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# AN APPROACH TO UNIQUE SOLUTIONS FOR CONSTANT-ELASTICITY COMMODITY MODELS

By Gerald E. Plato\*

Unique solutions are more difficult to guarantee for commodity models that have nonlinear simultaneous equations than for those with linear ones. The nonlinear case requires determination of uniqueness before a solution is attempted while uniqueness in the linear case is determined as a byproduct of the solution procedure. Unique solutions are important because they are necessary for unambiguous results (that is, results that can always be duplicated). This article explains an approach for guaranteeing unique solutions for commodity models specified with a nonlinear equation type often used in economics, the constant-elasticity equation. This choice allows researchers the option of using secondary data sources (parameter estimates) in developing commodity models.

**Keywords:** Nonlinear commodity models, unique solutions, Newton's method.

## INTRODUCTION

An ongoing task for economists is to explain the behavior of market prices and quantities, and to forecast and project them. The theory of general equilibrium, involving equations that simultaneously determine many prices and quantities, has been available for aiding in these functions since the 1870's. Shortcuts of obtaining the necessary data for these equations are now available for empirical applications (9, 6).<sup>1</sup> Also, high-speed computers are now available and can be used in conjunction with trial and error methods derived from those of Gauss and Newton for solving complex nonlinear models (8, 10). Thus, general equilibrium theory and recent technological advances provide researchers willing to assume constant elasticities the option of using secondary data sources in developing models that simultaneously determine many commodity prices and quantities.

These trial and error procedures do not guarantee unique solutions. Uniqueness has been a long-time concern in general equilibrium theory. It is well recognized that a system of excess demand (demand minus supply) equations describing one less than the total collection of markets has an equal number of independent equations and unknowns. Further, this equality is not sufficient to guarantee a unique equilibrium solution (7, p. 350; 11, p. 160). Recently, a number of economists have devel-

oped sufficient conditions for uniqueness of the general equilibrium model. Results provided by Arrow and Hahn (4) in this effort were used in developing an approach for guaranteeing unique solutions for a commodity projections component model. This model is specified with simultaneous and constant-elasticity equations. It is used in ERS to make commodity projections and it is part of the National Interregional Agricultural Projections Model (NIRAP) (5, p. 47).

The purpose of this article is to explain the approach used to guarantee unique solutions for this commodity projection component model. The first step in the approach involves a discussion of the determination of uniqueness in commodity models with linear simultaneous equations. Next, the use of a trial and error procedure, the Newton algorithm, in solving nonlinear commodity models is explained and its use for solving linear commodity models is discussed as being a special case. Finally, uniqueness in the linear case is extended to the commodity projections component model. This extension is based upon calculations used in finding an equilibrium solution by the Newton algorithm.

## LINEAR COMMODITY MODELS

Let the following equation represent a commodity model containing  $N$  commodities specified with linear-simultaneous equations:

$$BP + \Gamma X = 0. \quad (1)$$

The model is shown in the form of excess-demand (demand minus supply) equations.  $B$  is an  $N$  by  $N$  coefficient matrix for the commodity prices in the  $N$  by  $1$   $P$  vector.  $\Gamma$  is an  $N$  by  $K$  coefficient matrix for the exogenous variables in the  $K$  by  $1$   $X$  vector.<sup>2</sup> If the  $B$  matrix is nonsingular, the solution can be found by equation 2 below:

$$P = -B^{-1} \Gamma X. \quad (2)$$

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<sup>1</sup>Italicized numbers in parentheses refer to items in References at the end of this article.

<sup>2</sup>Since the model is not concerned with all possible commodities, the  $N$ th + 1 commodity is considered to be the numeraire or an aggregate of all the commodities omitted.

In the linear case, trial and error procedures are not required; one iteration produces the equilibrium solution. Also, if B is nonsingular, the solution is unique.<sup>3</sup> However, obtaining uniqueness like stability is more elusive for nonlinear models.

### SOLUTION PROCEDURE FOR NONLINEAR COMMODITY MODELS

The Newton algorithm can be used to solve nonlinear commodity models.<sup>4</sup> The algorithm proceeds by finding successive improvements in an initial guess of an equilibrium price vector.<sup>5</sup> An improvement is calculated by solving for P minus P<sub>0</sub> or Δ P in the following set of equations:

$$f(P_0, X) + B(P_0, X) (P - P_0) = 0 \quad (3)$$

where:

$f(P_0, X)$  equals the vector of commodity excess demands at the initial guess of equilibrium prices, P<sub>0</sub>, or at the prices calculated in the previous iteration, X represents the exogenous variables at preselected levels;

$B(P_0, X)$  is an N by N matrix of first partial derivatives of excess demand equations with respect to commodity prices evaluated at initial guesses of the equilibrium prices or at the prices calculated in the previous iteration and at the preselected levels of the exogenous variables;

and P equals the "improved" vector of commodity prices.

Equation (3) is based on the Taylor series expansion about the price vector P<sub>0</sub>. This expansion for the excess demand equations is shown below in equation (4):

$$\begin{aligned} f(P, X) &= f(P_0 + \Delta P, X) \\ &= f(P_0, X) + f'(P_0, X) (P - P_0) \\ &\quad + f''(P_0, X) (P - P_0)^2 + f'''(P_0, X) (P - P_0)^3 \\ &\quad + \dots \\ &= 0 \end{aligned} \quad (4)$$

<sup>3</sup> If equilibrium quantities are desired, they can be found by substituting the solution (equilibrium) prices into the commodity supply or demand equations.

<sup>4</sup> This method has been widely used in economics to solve nonlinear models (1, 2, 3, 8, 14).

<sup>5</sup> As with linear commodity models, the levels of the exogenous variables determine the equilibrium price vectors. For nonlinear models, a set of values for the exogenous variables may determine more than one solution vector.

The vector P of improved prices in equation (4) is assumed to be the equilibrium price vector (that is, the price vector that makes all the excess demand equations equal to zero). For equation (3), all squared and higher order terms shown in equation (4) are assumed equal to zero.

Solving the linear system in equation (3) would give a vector of prices that would eliminate excess demands if the system of equations followed the linear tangents rather than the underlying nonlinear curves. However, since the excess demand curves are nonlinear, the P vector of commodity prices that is calculated may not be close enough to an equilibrium solution. In fact the Newton algorithm may fail to move the commodity prices closer to equilibrium. If failure occurs, the price changes can be dampened by an arbitrary proportion or new starting values for the endogenous variables (i.e., the commodity prices) can be selected. When "close" to an equilibrium solution, the effects of the squared and higher order terms in equation (4) become small. The linear tangents then sufficiently approximate the underlying nonlinear curves and the Newton algorithm can proceed unaided to achieve the desired closeness to equilibrium. Prior to the second and following iterations by equation (3), vector P<sub>0</sub> is set equal to P.

If a linear model is solved using equation (3), only one improvement in prices (that is, one iteration) would be needed and the solution obtained would be the same as for equation (1). In the linear case, the initial guesses can be set equal to zero and the Δ P would equal the equilibrium prices.

As with linear simultaneous equations, uniqueness for nonlinear-simultaneous equations depends on the B matrix. In the remainder of this article, I discuss sufficient conditions to impose on the B matrix that will guarantee unique solutions for constant-elasticity commodity models.

### GUARANTEEING UNIQUENESS IN CONSTANT-ELASTICITY COMMODITY MODELS

Arrow and Hahn (4, p. 234) have shown that a unique solution is guaranteed if the matrix of partial derivatives with respect to prices B(P<sub>0</sub>, X) has a dominant diagonal for every conceivable set of commodity prices. A dominant diagonal matrix can be defined as:

$$|b_{ij}| d_j > \sum_{\substack{j=1 \\ i \neq j}}^N |b_{ij}| d_j \quad i = 1, 2, \dots, N \quad (5)$$

where  $i$ 's indicate rows,  $j$ 's indicate columns,  $b$ 's are matrix elements, and  $d$ 's represent a set of positive numbers (12, p. 311). It is shown below that one can guaran-

tee this kind of matrix and, hence, unique solutions for constant-elasticity commodity models.

Each element in the matrix of first partial derivatives (matrix B in equation (3)) is calculated by equation (6) in the constant-elasticity case:

$$b_{ij} = \frac{\partial \text{EXD}_i}{\partial P_j} = \left[ (\text{E.D.}_{ij})(d_i) - (\text{E.S.}_{ij})(s_i) \right] \frac{1}{P_j} \quad (6)$$

where:

- EXD<sub>i</sub> = excess demand equation for commodity *i*;
- P<sub>j</sub> = price of commodity *j*;
- E.D.<sub>ij</sub> = demand elasticity of commodity *i* with respect to price of commodity *j*;
- E.S.<sub>ij</sub> = supply elasticity of commodity *i* with respect to price of commodity *j*;
- d<sub>i</sub> = quantity of commodity *i* demanded; and
- s<sub>i</sub> = quantity of commodity *i* supplied.

The restrictions of weak gross substitutability (WGS) and degree zero homogeneity can be used to help insure that equation (6) will always produce a dominant diagonal matrix. WGS (4, p. 227) is defined by inequality (1):

$$\frac{\partial \text{EXD}}{\partial P_j} \geq 0, \quad i \neq j \quad \text{and} \quad \frac{\partial \text{EXD}_i}{\partial P_j} < 0 \quad i=j \quad (1)$$

This assumption excludes complementary price relationships. Because the demand and supply equations have constant elasticities, the homogeneity condition is guaranteed by assuring that:

$$\sum_{j=1}^{N+1} \text{E.D.}_{ij} + \text{E.I.}_i = 0 \quad \text{and} \quad \sum_{j=1}^{N+1} \text{E.S.}_{ij} = 0, \quad i=1, 2, \dots, N \quad (7)$$

where E.I.<sub>i</sub> is the income elasticity of demand for commodity *i*. (E.D.<sub>ij</sub> and E.S.<sub>ij</sub> are as defined in equation 6.)<sup>6</sup> The homogeneity restriction was imposed by Brandow (6, p. 13) and by George and King (9, p. 8) on their demand equations in this manner.

Inequalities (2) through (5) show how a dominant diagonal matrix can be guaranteed. First, inequality (2)

$$\left| \text{E.S.}_{ij} \right| > \sum_{\substack{j=1 \\ i \neq j}}^N \left| \text{E.S.}_{ij} \right| \quad i=1, 2, \dots, N \quad (2)$$

is guaranteed by WGS and degree zero homogeneity. Also, inequality (3)

$$\left| \text{E.D.}_{ij} \right| > \sum_{\substack{j=1 \\ i \neq j}}^N \left| \text{E.D.}_{ij} \right| \quad i=1, 2, \dots, N \quad (3)$$

is guaranteed by these two restrictions, if income elasticities are not negative. Negative income elasticities do not pose a problem for food commodity models. For example, George and King (9, p. 51) and Brandow (6, p. 17) have only one negative income elasticity among 49 and 24 food commodities, respectively.

Inequality (4)

$$\left| \text{E.D.}_{ij} - \text{E.S.}_{ij} \right| > \sum_{\substack{j=1 \\ i \neq j}}^N \left| \text{E.D.}_{ij} - \text{E.S.}_{ij} \right| \quad (4)$$

is guaranteed by inequalities (2) and (3). Also, inequality (5)

$$\left| \text{E.D.}_{ij}d_i - \text{E.S.}_{ij}s_i \right| > \sum_{\substack{j=1 \\ i \neq j}}^N \left| \text{E.D.}_{ij}d_i - \text{E.S.}_{ij}s_i \right| \quad (5)$$

is, in turn, guaranteed by inequality (4).

The left-hand side of inequality (5) is equal to the absolute value of the numerator in equation (6), when *i=j* (that is, the numerator for a diagonal element in the B matrix). The right-hand side of inequality (5) is equal to the summation of the absolute values of the numerators in equation (6) over *j* from 1 to N, excluding *i=j* (that is, the sum of the absolute values of the N-1 off-diagonal numerators for the same row in the B matrix).

If the B matrix is post-multiplied by a diagonalized matrix of the prices used in its calculation (that is, the diagonalized matrix of p<sub>j</sub>'s in equation (6) which correspond to the d's in equation (5)), inequality (5) is guaranteed for all *i*'s. Thus, the requirement for a dominant diagonal matrix is fulfilled. Consequently, uniqueness is assured.

Demand, supply, or both may be divided into components without reversing inequality (5) and, consequently, without destroying the guarantee for a dominant diagonal matrix. For example, in the commodity projections component of the NIRAP model, demand is divided into food demand, feed demand for livestock, export demand, and other use demand; and supply is divided into U.S. supply and imports.

## CONCLUDING REMARKS

The impetus for this article came from the need to determine how to guarantee unique equilibrium solutions in the commodity projections component of the NIRAP model. This component is used to make equi-

<sup>6</sup> See footnote 2 for an explanation of the Nth + 1 commodity.

librium projections of commodity prices and quantities under alternative scenarios that include prespecified levels of the required exogenous variables. The component model is also used in a comparative static manner to evaluate the effects of changing the level(s) of only one or several related exogenous variables (for example, see 13). Unique solutions in both of these uses are necessary for unambiguous projections; that is, projections that can always be duplicated.

Other procedures may be needed to prove uniqueness for nonlinear commodity models specified with different equation forms. A number of other sufficient conditions for guaranteeing uniqueness can be found in the literature on general equilibrium theory (in 4, for example). Perhaps some of these sufficient conditions can be fulfilled in other nonlinear commodity models, which would thereby guarantee unique solutions.

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