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ESTIMATING EXPONENTIAL UTILITY FUNCTIONS

By Steven T. Buccola and Ben C. French¹

In a recent study for the U.S. Department of Agriculture's Farmer Cooperative Service, we developed a framework in which a processing/marketing cooperative or other firm might evaluate alternative long-term contract provisions for final product sale and raw product purchase. The study focused on a California cooperative fruit and vegetable processor. Selection of alternative contract pricing arrangements for tomato paste sales and tomato purchases was treated as a problem in portfolio analysis.

- The exponential utility function for money has long attracted attention from theorists because it exhibits nonincreasing absolute risk aversion. Also, under certain conditions, it generates an expected utility function that is maximizable in a quadratic program. However, this functional form presents estimation problems. Logarithmic transformation of an exponential utility function does not conform to the Von Neumann-Morgenstern axioms. Hence, it cannot be used as a basis for best fit in statistical analysis. A criterion is described that can be used to select a best-fit exponential utility function, and its application in grower utility analysis is demonstrated.

■ Keywords: Exponential, utility, risk.

Bernoullian utility functions were estimated for a cooperative management spokesman and a board member, and for eight tomato growers, to identify contract portfolios that would maximize expected utility for growers or processors. An important issue in this identification process is the utility functional form employed, since this form influences the expected utility formulation that is the basis for portfolio choice.

In this article, we review some of the questions raised in selecting a utility functional form, suggest a method for fitting exponential forms to utility response data, and discuss several applications of this method.

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SELECTING A UTILITY FUNCTIONAL FORM

Since the development of the Bernoullian money utility function, the issue of its proper functional form has been discussed with no determinate conclusion in sight.² Early theorists and practitioners preferred the quadratic utility function:

$$U = a + bM - cM^2, \quad b, c > 0 \quad (1)$$

where U is utility and M is money, for three reasons. If properly constrained, the function conforms to the risk averter's requirements of a positively sloping, concave function; when combined with linear profit functions, it generates quadratic expected utility functions that are easily maximized with current programming routines; and it is easily fitted by OLS to utility questionnaire data.

Criticism of quadratic forms began with Arrow's and Pratt's identification of a coefficient of absolute risk aversion, $r_a(M) = -U''/U'$. If this coefficient is a declining function of M , the decisionmaker becomes more willing to accept a gamble with fixed probabilities of fixed "small" payoffs as his wealth increases (1, pp. 95-96).³ A rising coefficient implies decreased willingness, and a constant coefficient, unchanged willingness, to adopt this gamble. Intuition suggests that declining risk aversion ought to describe many persons' behavior, but coefficient $R_a(M)$ in quadratic utilities is $2c/(b-2cM)$, which instead rises with M .⁴

Alternative forms that are more acceptable according to the hypothesis of declining absolute risk aversion include the semilogarithmic

$$U = d + g \ln M, \quad g > 0, \quad (2)$$

² Utility functions may refer to money wealth or money profit, where the latter reflect changes from an initial wealth position. Functions discussed in this article may be applied to either wealth or profit. The empirical applications involve profit utilities.

³ Italicized numbers in parentheses refer to items in References at the end of this article.

⁴ Rising absolute risk aversion is consistent with a predilection for hoarding, and hence is not to be ruled out of hand. Besides, one may argue that the data and not researchers' expectations should determine functional form. We do not propose to judge the merits of the hypothesis of decreasing absolute risk aversion.

where $R_a(M) = 1/M$; and the negative inverse exponential (hereafter called exponential)

$$U = K - \Theta \exp[-\lambda M], \quad K, \Theta, \lambda > 0, \quad (3)$$

with $R_a(M) = \lambda$, a constant. In addition, Lin and Chang propose in this issue of the journal a polynomial specification with variable transformations on U or M ; in their article, $R_a(M)$ coefficients depend upon the values taken by transformation constants. Among the more traditional forms, (2) and (3) have not been widely used because they are not, as with the quadratic, associated with a quadratic and thus tractable expected utility function.

This presumably exclusive advantage of the quadratic was, however, undermined as early as 1956. At that time, R. Freund demonstrated that exponential utility, linear profit function, and normally distributed profit generate an expected utility model that is maximizable by operating with an associated quadratic function. Following Wiens' notation, exponential utility (3) and normally distributed profit $M \sim N(\mu, \sigma^2)$ produce expected utility⁵

$$E[U(M)] = K - \Theta \exp[-\lambda\mu + (\lambda^2/2)\sigma^2]. \quad (4)$$

Expression (4) is maximized by minimizing the exponent, a quadratic function. No such tractable solution procedure, other than use of the Taylor expansion with its associated error term, has been offered for the semilogarithmic form. For empirical researchers, this is an important disadvantage which overrides the hypothetical superiority of the semilog's declining absolute risk aversion.

There is no difficulty fitting quadratic or semilogarithmic forms to utility questionnaire data. In the latter case, for example, one merely expresses money values (positive only) in logs and regresses utility observations on these logs. A more complicated issue arises in fitting exponential forms. Treatment of this issue in the current article may be helpful to persons with theoretical objections to increasing risk aversion and with preference for conveniently maximizable expected utility.

⁵ Expected utility (4) is computed by appealing to the primitive form

$$E[U(M)] = \int_{-\infty}^{\infty} (K - \Theta \exp[-\lambda M]) \left\{ \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-(M-\mu)^2/2\sigma^2] \right\} dM.$$

Terms in the integral are combined and the square completed in the resultant exponent. The indicated form (4) then emerges upon appropriate cancellations.

THEORY OF ESTIMATING THE EXPONENTIAL PARAMETER

In general only utility parameters encountered as coefficients of income probability moments in an expected utility model have ultimate importance to the decision theorist or researcher. This observation may be inferred from the fact that the maximized expected utility model is the hypothesized basis of choice under risk, and that, under known programming methods, only coefficients of income probability moments affect optimal variable levels in these models. Since neither K nor Θ are coefficients of probability moments μ, σ^2 in (4), they are in themselves irrelevant to decisionmaking. Conversely, a decision is uniquely determined once λ, μ , and σ^2 are known.⁶

It would seem reasonable that a regression approach to estimating λ in (3) would first require λ 's removal from the exponent. Experience with Cobb-Douglas and other variable exponent functions suggests expressing (3) in log form to accomplish this. Subtracting K from both sides of (3),

$$U - K = -\Theta \exp[-\lambda M]. \quad (3)'$$

Taking natural logs of both sides,

$$\ln(U - K) = \ln(-\Theta \exp[-\lambda M]). \quad (5)$$

If utility in (3) is positive, $K > \Theta \exp[-\lambda M]$, so that $K > U$. Thus $(U - K) < 0$, $\ln(U - K)$ does not exist, and (5) cannot be estimated. However, multiplying (3)' by -1 ,

$$-U + K = \Theta \exp[-\lambda M] \quad (3)''$$

and

$$\ln(-U + K) = \ln \Theta - \lambda M \quad (5)'$$

Equation (5)' implies that λ is the negative of the observed coefficient of money if the natural log of $(-U + K)$ is regressed against money. Parameter Θ is

⁶ Lambda's sole importance for decisionmaking purposes does not rest on its status as the exponential utility's $R_a(M)$ coefficient. Under semilogarithmic utility (2) and normally distributed income $M \sim N(\mu, \sigma^2)$, expected utility is, by reference to the Taylor expansion, approximately $E[U(M)] = g \ln \mu - (g\sigma^2)/(2\mu^2)$. Here, a decision is uniquely determined when g, μ , and σ^2 are known, but g does not appear in the semilog utility's absolute risk aversion coefficient $1/M$.

found as the constant term's antilog. K must be determined in advance and it is equivalent to an additive adjustment to the original utility scale.

Because of the logarithmic transformation on $(-U+K)$, the estimated values of both Θ and λ depend upon the originally chosen utility scale U or the additive adjustment factor K . Presumably, a unique value of λ reflects the decisionmaker's true aversion to risk at interview time.⁷ One procedure for finding this value is to alter the utility scale or K and sequentially fit (5)' under each set of altered values, with the fit yielding the highest R^2 providing the best estimate of λ . This would appear to satisfy a "best-fit" criterion for selecting λ and remove the arbitrariness of utility scale selection.

The immediate difficulty with this procedure is that (5)' represents a nonlinear, though monotonic, transformation on (3)'. Von Neumann and Morgenstern have shown that the uniqueness of a utility function is preserved only under linear transformations $U^* = \alpha + \beta U$, where α and β are constants (5, pp. 24-25). More specifically, the value of λ providing the best fit to (5)' is not necessarily that providing the best fit to (3); furthermore if (3) conforms to the axioms of Bernoullian utility, (5)' does not.

The prohibition against nonlinear utility transformations tells us we cannot rely on (5)' as a specification for selecting a best-fit λ in (3). Goodness of fit must refer to equation (3) or a linear transformation of (3). This does not imply that a regression approach to estimating λ is futile. The following procedure, for example, might be used: (a) assign arbitrary values to K , Θ , λ in (3); (b) generate values of U for each of the money levels employed in the original utility questionnaire; (c) calculate vertical deviations of predicted points from those obtained in the interview; (d) sum the squares of these deviations, and select the set K , Θ , λ minimizing this sum.

In connection with the method suggested, note that K and Θ delimit the utility range where money income is positive. If $M = 0$, $U = K$ minus Θ ; and as $M \rightarrow \infty$, $U \rightarrow K$. The utility range of positive income M is Θ . Thus K , Θ merely serve to accommodate the original utility scale selected. Candidates $(K$ minus $\Theta)$ should be chosen so as to approximate the utility intercept as estimated from a look at the utility questionnaire plot, and K 's

⁷The requirement of a unique λ value derives from the one-to-one correspondence it bears to the optimal quadratic program $\max Z = -\lambda\mu + (\lambda^2/2)\sigma^2$, which in turn determines the exponential decisionmaker's maximum expected utility course of action. No such uniqueness is required of the quadratic utilist's coefficients b , c in (1). By reference to the Taylor expansion, the latter individual's expected utility is $E\{U(M)\} = b\mu - c(\mu^2 + \sigma^2)$, the optimal variables of which depend only upon the ratio c/b . Thus quadratic utility estimation involves discovering a best-fit ratio c/b only.

should be chosen so as to fall "somewhat" above the highest utility value assigned.

Steps (a) through (d) above essentially involve exploring the $S(K, \Theta, \lambda)$ response surface, where S is sum squared errors about the exponential fit. The intent is to discover the globally minimum value of S . Several procedures have, in the general nonlinear case, been proposed for finding this minimum value that do not require full factorial exploration of the response surface. These include utilization of linear Taylor series expansions of the nonlinear function, and methods of following the steepest negative gradient on the S surface. Draper and Smith note that such procedures are likely to converge slowly for exponential functions, which generally exhibit elongated or "ill-conditioned" equi- S ellipses in Θ and λ space (2, p. 284).

An alternative which avoids both full factorial exploration and multiparameter search procedures is to employ log specification (5)' in conjunction with steps (b) through (d) outlined above. The researcher need only select trial values of K , and for each value regress $\ln(-U+K)$ on M as indicated by (5)'. Calculated values Θ and λ are then substituted, along with associated K levels, into (3) and steps (b) through (d) are followed as described. Prior incorporation of K into the utility scale assures that associated sets Θ and λ will generate a function falling at least roughly within range of the original utility questionnaire responses. $S(K, \Theta, \lambda)$ surface exploration reduces to a single-dimension search since trial values Θ and λ are uniquely related to trial values K .

TWO GROWER UTILITY FUNCTIONS

Illustrations of the method presented above are provided by two of the grower utility functions estimated in the cooperative processor study. Table 1 shows growers' utility responses to a Von Neumann-Morgenstern type of questionnaire, in which dollar values refer

Table 1—Utility questionnaire responses of growers 1 and 2

Utility	Money values, grower 1	Money values, grower 2
Utilities	Thousand dollars	
100	700	1000
80	300	-25
60	200	-62
40	-50	-100
20	-150	-300
0	-300	-500

to prospective annual incomes. The patience of most respondents limited risk responses to four, which provided six data points. Regressions were fitted to these data according to specification (5)' and selected K values. The results are summarized in table 2 and plotted in figures 1 and 2.

Table 2—Coefficient estimates of grower exponential money utility function $U = K - \Theta \exp [-\lambda M]$

K	Θ	λ	$R^2(\log)^1$	SSE ²
<i>Grower 1</i>				
200	153.6	.000708	.978	131.78
160	117.3	.001002	.984	93.37
120	74.0	.001819	.984	150.57
101	46.2	.004458	.882	6,283.12
<i>Grower 2</i>				
200	146.0	.000443	.795	1,450.54
160	104.5	.000635	.818	1,335.23
120	60.0	.001194	.879	975.41
101	27.7	.003240	.966	2,227.19

¹These R^2 squares correspond to log fits (5)'. ²These sum squared errors correspond to original data fits (3).

Grower 1's responses well approximate an exponential shape, and relative goodness of fit among competing parameter values is slight. Parameter set $K = 120$, $\Theta = 74.0$, and $\lambda = .001819$ has the highest R^2 under log specification (5)'; but set $K = 160$, $\Theta = 117.3$, and $\lambda = .001002$ generates the least sum squared errors under original exponential specification (3). Hence, $\lambda = .001002$ is our best estimate of grower 1's risk aversion coefficient if utility is exponential.

Grower 2's responses do not well approximate an exponential fit but more nearly suggest a cubic shape. However, one's philosophical commitment to the hypothesis of nonincreasing absolute risk aversion, or the structure of the expected utility model which implements the utility information, may justify exponential estimation. The discrepancy in goodness-of-fit ranking between specifications (5)' and (3) is more marked here than in the first case. Set $K = 101$, $\Theta = 27.7$, and $\lambda = .003240$ provides the highest log fit R^2 , but it behaves poorest of the four alternatives as an approximation to the original data. Sum squared errors to the original data are minimized by set $K = 120$, $\Theta = 60$, $\lambda = .001194$, so that .001194 is our best estimate of grower 2's risk aversion coefficient if utility is exponential.

By way of comparison, quadratic forms were also fit to the utility response data in table 1. For grower

1, $U = 43.04 + .1265 M - .000064 M^2$; and for grower 2, $U = 66.13 + .1069 M - .0000725 M^2$ (all coefficients significant at the .01 confidence level). In both cases the quadratic function is more concave than the corresponding best-fit exponential function. As money values increase, the quadratic approaches the exponential from below, crosses it, then approaches the exponential again at high money values. In each case, coefficients of absolute risk aversion $R_a(M)$ under the quadratic specification are, below the point of intersection, lower than under exponential specification (λ itself). The coefficients are equal at or near intersection, and the quadratic's coefficient rises above the exponential's beyond the point of intersection. With grower 1, for example, $R_a(M)$ under quadratic specification is .000841 at minus \$200,000; .001066 at minus \$50,000 (the intersection point); and .001269 at plus \$200,000. At point plus \$500,000, the quadratic's $R_a(M)$ has risen to .002048, approximately double its value at intersection point and double the exponential parameter (.001002). In a research context, much of choice behavior under risk is determined by the absolute risk aversion coefficient. Thus researchers need to be wary of not only the utility functional form employed but also the feasible expected profit range of the set of risky ventures considered. In the current study, exponential and quadratic forms predicted similar choice behavior for expected profit ranges near the intersections of these functions, but highly divergent behavior elsewhere.

PROPERTIES OF THE ESTIMATOR

The method developed here for estimating the parameter λ of an exponential utility function minimizes sum squared errors. Hence it is a maximum likelihood estimator if utility response deviations about the regression line have zero mean and constant variance and if they are independently and normally distributed.⁸ On these assumptions, therefore, the estimator is asymptotically unbiased and efficient. However no evidence exists that it is unbiased in small samples such as those employed in this study.

Functions such as (5)' estimated under a log dependent variable develop concavity under the original, linear dependent variable scale (here, $-U + K$). This shape results because the first derivative of log values with respect to original values decreases as the original values themselves increase. However, the rate of decrease in the first derivative of log functions declines as

⁸ This hypothesis is proven for the general nonlinear case by observing that the likelihood function is a negative function of sum squared residuals (2, p. 265).

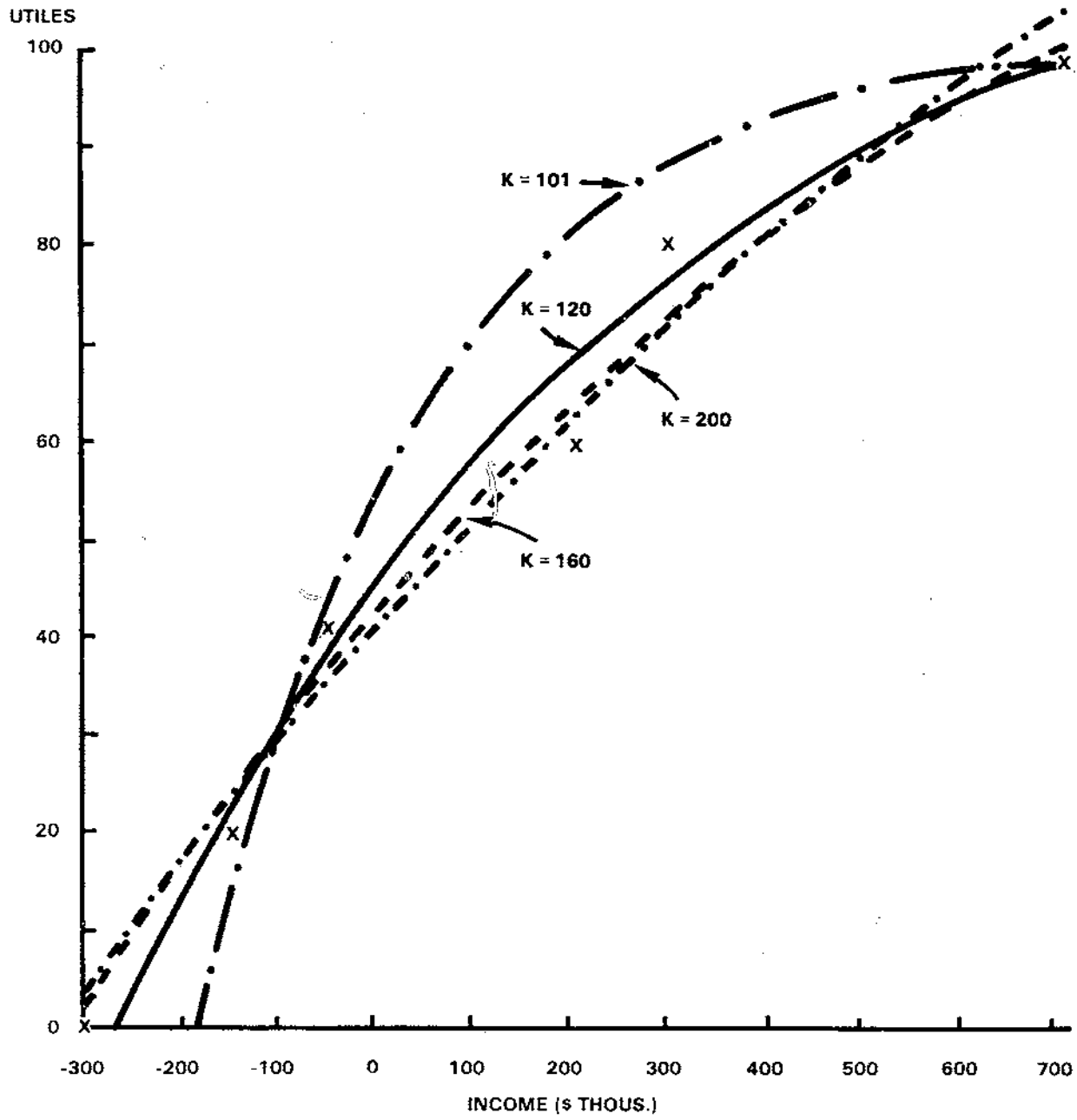


FIGURE 1. EXPONENTIAL FITS
TO UTILITY RESPONSES OF GROWER 1

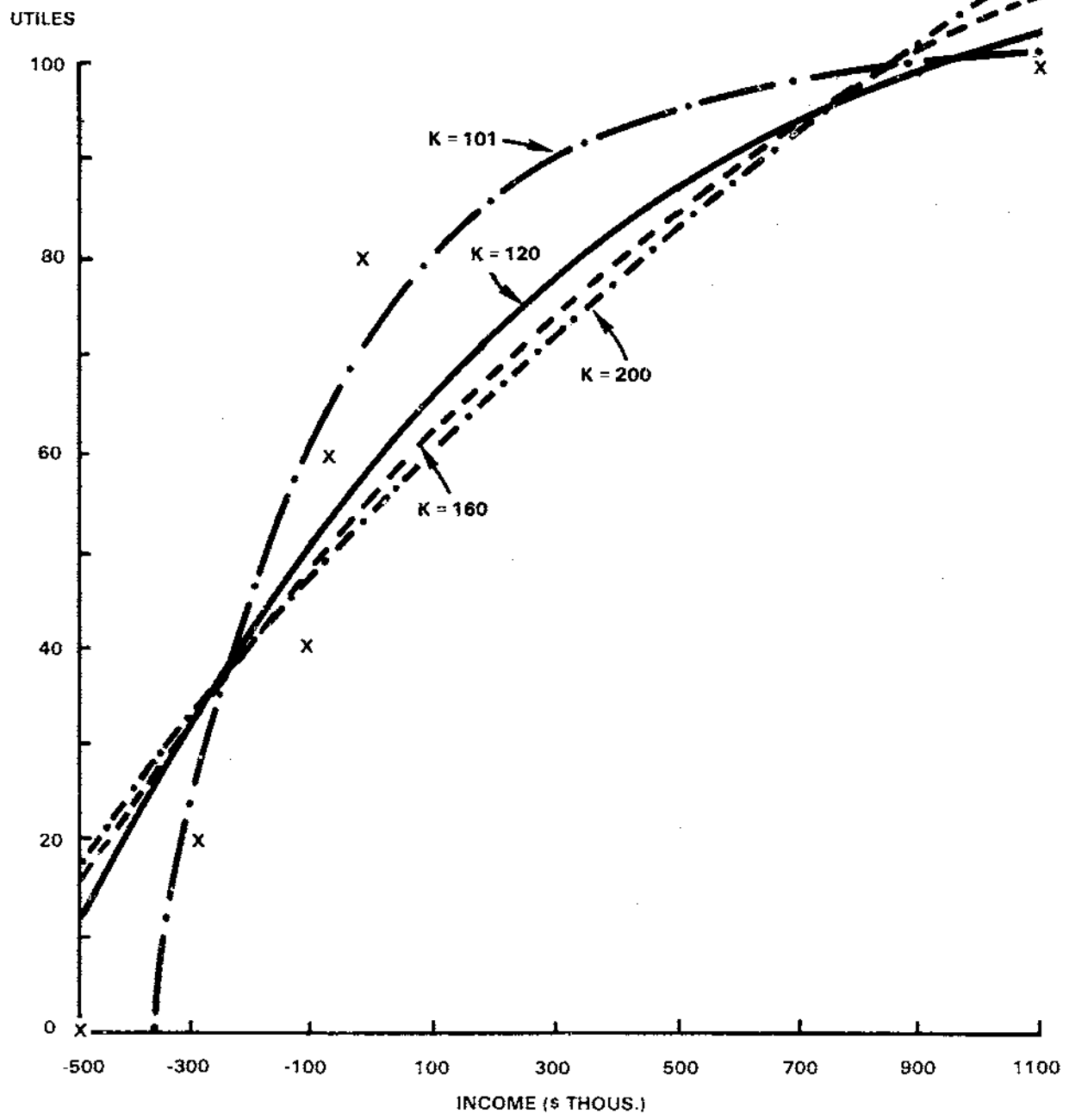


FIGURE 2. EXPONENTIAL FITS
TO UTILITY RESPONSES OF GROWER 2

larger numbers with constant differences are employed in the original scale.* Thus, larger values of K reduce concavity in functions estimated according to this procedure. Identical K values in the cases illustrated here generate identical dependent variable series and, hence, highly similar exponential functions.

To ensure that exponential utility estimates such as these do not depend upon the arbitrary utility scale U chosen, we note that any linear transformation $U^* = \alpha + \beta U$ on (3) produces

$$\begin{aligned} U^* &= \alpha + \beta U = \alpha + \beta (K - \Theta \exp [-\lambda M]) \\ &= \alpha + \beta K - \beta \Theta \exp [-\lambda M] \\ &= (\alpha + \beta K) - (\beta \Theta) \exp [-\lambda M] \end{aligned}$$

*The second derivative of a natural log function $y = \ln x$ is $-1/x^2$, the negative sign of which indicates the decreasing first derivative of y and the reciprocal $1/x^2$, the sign of which indicates the decreasing rate of this decrease.

K changes to $(\alpha + \beta K)$ and Θ to $(\beta \Theta)$, but λ is unaffected. The independence of λ to such utility scale changes is only maintained if (3) rather than (5) is utilized as a goodness-of-fit criterion.

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In Earlier Issues

"... Interest in land classification for the purposes of tax assessment has been a subject of recurring importance... the greatest interest and activity in this method of improving property-tax assessments has been centered in areas in which agricultural land accounts for a large part of total real estate values... Early attempts at classification netted little in the way of permanent improvement... local assessors usually classified the land or shifted this responsibility to the individual owner... Real progress has been made in recent years in assembling information on soil capabilities and records of farm production... Attempts to classify land on the basis of its use and annual average productivity should result in some general improvement in farm real estate assessments... New and unexplored possibilities for improving tax assessments on farm property appear to lie not in the direction of more accurate classification of land but in application of the concept of an income-producing entity to the farm.

Samuel L. Crockett
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