Estimating Land Use Patterns: A Separable Programing Approach

By Wen-yuan Huang and Howard C. Hogg

A model is described that projects land use patterns under both competitive and profit maximizing conditions. Separable programing is utilized to internalize price effects resulting from project production, and to allow extension of the basic model to include cross-elasticities. An empirical example, as used to illustrate the model, includes a detailed description of model formulation for solution on the IBM-MPS System.

Keywords: Separable programing, Project evaluation, Competitive-equilibrium land use, Profit-maximizing land use.

INTRODUCTION

The model described in this article can be used to predict land use patterns on newly developed projects when project production is expected to affect product price. Market demand and supply curves summarizing preproject conditions are included in the model for each market available to producers and each commodity that can be grown on the project. A linear programing analysis is used to predict land use patterns for the project. Project supplies are added to those of existing producers to assess demand and price implications. The project can be modeled as if it were a profit-maximizing monopoly, or, if appropriate, as if there were competitive equilibrium for each commodity, site, and market. A unique feature involves a method for approximating the competitive solution using ordinary linear programing. The model incorporates a constraint defining the marginal cost of commodities produced in the project, and it modifies the criterion function to compare average revenue with marginal cost. The model, as formulated, can be solved with the IBM-MPS (Mathematical Programing System).

There are a number of possible applications of the model. Land use patterns under competitive conditions, and, consequently, the direct effects of newly developed projects can be estimated. When production control is to be exercised by the developing entity, profit-maximizing land use patterns can also be estimated. This class of project exists when a private investor, producer cooperative, or Government agency desires to maximize project returns. The model presents an alternative to the point demand-minimum cost models widely used in interregional planning. In an application of this type, the model is formulated with demand curves for each crop and market but no explicit supply curves. The solution indicates the least-cost production pattern, by region and land class, to meet the specified demands. The main difference between the two approaches is that in the model discussed here, equilibrium product prices and quantities demanded are determined by both supply and demand.

MATHEMATICAL MODEL

The basic structure of the land use model can be expressed as follows:

Maximize:

$$Z = \sum_i \sum_j \sum_k (P_{ik}\Theta_{ij}X_{ijk} - C_{ijk}\,X_{ijk}) \qquad (1)$$

Subject to:

$$\sum_i \sum_k X_{ijk} \leqslant L_j \text{ for } j=1,\dots,J \qquad (2)$$

$$Q_{ik} = A_{ik} + B_{ik}P_{ik} \text{ for } i=1,\dots,I;\; k=1,\dots,K \qquad (3)$$

$$Q_{ik} = \Theta_{ij}X_{ijk}, \qquad (4)$$

$P_{ik} \geqslant 0$, and

$X_{ijk} \geqslant 0$ for $i=1,\dots,I$; $j=1,\dots,J$; $k=1.\dots,K$

Where:

- P_{ik} = Market price per unit of the ith crop in the kth market
- Θ_{ij} = Yield/acre of the ith crop on the jth land class
- X_{ijk} = Acreage of the ith crop planted on the jth land class and subsequently sold in the kth market
- C_{ijk} = Cost of production per acre of the ith crop on the jth land class and of transportation of the per acre product to the kth market
- L_j = Total available acreage of the jth land class
- Q_{ik} = Net market demand for the ith crop in the kth market to be supplied by the new project; that is:

$$P_{ik} = a_{ik} + b_{ik}\,q^d_{ik} \qquad (5)$$

and the current supply curve of the ith crop at the kth market is:

$$P_{ik} = c_{ik} + d_{ik} q_{ik}^{s} \qquad (6)$$

Then:

$$Q_{ik} = q_{ik}^{d} - q_{ik}^{s} \text{ or } Q_{ik} = A_{ik} + B_{ik} P_{ik} \qquad (7)$$

Where:

A_{ik} = The constant of the net demand curve

$$A_{ik} = \left[\frac{-a_{ik}}{b_{ik}} + \frac{c_{ik}}{d_{ik}}\right] \qquad (8)$$

B_{ik} = The price coefficient of the net demand curve

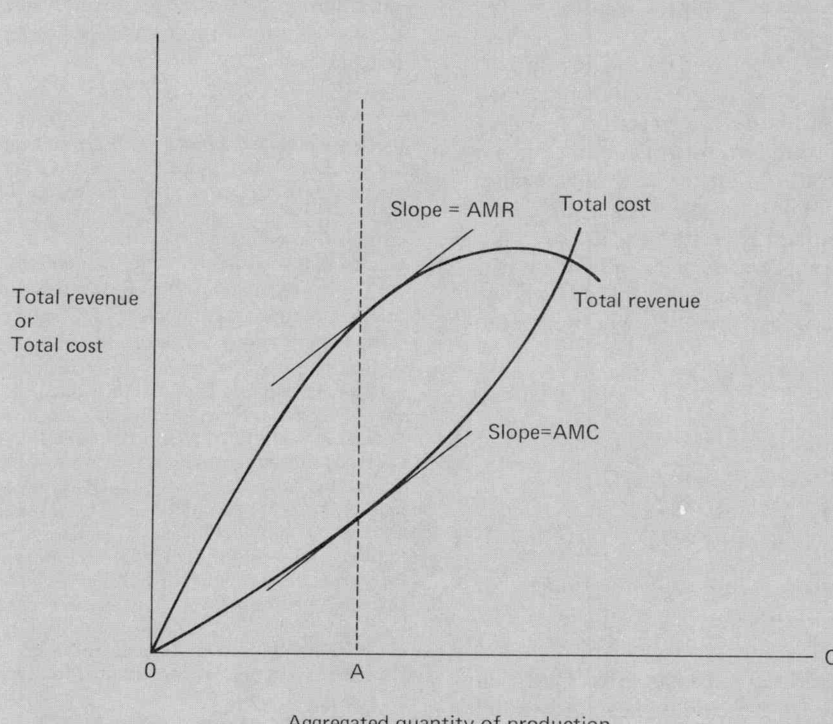
$$B_{ik} = \left[\frac{1}{b_{ik}} - \frac{1}{d_{ik}}\right] \qquad (9)$$

The model consists of sets of linear constraints and a quadratic objective function. That is, Z is a function of P_{ik} and X_{ijk}. For maximizing or minimizing a quadratic objective function subject to linear constraints, various solution procedures have been developed by Kuhn and Tucker (*7*), Beale (*1, 2*), Wolfe (*12*), Houthakker (*6*), and Hadley (*3*). A review of interregional quadratic formulations using some of these solution procedures is available from Heady and Hall (*4*).

A land use pattern under "monopoly" conditions can be estimated by maximizing the objective function (1) (assuming a concave function) subject to the constraint sets (2), (3), and (4). When land is limited, the solution will fall on the line OA shown in figure 1. When land is relatively abundant, the solution is point A. At this point, aggregate marginal revenue is equal to aggregate marginal cost.

By setting price constraints, the model can be used to estimate the land use pattern under competitive equilibrium conditions. Such a condition is reached when the last piece of land entering production will give the same marginal rent for any crop planted. Under the assumption of a concave objective function, additional constraints are required to solve the model under two different land availabilities. In the first case, quantities of each crop demanded by each market can always be provided

Optimal Solution for Land Use Under Monopoly Conditions (MR=MC)



Note: Point A is where aggregated marginal revenue (AMR) equals aggregated marginal cost (AMC).

Figure 1

because land is not a limiting factor. The last piece of land entering production earns no rent for any crop planted. The constraints to be added are:

$$P_{ik} = \overset{(1)}{U}_{ik} \cdot \overset{(1)}{W}_{ik} + \overset{(2)}{U}_{ik} \cdot \overset{(2)}{W}_{ik} + \ldots + \overset{(L)}{U}_{ik} \cdot \overset{(L)}{W}_{ik} \quad (10)$$

$$= \sum_{j=1}^{L} \overset{(j)}{U}_{ik} \cdot \overset{(j)}{W}_{ik} \text{ for } i=1, \ldots, I$$

$$k=1, \ldots, K$$

Where:

$\overset{(j)}{U}_{ik} = \dfrac{C_{ijk}}{\Theta_{ij}}$, the marginal cost with $\overset{(j)}{U}_{ik} < \overset{(j+1)}{U}_{ik}$

for $j=1, \ldots, L-1$, and

$\overset{(j)}{W}_{ik}$ for $j=1, \ldots L$ constituting logical variables.

Each logical variable will take a value of 0 to 1 and

$$\sum_{j=1}^{L} \overset{(j)}{W}_{ik} = 1.$$

Figure 2 shows the prices P_{ik} (in the solution) will be between $\overset{(1)}{U}_{ik}$ and $\overset{(m)}{U}_{ik}$, depending on the position of the demand curve Q_{ik}. The solution is obtained at the point where the demand curve intersects the marginal cost curve MC_{ik}. This marginal cost is the cost of producing a unit of the *i*th crop on the last unit of land entering production and of selling it in the *k*th market.

In the second case, land is limited and all of it is brought into production. The last piece of land entering production will earn the same marginal rent for any crop planted. Figure 3 shows that price P_{ik} will be greater than or equal $\overset{(m)}{U}_{ik}$. The constraints needed are:

$$P_{ik} - \sum_{L=1}^{m} \overset{(L)}{U}_{ik} \overset{(L)}{W}_{ik} = R \text{ for } i=1, \ldots, I \quad (11)$$

$$k=1, \ldots, K$$

Where:

R = Marginal net price, an internally determined constant

Note that by setting $R = 0$ in constraint (11), the constraint (10) becomes a special case of (11). An equilibrium solution is reached only when the minimum marginal rent is obtained, but all land is brought into production. Given the form of the objective function (1), it is likely that many values of R will satisfy relationship (11). However, only one value of R is a minimum for all possible R's and provides an equilibrium solution.[1] Here, R must be added to the objective function:

$$Z = \sum_i \sum_j \sum_k P_{ik} \, \Theta_{jk} \cdot X_{ijk} - C_{ijk} X_{ijk} - \propto R \quad (12)$$

Where:

$\propto$ = is an arbitrarily large nonnegative value, and $R \geqslant 0$

A solution under competitive conditions can thus be obtained by maximizing the objective function (12) subject to the constraint sets (2), (3), (4), and (11). This solution represents an equilibrium in the crop sector and in allocating products from new production areas to various markets. The solution is not the equivalent of Samuelson's equilibrium trade solution, which provides a longrun trade equilibrium (*10*). The competitive solution of our land use model can be interrupted as a longrun equilibrium because R is the opportunity cost of retaining land in a particular use. When a trade equilibrium is desired, the model can be extended to provide it.[2]

SEPARABLE PROGRAMING FORMULATION

In this article, the solution procedure of separable programing, described by Hadley, is used (*3*). Separable programing is a technique for handling a nonlinear objective function or nonlinear constraints that can be written as:

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[1] As a simple example of the equilibrium solution, assume one market for two crops. If at an arbitrary production level, marginal rent R_{11} from crop 1 is greater than marginal rent R_{21} from crop 2, the land used by crop 2 will be reallocated to crop 1. Consequently, R_{11} will decrease, while R_{21} will increase because of the downward-sloping net demand curves for the two crops. Land reallocation will continue until an equilibrium condition, R_{11} equals R_{21}, is reached. In other words, marginal rent from planting either crop 1 or crop 2 is the same.

The term $-\propto R$, wherein $\propto$ is a positive constant and R is rent, is to be maximized. The smallest value of R which makes R_{11} equal R_{21} equal R will be found. Thus, by employing the constraints (equation (11)) and the objective function (equation (12)), equilibrium in the crop sector can be obtained.

[2] In his net social payoff model, Samuelson assumes that the markets are interdependent; trade between two markets is allowed. (The model presented here assumes no trade between markets.) The equilibrium solution is obtained when the price difference of a commodity between any two markets is less than the transportation cost of moving a unit of commodity from one market to the other. When this condition is reached, net social payoff is maximized. Takayama and Judge (*1*) and Plessner and Heady (*9*) formulated constraints for an equilibrium in trade as:

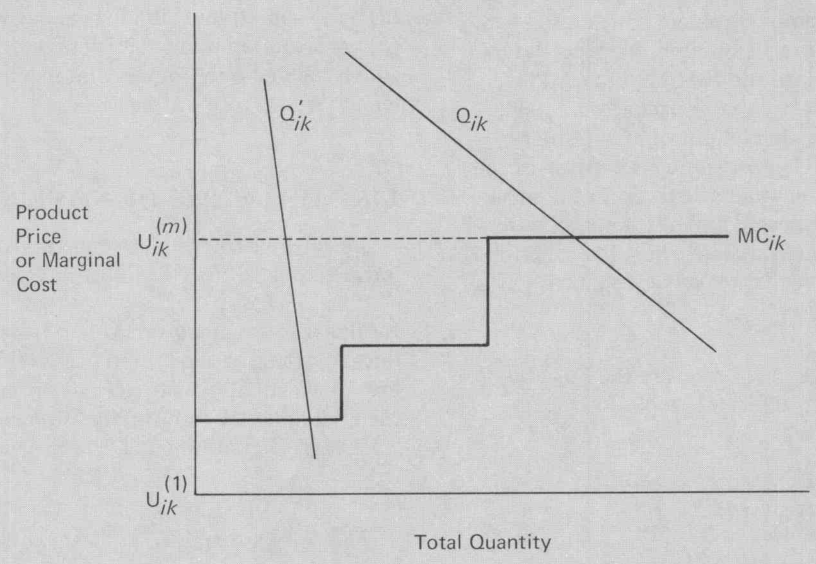
$$P_{ik} - P_{ih} \leqslant T_{kh} \text{ for all } i, k, \text{ and } h$$

Where:

T_{kh} is the transportation cost between k and h markets

This set of constraints can be added to our model so that an equilibrium solution in both the crop sector and trade can be achieved.

Equilibrium Solution (P_{ik} - MC_{ik}) with Abundant Land



Note: Q_{ik} and Q'_{ik} are two possible positions of demand curve for ith crop on kth market.

Figure 2

Equilibrium Solution (P_{ik} = MC_{ik} = constant) with Limited Land

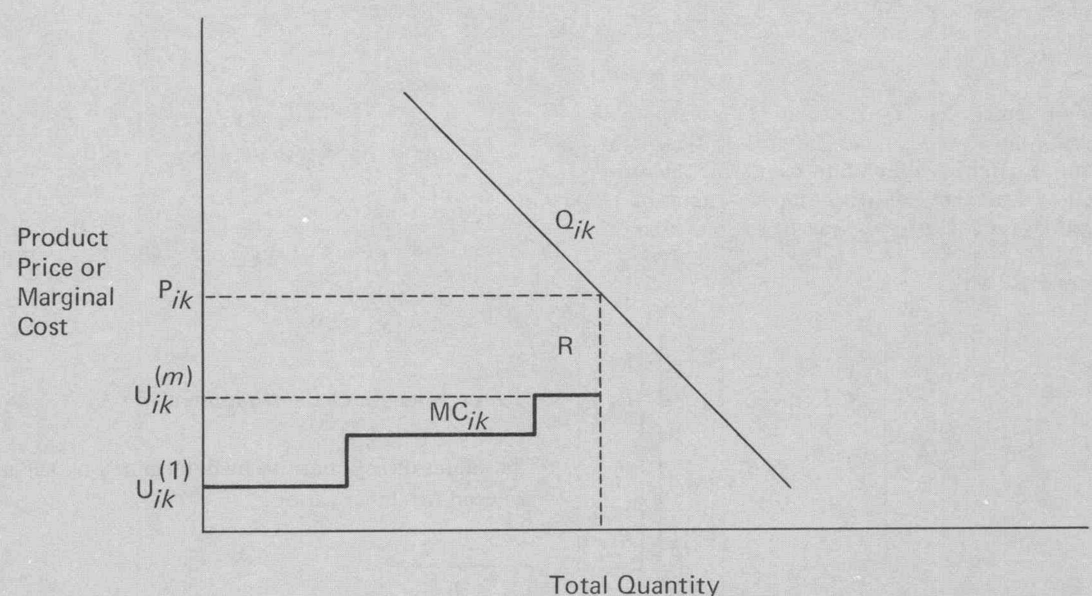


Figure 3

$$f(X_1, X_2, \ldots, X_n) = f_1(X_1) + \ldots f_n(X_n).$$

In each nonlinear function $f_i(X_i)$, i=1, . . . , n is a function of only one variable and it is approximated by a piecewise linear function. The nonlinear problem thus becomes a linear programing problem. One reason for using separable programing is because the procedure has been incorporated into the IBM-MPS (*8*). MPS, one of the most flexible computer packages available, provides great efficiency in computation. Another reason for using separable programing is because of the nature of the model. The quadratic terms in the objective function can be expressed as a linear combination of a function of only one variable. Thus, the function Z equals $f(P_{ik}, X_{ijk})$ can be expressed as Z equals $f_1(P_{ik})$ plus $f_2(X_{ijk})$ as follows:

1. Rearrange the nonlinear terms in the objective function:

$$\sum_i \sum_j \sum_k P_{ik}\, \Theta_{ij}\, X_{ijk}$$

as

$$\sum_i \sum_k P_{ik} \left(\sum_j \Theta_{ij}\, X_{ijk}\right)$$

2. Since $\sum_j \Theta_{ij} X_{ijk}$ is equal to Q_{ik}, it is valid to substitute $(A_{ik} + B_{ik}P_{ik})$ for $\sum_j \Theta_{ij} X_{ijk}$ in the objective function. The objective function (12) becomes:

$$Z = \sum_i \sum_k P_{ik} \cdot A_{ik} + \sum_i \sum_k B_{ik} P_{ik}^2 - \sum_i \sum_j \sum_k C_{ijk} \cdot X_{ijk} - \alpha R \qquad (13)$$

Because the original objective function (1) is a separable function and can be expressed as function (13), separable programing can be employed to search for the optimum solution. Furthermore, with objective function (13), the quadratic term $\sum_i \sum_k B_{ik} P^2_{ik}$ can be expressed in the matrix form $P'BP$ as:

$$\begin{bmatrix} P_{11}\, P_{12} \cdots P_{ik} \cdots P_{IK} \end{bmatrix} \begin{bmatrix} B_{11} & & & & \\ & B_{12} & & & \\ & & \cdots & & \\ & & & B_{ik} & \\ & & & \cdots & \\ & & & & B_{IK} \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{12} \\ \cdots \\ P_{ik} \\ \cdots \\ P_{IK} \end{bmatrix}$$

The matrix B is a diagonal one. If all values of B_{ik} for i=1, . . . , I, K=1, . . . , K are negative, the matrix B is negative definite and function (13) is concave. In this case, a unique solution exists for maximization.

To test to determine if a solution is possible, three steps must be taken before preparing the computer input data. These are: (1) linearization of the nonlinear term; (2) approximation of the logical constraint (11); and (3) construction of an MPS data matrix. The second step is needed only for a competitive equilibrium problem.

LINEARIZATION OF NONLINEAR TERMS

Before constructing the constraint matrix, the nonlinear term $\sum_i \sum_k B_{ik} P^2_{ik}$ must be linearized. Procedures for linearization are given in (*8*). However, an example of linearization, for the term $B_{11}P^2_{11}$ (i=1, k=1) appears below. Two equations are needed for each nonlinear term: the grid equation, and the functional equation.

The grid equation is:

$$P_{11} = X_{11}^{(0)} + D_{11}^{(1)} X_{11}^{(1)} + D_{11}^{(2)} X_{11}^{(2)} + \ldots + D_{11}^{(n_{11})} X_{11}^{(n_{11})} \qquad (14)$$

Where:

$X_{11}^{(0)}$ = The value of P_{11} at the beginning of the first interval

$X_{11}^{(r)}$ for r=1, . . . n are special variables for separable variable P_{11}

$0 \leqslant X_{11}^{(r)} \leqslant 1$, and if $X_{11}^{(1)} \ldots X_{11}^{(r)}$ are used to compute the P_{11} value, then

$$X_{11}^{(1)} = X_{11}^{(2)} = \ldots = X_{11}^{(r-1)} = 1 \text{ and } X_{11}^{(r+1)} = X_{11}^{(r+2)} = X_{11}^{(n_{11})} = 0$$

$D_{11}^{(r)}$ = length of the rth interval, r = 1, . . . , n_{11}

The values of $D_{11}^{(r)}$ have to be determined by the user and are used for linearization.[3]

[3] $D_{11}^{(L)}$ is an arbitrarily small increment of the price P_{11}. The magnitude of $D_{11}^{(r)}$ should be small if the value of the nonlinear term $B_{11}P_{11}^2$ is to be sensitive to the change in price P_{11}.

The functional equation is:

$$B_{11}P_{11}^{2} = Y_{11}^{(0)} + E_{11}^{(1)} X_{11}^{(1)} + \ldots + E_{11}^{(n_{11})} X_{11}^{(n_{11})} \quad (15)$$

Where:

$Y_{11}^{(0)}$ = is the value of $B_{11}P_{11}^{2}$ at the beginning of the first interval

$E_{11}^{(r)}$ = is the change in the value of $B_{11}P_{11}^{2}$ in the rth interval for r=1, . . . , n_{11}

Figure 4 illustrates the linearization process and the relationship between the two functions. For example, the value of $B_{11}P_{11}^{2}$ at the point (X, Y) is $\left(B_{11}E_{11}^{(1)} + B_{11}E_{11}^{(2)}\right)$, and the corresponding value for P_{11} is $\left(D_{11}^{(1)} + D_{11}^{(2)}\right)$. To obtain these values from equations (14) and (15), the special variables $X_{11}^{(1)}$ and $X_{11}^{(2)}$ are set equal to 1 and the other values of $X_{11}^{(r)}$ are set equal to 0.

Similar procedures can be employed for linearizing each of the other terms, $B_{ik} P_{ik}^{2}$ for all i not equal to 1 and k not equal to 1. Once the linearization procedures are completed, the constraint matrix can be constructed.

APPROXIMATION OF THE LOGICAL CONSTRAINT FOR THE COMPETITIVE EQUILIBRIUM CONDITION

Employing the special variables described earlier, the constraint (11) can be reformulated as:

$$P_{ik} - M_{ik}^{(1)} X_{ik}^{(1)} - M_{ik}^{(2)} X_{ik}^{(2)} - \ldots - M_{ik}^{(n)} X_{ik}^{(n)} = R$$

for i=1, . . . , I
k=1, . . . , K (16)

Where:

$$M_{ik}^{(1)} = U_{ik}^{(1)}$$

$$M_{ik}^{(2)} = U_{ik}^{(2)} - U_{ik}^{(1)}$$

$$\vdots$$

$$M_{ik}^{(n)} = U_{ik}^{(n)} - U_{ik}^{(n\text{-}1)}$$

$X_{ik}^{(r)}$ = The special variables defined in equation (14), r=1, n. Constraint (16) is an approximation of constraint (11). The approximation error will be small if the magnitudes of $M_{ik}^{(2)}, M_{ik}^{(3)}, \ldots$ $M_{ik}^{(n)}$ are small.

CONSTRAINT MATRIX

The objective function (13) and the constraint sets (2), (3), (4) and (16) for three crops (I=3), two land classes (J=3) and two markets (J=2) are constructed in matrix form in table 1. Each row is a constraint and each column, a structural variable. The matrix, which represents the basic structure of the model, can be extended for a relatively large number of crops, land classes, and markets. Additional constraints may be added for other types of resources. Because P_{ik} and X_{ijk} are shown as structural variables, cross-elasticities between crops can be built into the model by restating the relationship between P_{ik} and X_{ijk} (see appendix).

USE OF THE MODEL

In 1968, a paper published in this journal described an iterative model similar to the one developed here (5). The illustrative example comes from that article. Tables 3 and 4 contain all of the basic data which, except for the net demand curves in table 3, are identical to the data required in the earlier formulation. Both models are project oriented but ours is more general because cross-elasticities and constraints other than land and commodity demand can be accommodated. Table 2 illustrates how the input data is entered into the matrix shown in general form in table 1. In table 2, the coefficients A_{ik} and B_{ik} are from table 3, and C_{ijk} and Θ_{ij} are from table 4 (the value for $\propto$ is 1,000,000). The coefficients Δ's and M's are computed by using equation 15 and these are as follows:

Δ_1 = 843325	Δ_{17} = 50150	M_1 = 140
Δ_2 = 371063	Δ_{18} = 150450	M_2 = 4.444
Δ_3 = 438529	Δ_{19} = 250750	M_3 = 5.556
Δ_4 = 505995	Δ_{20} = 351050	M_4 = 88.888
Δ_5 = 573461	Δ_{21} = 33025	M_5 = 1.112
Δ_6 = 640927	Δ_{22} = 99075	M_6 = .909
Δ_7 = 708393	Δ_{23} = 165125	M_7 = 175
Δ_8 = 775859	Δ_{24} = 231175	M_8 = 41.666
Δ_9 = 843325	Δ_{25} = 83825	M_9 = 83.34
Δ_{10} = 910791	Δ_{26} = 251475	M_{10} = 100
Δ_{11} = 978257	Δ_{27} = 422875	M_{11} = 5.55
Δ_{12} = 1045723	Δ_{28} = 592025	M_{12} = 6.945
Δ_{13} = 1113189	Δ_{29} = 85825	M_{13} = 110.53
Δ_{14} = 1180655	Δ_{30} = 257475	M_{14} = .5850
Δ_{15} = 1248121	Δ_{31} = 499125	M_{15} = .6530
Δ_{16} = 1315587	Δ_{32} = 600775	

The formulations represent two alternatives: (1) the land use pattern expected on a newly developed project under competitive conditions, and (2) the profit-maximizing land use pattern when production control, as opposed to free choice, is exercised. Tables 5 and 6 compare the original solution with that of the present model. The example uses five crops, three land classes, and one market.

All lands in the project enter production under competitive conditions in the example. In the "monopoly" solution, only land class 2 is utilized in its entirety, resulting in a zero opportunity rent for land. The "monopoly" prices are, of course, higher than those indicated for the competitive solution. The material presented in tables 5 and 6 is available directly from model output. In addition, certain other impacts of the project can be quantified from the model. For example, the quantities grown by project producers can be obtained by solving the original supply curves with the computed equilibrium prices. Performing the same operation on the demand curves results in an estimate of total market supply. The difference between solution values for the iterative and separable procedures results from the linearization of equation (7) and the approximation of equation (11) by (16). Objective function values for the competitive solution were \$307,481 (separable) and \$310,183 (iterative). A value of \$1,052,032 was obtained for the monopoly solution.

Some users may be concerned that the budgeted production costs are based on factor use under a price level different from the final equilibrium prices. This inconsistency can be resolved by rebudgeting at the new prices. If optimum input combinations change, the model can be rerun with the newly budgeted costs. Several attempts at this iterative process will isolate a price range that brackets the equilibrium position. An acceptable range depends on the linearization precision, data accuracy, and the relative magnitude of the range of prices. In most applications, it should not be necessary to undertake this step as the difference between the budget price and final equilibrium price will be small.

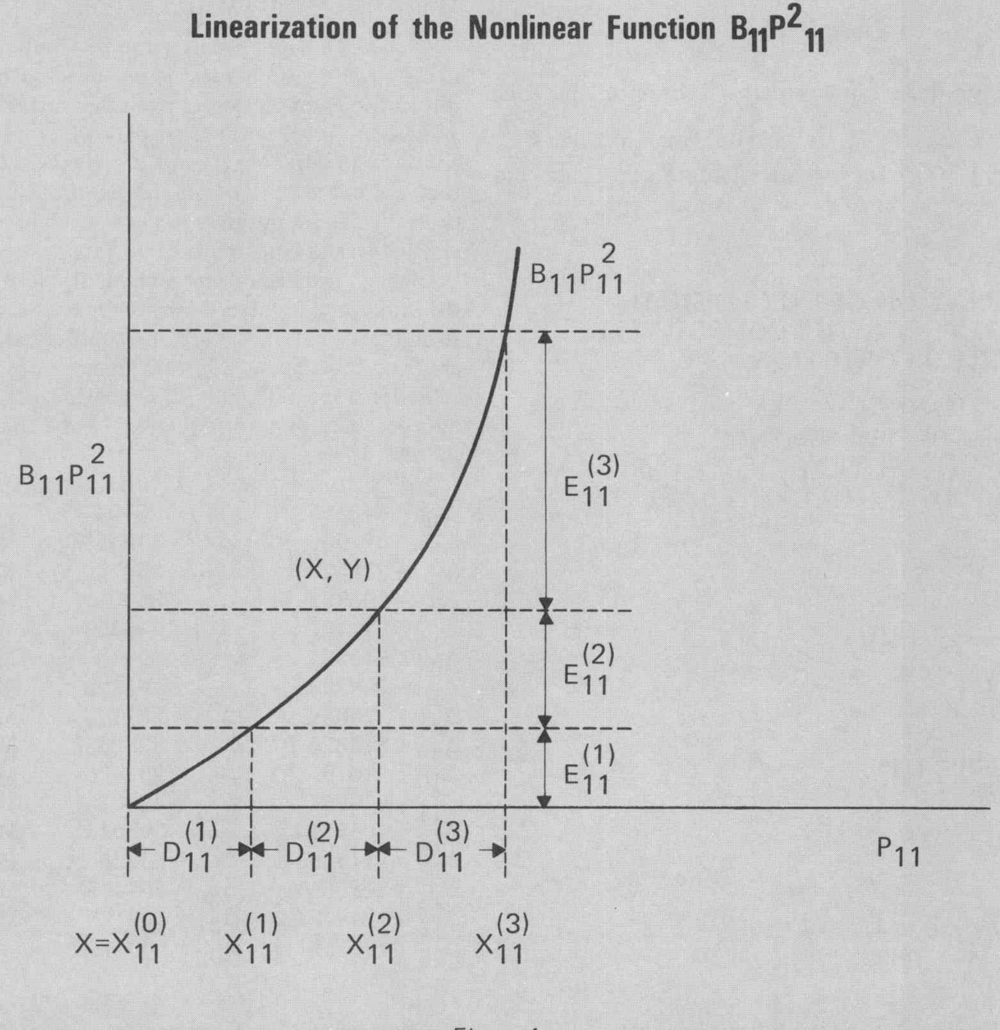


Figure 4

Table 1. Data matrix for land use model

ROW \ COLUMN	ACTIVITY LEVEL																								
	P^2_{11}	P^2_{21}	P^2_{31}	P^2_{12}	P^2_{22}	P^2_{32}	P_{11}	P_{21}	P_{31}	P_{12}	P_{22}	P_{32}	X_{111}	X_{211}	X_{311}	X_{121}	X_{221}	X_{321}	X_{112}	X_{212}	X_{312}	X_{122}	X_{222}	X_{322}	R
Z	B_{11}	B_{21}	B_{31}	B_{12}	B_{22}	B_{32}	A_{11}	A_{21}	A_{31}	A_{12}	A_{22}	A_{32}	$-C_{111}$	$-C_{211}$	$-C_{311}$	$-C_{121}$	$-C_{221}$	$-C_{321}$	$-C_{112}$	$-C_{212}$	$-C_{312}$	$-C_{122}$	$-C_{222}$	$-C_{322}$	
L1													1	1	1				1	1	1				
L2																1	1	1				1	1	1	
D11							$-B_{11}$						Θ_{11}			Θ_{12}									
D21								$-B_{21}$						Θ_{21}			Θ_{22}								
D31									$-B_{31}$						Θ_{31}			Θ_{32}							
D12										$-B_{12}$									Θ_{11}			Θ_{12}			
D22											$-B_{22}$									Θ_{21}			Θ_{22}		
D32												$-B_{32}$									Θ_{31}			Θ_{32}	
N11	-1																								
N21		-1																							
N31			-1																						
N12				-1																					
N22					-1																				
N32						-1																			
G11							-1																		
G21								-1																	
G31									-1																
G12										-1															
G22											-1														
G32												-1													
P11							-1																		1
P21								-1																	1
P31									-1																1
P12										-1															1
P22											-1														1
P32												-1													1
Upper Bound																									
Lower Bound																									

ROW \ COLUMN	SPECIAL VARIABLES																							
	X^1_{11}	X^2_{11}	X^3_{11}	X^4_{11}	X^1_{21}	X^2_{21}	X^3_{21}	X^4_{21}	X^1_{31}	X^2_{31}	X^3_{31}	X^4_{31}	X^1_{12}	X^2_{12}	X^3_{12}	X^4_{12}	X^1_{22}	X^2_{22}	X^3_{22}	X^4_{22}	X^1_{32}	X^2_{32}	X^3_{32}	X^4_{32}
Z																								
L1																								
L2																								
D11																								
D21																								
D31																								
D12																								
D22																								
D32																								
N11	Δ_{111}	Δ_{112}	Δ_{113}	Δ_{114}																				
N21					Δ_{211}	Δ_{212}	Δ_{213}	Δ_{214}																
N31									Δ_{311}	Δ_{312}	Δ_{313}	Δ_{314}												
N12													Δ_{121}	Δ_{122}	Δ_{123}	Δ_{124}								
N22																	Δ_{221}	Δ_{222}	Δ_{223}	Δ_{224}				
N32																					Δ_{321}	Δ_{322}	Δ_{323}	Δ_{324}
G11	d_{111}	d_{112}	d_{113}	d_{114}																				
G21					d_{211}	d_{212}	d_{213}	d_{214}																
G31									d_{311}	d_{312}	d_{313}	d_{314}												
G12													d_{121}	d_{122}	d_{123}	d_{124}								
G22																	d_{221}	d_{222}	d_{223}	d_{224}				
G32																					d_{321}	d_{322}	d_{323}	d_{324}
P11																								
P21																								
P31																								
P12																								
P22																								
P32																								
Upper Bound	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Lower Bound	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

ROW \ COLUMN	SPECIAL VARIABLES (cont.)																		RHS	
	Y^1_{11}	Y^2_{11}	Y^3_{11}	Y^1_{21}	Y^2_{21}	Y^3_{21}	Y^1_{31}	Y^2_{31}	Y^3_{31}	Y^1_{12}	Y^2_{12}	Y^3_{12}	Y^1_{22}	Y^2_{22}	Y^3_{22}	Y^1_{32}	Y^2_{32}	Y^3_{32}		
Z																			N	∞
L1																			L	L_1
L2																			L	L_2
D11																			E	A_{11}
D21																			E	A_{21}
D31																			E	A_{31}
D12																			E	A_{12}
D22																			E	A_{22}
D32																			E	A_{32}
N11																			E	Y^o_{11}
N21																			E	Y^o_{21}
N31																			E	Y^o_{31}
N12																			E	Y^o_{12}
N22																			E	Y^o_{22}
N32																			E	Y^o_{32}
G11																			E	X^o_{11}
G21																			E	X^o_{21}
G31																			E	X^o_{31}
G12																			E	X^o_{12}
G22																			E	X^o_{22}
G32																			E	X^o_{32}
P11	M_{111}	M_{112}	M_{113}																E	0
P21				M_{211}	M_{212}	M_{213}													E	0
P31							M_{311}	M_{312}	M_{313}										E	0
P12										M_{121}	M_{122}	M_{123}							E	0
P22													M_{221}	M_{222}	M_{223}				E	0
P32																M_{321}	M_{322}	M_{323}	E	0
Upper Bound	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		
Lower Bound	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		

NOTE: i = Crop, j = Land class, k = Market

$M_{ikr} = M_{ik}^{(r)}$ in equation (16)

$\Delta_{ikr} = E_{ik}^{(r)}$ in equation (15)

$d_{ikr} = D_{ik}^{(r)}$ in equation (14)

$X^r_{ik} = X_{ik}^{(r)}$ in equation (14)

$Y^r_{ik} = X_{ik}^{(r)}$ in equation (16)

Table 2. Data matrix for land use model

ACTIVITY LEVEL

Row	P^2_{11}	P^2_{21}	P^2_{31}	P^2_{41}	P^2_{51}	P_{11}	P_{21}	P_{31}	P_{41}	P_{51}	X_{111}	X_{211}	X_{311}	X_{411}	X_{511}	X_{121}	X_{221}	X_{321}	X_{421}	X_{521}	X_{131}	X_{231}	X_{331}	X_{431}	X_{531}	R	RHS	
Z	B_{11}	B_{21}	B_{31}	B_{41}	B_{51}	A_{11}	A_{21}	A_{31}	A_{41}	A_{51}	$-C_{111}$	$-C_{211}$	$-C_{311}$	$-C_{411}$	$-C_{511}$	$-C_{121}$	$-C_{221}$	$-C_{321}$	$-C_{421}$	$-C_{521}$	$-C_{131}$	$-C_{231}$	$-C_{331}$	$-C_{431}$	$-C_{531}$	$-\alpha$	N	∞
L1											1	1	1	1	1												L	550
L2																1	1	1	1	1							L	100
L3																					1	1	1	1	1		L	100
D11						$-B_{11}$					50					45					40						E	A_{11}
D21							$-B_{21}$					55					50					45					E	A_{21}
D31								$-B_{31}$					40					30					20				E	A_{31}
D41									$-B_{41}$					100					90					80			E	A_{41}
D51										$-B_{51}$					95					90					85		E	A_{51}
N11	-1																										E	0
N21		-1																									E	0
N31			-1																								E	0
N41				-1																							E	0
N51					-1																						E	0
G11						-1																					E	0
G21							-1																				E	0
G31								-1																			E	0
G41									-1																		E	0
G51										-1																	E	0
P11						-1																				1	E	0
P21							-1																			1	E	0
P31								-1																		1	E	0
P41									-1																	1	E	0
P51										-1																1	E	0
Upper Bound																												
Lower Bound																												

SPECIAL VARIABLES

Row	X^1_{11}	X^2_{11}	X^3_{11}	X^4_{11}	X^5_{11}	X^6_{11}	X^7_{11}	X^8_{11}	X^9_{11}	X^{10}_{11}	X^{11}_{11}	X^{12}_{11}	X^{13}_{11}	X^{14}_{11}	X^{15}_{11}	X^{16}_{11}
Z																
L1																
L2																
L3																
D11																
D21																
D31																
D41																
D51																
N11	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}
N21																
N31																
N41																
N51																
G11	50	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10
G21																
G31																
G41																
G51																
P11																
P21																
P31																
P41																
P51																
Upper Bound	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Lower Bound	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Row	X^1_{21}	X^2_{21}	X^3_{21}	X^4_{21}	X^1_{31}	X^2_{31}	X^3_{31}	X^4_{31}	X^1_{41}	X^2_{41}	X^3_{41}	X^4_{41}	X^1_{51}	X^2_{51}	X^3_{51}	X^4_{51}
Z																
L1																
L2																
L3																
D11																
D21																
D31																
D41																
D51																
N11																
N21	λ_{17}	λ_{18}	λ_{19}	λ_{20}												
N31					λ_{21}	λ_{22}	λ_{23}	λ_{24}								
N41									λ_{25}	λ_{26}	λ_{27}	λ_{28}				
N51													λ_{29}	λ_{30}	λ_{31}	λ_{32}
G11																
G21	50	50	50	50												
G31					50	50	50	50								
G41									50	50	50	50				
G51													50	50	50	50
P11																
P21																
P31																
P41																
P51																
Upper Bound	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Lower Bound	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Row	Y^1_{11}	Y^2_{11}	Y^3_{11}	Y^1_{21}	Y^2_{21}	Y^3_{21}	Y^1_{31}	Y^2_{31}	Y^3_{31}	Y^1_{41}	Y^2_{41}	Y^3_{41}	Y^1_{51}	Y^2_{51}	Y^3_{51}
Z															
L1															
L2															
L3															
D11															
D21															
D31															
D41															
D51															
N11															
N21															
N31															
N41															
N51															
G11															
G21															
G31															
G41															
G51															
P11	M_1	M_2	M_3												
P21				M_4	M_5	M_6									
P31							M_7	M_8	M_9						
P41										M_{10}	M_{11}	M_{12}			
P51													M_{13}	M_{14}	M_{15}
Upper Bound	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Lower Bound	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Note: $A_{11}=78833$, $A_{21}=4039$, $A_{31}=4184$, $A_{41}=7583$, $A_{51}=7500$/
$B_{11}=-337.33$, $B_{21}=-20.06$, $B_{31}=-13.21$, $B_{41}=-33.83$, $B_{51}=-34.33$ /
$\alpha = 10^5$

$C_{111}=7000$, $C_{211}=5000$, $C_{311}=7000$, $C_{411}=10000$, $C_{511}=10500$; $C_{121}=6500$, $C_{221}=4500$, $C_{321}=6500$, $C_{421}=9500$, $C_{521}=10000$; $C_{131}=6000$, $C_{231}=4000$, $C_{331}=6000$, $C_{431}=9000$, $C_{531}=9500$

Table 3. Market demand and supply functions for existing producers[a]

Crop	Demand functions		Supply functions		Net demand curves	
	Price intercept	Slope	Cost intercept	Slope	Quantity intercept	Slope
	a_{ik}	b_{ik}	c_{ik}	d_{ik}	A_{ik}	B_{ik}
1	250	-0.003	-1,125	0.25	78,833	-337.33
2	300	-.05	-33,330	17.0	4,039	-20.06
3	400	-.08	-1,143	1.4	4,184	-13.21
4	250	-.03	-1,500	2.0	7,583	-33.83
5	300	-.03	-2,500	1.0	7,500	-34.33

[a]Prices are estimated in dollars per 1,000 pounds and quantities are in 1,000 pound units. In this example the demand and supply curves are linear.

Table 4. Production costs and yields by land class[a]

Crop	Production costs per acre C_{ijk}			Per acre yield Θ_{ij}		
	Land class 1	Land class 2	Land class 3	Land class 1	Land class 2	Land class 3
		Dollars			*1,000 lbs.*	
1	7,000	6,500	6,000	50	45	40
2	5,000	4,500	4,000	55	50	45
3	7,000	6,500	6,000	40	30	20
4	10,000	9,500	9,000	100	90	80
5	10,500	10,000	9,500	95	90	85

[a]In this example, land classes 1, 2, and 3 show progressively lower yields for all crops, but this progression is not a requirement of the program.

Table 5. Final acreages X_{ijk} for alternative problem solutions

Crop	Original iterative solution			Separable competitive solution			Separable monopoly solution		
	Land class 1	Land class 2	Land class 3	Land class 1	Land class 2	Land class 3	Land class 1	Land class 2	Land class 3
					Acres				
1	467.2	100.0	6.9	473.3	100.0	2.6	295.3	---	---
2	---	---	50.0	---	---	49.3	---	---	22.9
3	42.5	---	---	38.5	---	---	38.5	---	---
4	40.3	---	---	38.1	---	---	25.6	---	---
5	---	---	43.0	---	---	43.0	24.7	---	---
Total	550.0	100.0	100.0	550.0	100.0	94.9	384.1	---	22.9

Table 6. Equilibrium product prices and opportunity rents from alternative solutions

Crop	Equilibrium product prices			Land class	Opportunity rents		
	Iterative solution	Competitive solution	Monopoly solution		Iterative solution	Competitive solution	Monopoly solution
		Dollars				*Dollars*	
1	150.28	150.00	190.00	1	514.15	493.67	0
2	89.14	90.91	150.00	2	262.73	254.43	0
3	187.85	200.00	200.00	3	11.32	0	0
4	105.14	112.50	150.00				
5	111.90	111.77	150.00				

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APPENDIX

A simple single-market example appears below. Assuming that the cross elasticity between the demand for crop 1, q_{11} and the price, P_{21}, of crop 2 in market 1 is constant, the cross elasticity can be expressed as:

$$e_{12} = \frac{dq_{11}}{q_{11}} \Big/ \frac{dP_{21}}{P_{21}}$$

Rewrite as:

$$e_{12} \cdot \frac{dP_{21}}{P_{21}} = \frac{dq_{11}}{q_{11}}$$

The solution for this differential equation is:

$$q_{11} = \frac{q^o_{11}}{P^o_{21}} \; P_{21}^{e_{12}} \qquad \text{(a)}$$

Where q^o_{11} and P^o_{21} are the equilibrium (intersection of demand and supply) quantity and price on the market before the new project starts.

Since $Q_{11} = q_{11} - q^o_{11}$ (b)

equation (a) can be expressed as:

$$Q_{11} + q^o_{11} = \frac{q^o_{11}}{P^o_{11}} \cdot P_{21}^{e_{12}}$$

Substituting $\sum_j \Theta_{1j} X_{1j1}$ for Q_{11},

equation (b) becomes

$$\Sigma\, \Theta_{1j} X_{1j1} + q_{11} = \frac{q^{o}_{11}}{P^{o}_{21}} \; P_{21}^{e_{12}} \qquad \text{(c)}$$

Incorporating the cross-elasticity in the model means adding constraint (c). Since $P_{21}^{e_{12}}$ is a nonlinear term, linearization of $P_{21}^{e_{12}}$ is required. The procedures for linearization are those described in the text. To add a cross-elasticity constraint to figure 5 requires the addition of 1 column (Structural Variable) for $P_{21}^{e_{12}}$ and 2 rows—1 row for the cross-elasticity constraint and 1 for the functional equation. Additional grid equations and special variables employed in the linearization of P_{ik}^{2} can be used for the linearization of $P_{21}^{e_{12}}$