(i + \tfrac{1}{2}\frac{(r-i)^2}{(1-\alpha)\sigma^2}\right)\right)\left(\frac{iW - p\gamma}{(1-\alpha)i}\right);$$

$$R = \frac{(r-i)}{(-U_{WW}/U_W)\sigma^2} = \frac{(r-i)}{\sigma^2}\left(\frac{iW - p\gamma}{(1-\alpha)i}\right)$$

Finally, substitute the partial derivatives and the optimal consumption and investment into the right-hand side of the Hamilton-Jacobi-Bellman equation.

$$-\rho U + \frac{e^{-\rho t}\beta}{p^{\alpha}}\left(\rho - \alpha\left(i + \tfrac{1}{2}\frac{(r-i)^2}{(1-\alpha)\sigma^2}\right)\right)^{\alpha}\left(\frac{iW - p\gamma}{(1-\alpha)i}\right)^{\alpha}$$

$$+\frac{\alpha i}{iW - p\gamma}U\left(iW - p\gamma - \left(\rho - \alpha\left(i + \tfrac{1}{2}\frac{(r-i)^2}{(1-\alpha)\sigma^2}\right)\right)\left(\frac{iW - p\gamma}{(1-\alpha)i}\right) + \frac{(r-i)^2}{\sigma^2}\left(\frac{iW - p\gamma}{(1-\alpha)i}\right)\right)$$

$$+\tfrac{1}{2}\frac{\alpha(\alpha-1)i^2}{(iW - p\gamma)^2}U\frac{(r-i)^2}{\sigma^2}\left(\frac{iW - p\gamma}{(1-\alpha)i}\right)^2 \overset{?}{=} 0.$$

Simplifying shows that it equals zero, as required.

$$-\rho U+\frac{1}{(1-\alpha)}\left(\rho-\alpha\left(i+\tfrac{1}{2}\frac{(r-i)^2}{(1-\alpha)\sigma^2}\right)\right)U$$
$$+\alpha i U-\frac{\alpha}{(1-\alpha)}\left(\rho-\alpha\left(i+\tfrac{1}{2}\frac{(r-i)^2}{(1-\alpha)\sigma^2}\right)\right)U+\frac{\alpha}{(1-\alpha)}\cdot\frac{(r-i)^2}{\sigma^2}U$$
$$-\tfrac{1}{2}\frac{\alpha}{(1-\alpha)}\cdot\frac{(r-i)^2}{\sigma^2}U$$
$$=-\left(\rho-\alpha\left(i+\tfrac{1}{2}\frac{(r-i)^2}{(1-\alpha)\sigma^2}\right)\right)U+\frac{(1-\alpha)}{(1-\alpha)}\left(\rho-\alpha\left(i+\tfrac{1}{2}\frac{(r-i)^2}{(1-\alpha)\sigma^2}\right)\right)U=0.$$

Thus, expected utility in equation (7) is a closed form solution which has the coefficient of absolute risk aversion in equation (8) and optimal consumption and investment in equations (9) and (10). As a special case, utility without risk in equation (3) is also a closed-form solution which has optimal consumption is equation (4).

Transition density for risky investments

The transition density in equation (12) must satisfy Kalmogorov's forward, also called the Fokker-Planck, equation (Gard, p. 30).

$$\frac{\partial p}{\partial \tau}+\frac{\partial p}{\partial R_\tau}(r-i)R_\tau-\tfrac{1}{2}\frac{\partial^2 p}{\partial R_\tau^2}\sigma^2 R_\tau^2=0.$$

Partially differentiate the transition density once with respect to the future time and twice with respect to the future value of the risky investment.

$$\frac{\partial p}{\partial \tau}=-\tfrac{1}{2}\frac{p}{(\tau-t)}+\frac{\left(\ln R_\tau-\left(\ln R_t+(r-i)(\tau-t)+\tfrac{1}{2}\sigma^2(\tau-t)\right)\right)\left((r-i)+\tfrac{1}{2}\sigma^2\right)}{\sigma^2(\tau-t)}p$$
$$+\tfrac{1}{2}\frac{\left(\ln R_\tau-\left(\ln R_t+(r-i)(\tau-t)+\tfrac{1}{2}\sigma^2(\tau-t)\right)\right)^2}{\sigma^2(\tau-t)^2}p;$$

$$\frac{\partial p}{\partial R_\tau}=-\frac{\left(\ln R_\tau-\left(\ln R_t+(r-i)(\tau-t)+\tfrac{1}{2}\sigma^2(\tau-t)\right)\right)}{R_\tau\sigma^2(\tau-t)}p;$$

$$\frac{\partial^2 p}{\partial R_\tau^2}=-\frac{p}{R_\tau^2\sigma^2(\tau-t)}+\frac{\left(\ln R_\tau-\left(\ln R_t+(r-i)(\tau-t)+\tfrac{1}{2}\sigma^2(\tau-t)\right)\right)}{R_\tau^2\sigma^2(\tau-t)}p$$
$$+\frac{\left(\ln R_\tau-\left(\ln R_t+(r-i)(\tau-t)+\tfrac{1}{2}\sigma^2(\tau-t)\right)\right)^2}{R_\tau^2\sigma^4(\tau-t)^2}p.$$

Substitute the derivatives into the left-hand side of Kalmogorov's forward equation.

$$-\tfrac{1}{2}\frac{p}{(\tau-t)}+\frac{\left(\ln R_\tau-\left(\ln R_t+(r-i)(\tau-t)+\tfrac{1}{2}\sigma^2(\tau-t)\right)\right)\left((r-i)+\tfrac{1}{2}\sigma^2\right)}{\sigma^2(\tau-t)}p$$

$$+\tfrac{1}{2}\frac{\left(\ln R_\tau-\left(\ln R_t+(r-i)(\tau-t)+\tfrac{1}{2}\sigma^2(\tau-t)\right)\right)^2}{\sigma^2(\tau-t)^2}p$$

$$-\frac{\left(\ln R_\tau-\left(\ln R_t+(r-i)(\tau-t)+\tfrac{1}{2}\sigma^2(\tau-t)\right)\right)}{R_\tau\sigma^2(\tau-t)}p(r-i)R_\tau$$

$$+\tfrac{1}{2}\frac{p}{R_\tau^2\sigma^2(\tau-t)}\sigma^2R_\tau^2-\tfrac{1}{2}\frac{\left(\ln R_\tau-\left(\ln R_t+(r-i)(\tau-t)+\tfrac{1}{2}\sigma^2(\tau-t)\right)\right)}{R_\tau^2\sigma^2(\tau-t)}p\sigma^2R_\tau^2$$

$$-\tfrac{1}{2}\frac{\left(\ln R_\tau-\left(\ln R_t+(r-i)(\tau-t)+\tfrac{1}{2}\sigma^2(\tau-t)\right)\right)^2}{R_\tau^2\sigma^4(\tau-t)^2}p\sigma^2R_\tau^2\overset{?}{=}0.$$

Simplifying shows the Kalmogorov's forward equation to be satisfied.

$$\frac{\partial p}{\partial\tau}+\tfrac{1}{2}\frac{p}{(\tau-t)}-\frac{\left(\ln R_\tau-\left(\ln R_t+(r-i)(\tau-t)+\tfrac{1}{2}\sigma^2(\tau-t)\right)\right)\left((r-i)+\tfrac{1}{2}\sigma^2\right)}{\sigma^2(\tau-t)}p$$

$$-\tfrac{1}{2}\frac{\left(\ln R_\tau-\left(\ln R_t+(r-i)(\tau-t)+\tfrac{1}{2}\sigma^2(\tau-t)\right)\right)^2}{\sigma^2(\tau-t)^2}p$$

$$=\frac{\partial p}{\partial\tau}-\frac{\partial p}{\partial\tau}=0.$$

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