OPTIMAL LEVERAGE FOR THE UTILITY MAXIMIZING FIRM

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This paper investigates the behavior of a utility maximizing firm which chooses both its financial structure and its technique of production. It is assumed that the cost of borrowing depends on the amount that is borrowed. Unlike Sandmo's utility maximizing firm which is excluded from financial markets, risk attitudes do not affect the production choice directly. Risk aversion shifts the demand curve for credit. This demand shift affects the marginal interest rate faced by the firm, and through the interest rate there is an effect on production. A demand for credit curve is derived explicitly from the production technology and risk attitudes; its shape is quite variable and it may possess a backward bending section. Interaction of credit supply and demand schedules may segment the population into groups with very different responses to risk or credit market policy interventions. JEL classifications 0223,3153,7140.

OPTIMAL LEVERAGE FOR THE UTILITY MAXIMIZING FIRM

Considerable attention has been paid to the behaviour of the utility maximizing firm under uncertainty (see Sandmo (1971), Hawawini (1978), Meyer (1987)). It has been shown for example that under price uncertainty the supply curve is shifted to the left by risk aversion, that the optimal output for a risk averse firm depends upon fixed costs, and that (under the assumption of decreasing absolute risk aversion) an increase in price risk will reduce the optimal level of output. In these typical studies, firms have a fixed capital base and choose operating levels which maximize expected utility. They do not lend, borrow or raise equity, and are effectively excluded from capital markets.

This paper investigates the behaviour of a utility maximizing firm which is allowed to choose its financial structure as well as its operating level and technique of production. For such a firm the response to risk may be quite different to that described by Sandmo et al. The interaction between risk aversion and borrowing is of particular interest because risk aversion manifests itself in production choices only if markets are incomplete. Correctly specifying these markets is important if risk response is to be modeled accurately. Given a complete set of Arrow-Debreu contingent securities, all firms will maximize profits. Differences in risk aversion will be accommodated by purchasing insurance. In practice most firm specific risk is not insurable, so there is little access to insurance markets, but almost all firms have some access to borrowing opportunities. Even without access to insurance markets, income smoothing through time by lending and borrowing can have an important effect on how firms react to risk. Newbery and Stiglitz

(1981, Chapter 14.3) show how important capital markets can be, but they do so under unrealistic assumptions that any amount can be loaned or borrowed at a fixed interest rate. Most of the large literature on the empirical estimation of attitudes to risk ignores the fact that adjustment of financial leverage may just as important as choosing a production technique (one exception is the study by Bardsley and Harris 1987). A similar criticism applies to a great deal of policy analysis, particularly in the area of commodity price stabilization, which assumes that risk attitudes derived from a laboratory study of the curvature of the consumption function can be transferred to an economic environment with hardly any consideration of how rich or how poor is the set of markets in that environment.

This paper attempts to model the effect of risk on production while at the same time paying careful attention to the interaction with other markets.

If a firm can raise both debt and equity finance then it is well known that management will maximize the market value of the firm, acting as a risk neutral profit maximizer. Ignoring second order effects, such as taxation, bankruptcy and agency costs, the financial structure is indeterminate (Modigliani and Miller 1958). For such firms the hypothesis of utility maximization has little to offer.

Utility maximization is relevant, however, to firms where ownership is concentrated and closely linked to control. Small business of all sorts comes under this description, with agriculture being the most extensive example. Such firms are typically excluded from broadly based equity markets because of information costs and the difficulty of monitoring and control. However such firms can and do borrow to finance their operations. There are many examples of both commercial and government institutions which specialize in lending to small business, especially to agriculture. This paper is concerned with this large class of utility maximizing firms which are excluded from broadly based equity markets, but are not excluded from all capital markets as in the studies of Sandmo et al.

I. Definitions

In this paper a model of interlinked production and borrowing decisions is is developed for a utility maximizing firm. The model is kept simple enough to solve explicitly, revealing the sequential nature of the decision process which is intrinsic to the problem, and leading to explicit formulae for the elasticity of demand for credit.

The model is of a two period nature. In period one the firm, endowed with a certain initial wealth, decides how much to borrow and chooses a risky production technique. In period two, the profits are revealed, the debt is repaid, and the firm's terminal wealth is realized. The model is thus comparative static in nature, dealing with the desired or optimal financial leverage and production decisions. It has nothing to say about the adjustment path by which these desired levels are approached.

It is assumed that the cost of borrowing depends on the amount that is borrowed, since the risk of default rises with increasing financial leverage. The lender can observe the borrower's initial wealth and is familiar with the available technology, but cannot monitor the choice of technique once the loan has been agreed. The cost of borrowing may rise smoothly or it may be a step function, with the borrower running into a credit limit after a certain point.

Let the firm's initial wealth be W, let D be the amount borrowed, and let $K = W + D$ be the working capital available for investment. D can of course be negative, in which case the firm is diversifying by lending outside the business. Let I(D) be the total cost of borrowing and let $\lambda = K/W$ be a measure of financial leverage. It can be assumed without loss of generality that the initial wealth W is normalized to unity, and this will be done henceforth in order to simplify the notation.

Debt financing affects the shape of the probability distribution of income in two ways. Firstly, the interest payment on the loan must be met irrespective of the uncertain outcome of production (bankruptcy and default are excluded from this model). This fixed cost translates the income distribution to the left by a fixed amount. Secondly, borrowing changes the ratio of working capital to owner's equity and scales up the income distribution (prior to the payment of interest) by the same amount. The income distribution is thus derived from the rate of return distribution by a change of location and scale parameters. It is then natural to assume following Meyer (1987) that the rate of return is drawn from a two parameter family of probability distributions which is invariant under changes of scale and location. This assumption will be met, for example, if the rate of return is normally distributed. The income distribution will then belong to the same two parameter family, and all probability distributions can be fully specified by giving the mean and standard distribution.

Let T be the set of available techniques (the technology set), and let $\tau \in T$ be the chosen technique. Let $r(\tau)$ be the realized rate of return given technique τ (constant returns to scale at the firm level will be assumed so that this rate of return is well defined irrespective of scale) This is a random variable with mean $\pi = \pi(\tau)$ and standard deviation $\sigma = \sigma(\tau)$. The uncertainty about r may include output uncertainty as well as price uncertainty about outputs or inputs. The technology set can be represented geometrically by a region in the mean-standard deviation plane (see Figure 1 below). If it is assumed that a risk efficient technique is always chosen (minimum standard deviation for a given mean rate of return), then π can be written as a function of σ:

(1) $$\pi = \pi(\sigma).$$

This is the equation of the efficiency frontier or the risk constraint.

INSERT FIGURE 1 ABOUT HERE

The geometry of the efficiency frontier will be important in what follows. Two parameters which will be needed below are its slope, measured in dimensionless form as an elasticity $\varepsilon = (\sigma/\pi)\pi'$, and its curvature measured by $\theta = -\sigma\pi''/\pi'$ (note the analogy with the Arrow-Pratt measure of relative risk aversion which describes the curvature of a utility function).

Before developing the model in detail, it may be of interest to discuss how some existing approaches to modeling risk in production fit into this framework. Two models, one by Sandmo (1971) and the other set out in Newbery and Stiglitz (1981, Chapter 12.2), will be discussed.

Consider Sandmo's model first. Output x is produced using a technology with cost function $B + C(x)$. There is no production uncertainty. Output is sold at an uncertain price p with mean $\bar{p}$ and standard deviation σ_p. Expected profit is $\pi = \bar{p}x - C(x) - B$, the standard deviation of profit is $\sigma = x\sigma_p$, and the equation of the risk constraint is $\pi = \bar{p}(\sigma/\sigma_p) - C(\sigma/\sigma_p) - B$. The curvature of the risk constraint in this model is intimately related to the curvature of the cost function.

Now consider the Newbery and Stiglitz example, which is concerned with output uncertainty rather than price uncertainty. A farmer plants a fixed area (normalized to unity) with two risky crops. He plants an area α of crop 1, which has expected yield y_1 with standard deviation σ_1, and an area $\beta = 1 - \alpha$ with crop 2, which has expected yield y_2 with standard deviation σ_2. Output is sold at a certain price, normalized at 1. Production cost for both crops is the same and may without loss of generality be assumed zero. Expected profit is $\pi = \alpha y_1 + \beta y_2$, standard deviation of profit is $\sigma^2 = \alpha^2\sigma_1^2 + \beta^2\sigma_2^2$. The equation of the risk constraint is, in implicit form,

$$\sigma = \sqrt{\frac{(\pi-y_2)^2\sigma_1^2 + (y_1-\pi)^2\sigma_2^2}{(y_1 - y_2)^2}}.$$

The curvature of the risk constraint is in this case related to the variation (and possibly covariation) in yields.

II. The Model

Having chosen technique $\tau\epsilon T$ and leverage λ, income is a random variable $Y = r\lambda - I(\lambda)$ with mean $\bar{Y} = \pi\lambda - I(\lambda)$ and standard deviation $\sigma_Y = \sigma\lambda$. It will be assumed that the firm maximizes expected utility EU(Y) subject to the risk constraint (1). Since initial wealth is fixed, it is convenient to write utility in terms of income rather than terminal wealth. The curvature of U should thus be interpreted as measure of partial risk aversion (see Menezes and Hanson (1970), Zeckhauser and Keeler (1970)). The Lagrangean which describes the firm's problem is

$$(2) \qquad \mathcal{L} \quad = \quad E[\ U(\pi(\sigma) - I + \sigma\lambda z)\ ] + A(I - I(\lambda)),$$

where A is a Lagrange multiplier, z is a standardized random variable (drawn from the two parameter family of probability distributions) with mean zero and unit variance, and the expectation is taken with respect to z.

Before considering the optimal decisions implied by this formula it is useful to review some of the geometrical properties of the mean - standard deviation diagram which were established by Sandmo and Meyer. Figure 2 shows a map of indifference curves in the mean - standard deviation diagram. Let $V(\sigma_Y,\bar{Y}) = EU(Y)$ be the induced preferences over the parameters $\bar{Y}$ and σ_Y. Meyer shows that (writing $z = (Y - \bar{Y})/\sigma_Y$)

$$V_1 = E[\ z\ U'(Y)\ ],$$

$$V_2 = E[\ U'(Y)\ ],$$

$$S(\sigma_Y,\bar{Y}) = -V_1/V_2,$$

where S is the slope of the indifference curve. If $U'\geq 0$ and $U''\leq 0$ then $V_1\leq 0$, $V_2\geq 0$, $S\geq 0$, and V is a concave function of σ_Y and $\bar{Y}$. If absolute risk aversion is decreasing then $S_{\bar{Y}}\leq 0$ and if if relative risk aversion is increasing then $\bar{Y}S_{\bar{Y}} + \sigma_Y S_{\sigma_Y} \geq 0$ (with equality holding under constant relative risk aversion).

To simplify the presentation, write $i = I'(\lambda)$ for the marginal interest rate and $\bar{i} = I(\lambda)/(\lambda-1)$ for the average interest rate. The first order conditions implied by the Lagrangean (2) are

$$(3) \qquad \pi = i + \sigma\pi_{\sigma}$$

$$(4) \qquad \pi_{\sigma} = S.$$

First consider equation (3), which describes the choice of technique; this is shown in Figure 1. The tangent at (σ,π) to the risk constraint cuts the π axis at the marginal interest rate i. For a given i equations (1) and (2) have a unique solution $\sigma^{*}(i)$, $\pi^{*}(i)$ provided that the risk constraint is convex as has been assumed here. This leads to the first major implication of this model. The choice of technique $\tau \in T$ depends only on the technology (the risk constraint) and the marginal interest rate. Given i, all firms adopt the same technique irrespective of their degree of risk aversion. The following Proposition describes how the choice of technique changes when the marginal interest rate is varied.

PROPOSITION 1: *Let $\sigma^{*}(i)$ and $\pi^{*}(i)$ be the optimal production risk and expected rate of profit, given the marginal interest rate* i. *Then*

(i) $$\frac{\partial\sigma^{*}(i)}{\partial i} = \frac{1}{\sigma\varepsilon\theta} \geq 0;$$

(ii) $$\frac{\partial\pi^{*}(i)}{\partial i} = \frac{1}{\theta} \geq 0.$$

PROOF: This is immediate, after differentiating equation (3) with respect to i.

INSERT FIGURE 2 ABOUT HERE

Now consider the choice of scale, λ. Given the optimal choice of technique (that is to say of $\sigma^{*}(i)$ and $\pi^{*}(i)$), leverage is determined by choosing σ_Y, and hence $\lambda = \sigma_Y/\sigma^*$, to maximize $V(\sigma_Y,\bar{Y})$ along the line

$$\bar{Y} = \left(\pi_\sigma^* + \frac{i-\bar{i}}{\sigma^*}\right)\sigma_Y + \bar{i}. \tag{5}$$

This is shown in Figure 2. V is a concave function of σ_Y under the assumption of risk aversion, so this problem has a unique solution $\lambda^*(i,\bar{i})$. Optimal leverage depends on both the marginal and the average interest rate. The following Proposition describes how the optimal leverage changes when the marginal or average interest rate is varied.

PROPOSITION 2: *Let $\lambda^*(i,\bar{i})$ be the optimal leverage given the marginal interest rate i and the average rate $\bar{i}$. Then*

(i) $\dfrac{\partial\lambda^*(i,\bar{i})}{\partial\bar{i}} = 0$ *under constant absolute risk aversion;*

(ii) $\dfrac{\partial\lambda^*(i,\bar{i})}{\partial\bar{i}} = 0$ *if $\lambda = 1$;*

(iii) $\dfrac{\partial\lambda^*(i,\bar{i})}{\partial\bar{i}} \leq 0$ *if $\lambda > 1$, absolute risk aversion is decreasing and relative risk aversion is constant or increasing;*

(iv) $\dfrac{\partial\lambda^*(i,\bar{i})}{\partial\bar{i}} \geq 0$ *if $\lambda < 1$, absolute risk aversion is decreasing and relative risk aversion is constant or increasing;*

(v) If agent 1 has preferences which are everywhere more risk averse than those of agent 2, and if agents 1 and 2 choose optimal leverage $\lambda_1^*(i,\bar{i})$ and $\lambda_2^*(i,\bar{i})$ respectively, then $\lambda_1^*(i,\bar{i}) < \lambda_2^*(i,\bar{i})$.

PROOF: Substitute $\sigma_Y = \sigma\lambda$ and $\bar{Y} = (\pi-\bar{i})\lambda+\bar{i}$ into equation (4) and differentiate with respect to $\bar{i}$. This gives

$$\frac{\partial\lambda}{\lambda\partial\bar{\imath}} = \frac{(\lambda-1)S_{\bar{Y}}}{\left(\bar{Y}S_{\bar{Y}} + \sigma_Y S_{\sigma_Y}\right) - \bar{\imath}\, S_{\bar{Y}}}.$$

Properties (i) to (iv) then follow from the properties of S set out previously.To prove (v), let λ_1^* and $\sigma_Y^* = \sigma^*\lambda_1^*$ be the optimal choice of agent 1. Then the indifference curves of V are tangent to the line (5) at this point. If agent 2 is more risk averse, then the indifference curves for agent 2 are steeper (more positively sloped) than those of agent 1 (Meyer 1987 Property 7). Given the convexity of preferences the point of tangency for agent 2 will thus be to the left of that for agent 1.

The second order necessary conditions are as follows:

(6) $\quad \pi_{\sigma\sigma} \le S_\sigma$

(7) $\quad I''(\lambda) \ge -\sigma S_\lambda$

(8) $\quad (S_\sigma - \pi_{\sigma\sigma})(I''(\lambda) + \sigma S_\lambda) \ge \lambda S_\lambda^{\sim}$

The first of these states that the risk constraint must be either convex downwards or at least must not curve upwards too rapidly (it will be noted in Lemma 1 below that under realistic circumstances both S_λ and S_σ may be positive). The possibility that an equilibrium may exist under constant or mildly increasing returns to scale was noted by Sandmo in relation to his model. The second condition states that either the marginal interest rate increases with leverage or, if it is decreasing then it does not decrease too rapidly. The third condition is not so easy to interpret. It says roughly that equality cannot hold simultaneously in (6) and (7). The following Lemma is useful in signing these second order effects.

LEMMA 1: *Let ι be the marginal interest rate, let* $\bar{\imath}$ *be the average interest* *rate , and let* $\sigma = \sigma^*(i)$, $\pi = \pi^*(i)$, $\lambda = \lambda^*(i,\bar{I})$ *be the optimal production and financing decisions. Let* $m = \bar{Y}S_{\bar{Y}} + \sigma_Y S_{\sigma_Y}$ *and let G by the gradient of* $S(\sigma_Y,\bar{Y})$ *in the direction with slope* π_σ *(see Figure 3). Then:*

(i) $$\lambda S_\lambda = \sigma S_\sigma\ ;$$

(ii) $$S_\sigma = \sigma\sqrt{1+\pi_\sigma^2}\ G;$$

(iii) $$S_\sigma = \frac{m}{\sigma} + \lambda\left[\frac{\bar{Y}}{\sigma_Y} - \pi_\sigma\right] S_{\bar{Y}}.$$

Assume also that the marginal interest rate is greater than or equal to the average interest rate and that the average interest rate is positive. Then

(iv) $$\pi_\sigma \le \frac{\bar{Y}}{\sigma_Y}\ ;$$

(v) *under constant relative risk aversion*

$$S_\lambda \ge 0$$
$$S_\sigma \ge 0;$$

(vi) *under increasing relative risk aversion and decreasing absolute risk aversion*

$$S_\lambda \ge 0$$
$$S_\sigma \ge 0.$$

PROOF: To prove the symmetry condition $\lambda S_\lambda = \sigma S_\sigma$, calculate the cross derivatives $\frac{\partial^2 V}{\partial\lambda\partial\sigma} = \frac{\partial^2 V}{\partial\sigma\partial\lambda}$ and substitute in the first order conditions. To prove (ii), let $\varphi(\sigma) = (\sigma\lambda,\pi\lambda-I)$. Then $\varphi'(\sigma) = (\lambda,\lambda\pi_\sigma)$, which is a vector of length $\lambda\sqrt{1+\pi_\sigma^2}$. Thus $G = (dS(\varphi(\sigma))/d\sigma)/(\lambda\sqrt{1+\pi_\sigma^2})$, which is equation (ii). To prove (iii), note that $S_\sigma = \lambda\sigma S_{\sigma_Y} + \lambda\pi_\sigma S_{\bar{Y}}$. The result follows if the definition of m is used to eliminate S_{σ_Y}. The proof of (iv) is immediate from the definitions and the first order conditions.

The inequalities (iv) and (v) follow from (iii) and (iv) and the results of Meyer (1987), who proves first of all that the derivative $S_{\bar{Y}}$ is negative, (zero, positive) according to whether absolute risk aversion is decreasing, (constant, increasing), and second that the derivative derivative m is negative, (zero, positive) according to whether relative risk aversion is decreasing, (constant, increasing).

The geometry of Figure 3 may help to make more intuitive the implications of the Lemma. The sign of the derivatives of S in the vertical and the radial directions are known from the results of Meyer in terms of absolute and relative risk aversion coefficients. Under decreasing absolute risk aversion (DARA) and constant or increasing relative risk aversion (CRRA or IRRA) the gradient of S lies in the shaded cone. Both S_{σ} and S_{λ} are a positive constant times the derivative of S in the direction with slope π_{σ} ((i) and (ii)). This derivative is just the projection of the gradient in the direction π_{σ}. The slope π_{σ} is positive and from (iv) the direction with slope π_{σ} lies to the right of the radial direction. Thus the projection on the gradient of S is positive.

INSERT FIGURE 3 ABOUT HERE

From Proposition 2 it follows that leverage depends on both marginal and average interest rates so strictly speaking there is no demand curve for credit. There are two cases where demand depends only on the marginal interest rate and a demand curve can be properly defined. These are

(i) if risk preferences display constant average risk aversion (Proposition 2(*i*)), or

(ii) if the marginal interest rate is everywhere equal to the average interest rate.

The complexities of the general case will not be explored any further in this paper and the following simplifying assumption will be made.

ASSUMPTION 1: *It will henceforth be assumed that the average and marginal interest rate are the same.*

This assumption is not quite as restrictive as may at first appear. Figure 3 shows a supply curve for credit which satisfies the assumption and which has the following features. If $\lambda<1$ then the firm is a net lender and faces a constant interest rate i_0. This would seem to be quite a natural assumption. If $1<\lambda<\lambda_1$ then the firm faces a constant interest rate $i_1>i_0$. This is not so attractive an assumption, but it is actually not wildly at variance with the stylized facts which are observed in many circumstances. The interest rate may be held constant either because of the regulatory environment or because of the transactions cost of calculating a price for each customer. For $\lambda>\lambda_1$ credit rationing occurs and both the marginal and average interest rate is effectively infinite.

III. Interpretation of the model

In Sandmo's model risk aversion is reflected directly in the firm's choice of technique. Risk averse firms reduce variable inputs and output. In contrast, in the model developed here, risk has no direct impact on the choice of technique. Equation (3) shows that for a given marginal interest rate all firms make the same production decision, irrespective of their attitude to risk. Risk attitudes enter the model only through the credit market and the financing decision. They affect the production decision indirectly through the interest rate. Figure 3 shows how the effect of risk aversion on the production decision is filtered through the credit market. Given Assumption 1, a demand curve for credit is well defined as a function of the interest rate i. Assume for the moment that the demand curve is everywhere downward sloping (this issue will be explored later in the paper). By Proposition 1, the demand curve shifts leftwards as preferences become more risk averse. Depending upon risk preferences the demand curve may cut the supply curve on a horizontal segment or on a vertical segment.

INSERT FIGURE 4 ABOUT HERE

In discussing the effect of risk aversion it is useful to distinguish between the choice of technique (that is the choice of σ) and the choice of scale (that is the choice of λ). Four cases can be distinguished. Refer to Figure 3, in which possible demand curves are labeled D_1 to D_4.

(i) Demand D_1. Highly risk averse individuals will be net lenders and will face a perfectly elastic supply curve. All of these individuals will face the same marginal interest rate and will choose identical techniques of production despite their differing risk preferences. They will choose to adjust their risk exposure by adjusting their financial position rather than their production technique.

(ii) Demand D_2. The next group of firms, who are slightly less risk averse, face a completely inelastic credit supply. They neither lend nor borrow, and are completely insulated from small changes in credit market conditions This is precisely the case analyzed by Sandmo. For these firms the interest rate i should be interpreted as a shadow price, lying between i_0 and i_1. As risk aversion decreases the demand curve shifts right and i increases. In Figure 1, as i increases the point of tangency shifts to the right. A riskier but more profitable technique is chosen.

(iii)Demand D_3. As risk aversion decreases further, the credit supply curve again becomes elastic, but at the higher interest rate i_1. These firms all choose the same technique and differentiate themselves by their choice of scale.

(v) Demand D_4. Finally, at the lowest levels of risk aversion credit is rationed, adjustment of scale is impossible, and risk attitudes are again expressed through the choice of technique.

Simple though it is, this model is consistent with a wide variety of different behaviour patterns. In some cases there may be a uniform choice of technique, despite differing risk attitudes. In other cases, risk attitudes may be closely reflected by changes in the choice of technique. Some firms reveal their type (their degree of risk aversion) by their demand for credit. Others (those who are constrained) do not.

Different subgroups of the population may respond quite differently to the same policy intervention, whether that intervention be in the credit market or through price stabilization. The size of these groups depends on both the distribution of attitudes to risk within the population (see Antle (1987) for a discussion of this) and on the demand elasticity. If credit demand is very elastic then it is more likely to intercept the supply curve in a vertical segment, and conversely if demand is inelastic then it is more likely that it will intersect an elastic portion of the supply curve. Is this heterogeneity likely to be seen in practice? Clearly this will vary from case to case but it may be worth noting that, at least for the case of Australian agriculture, there is considerable variation in debt. Around half of Australian farms are net lenders rather than borrowers while there is always a small group at any time who are highly levered and probably credit constrained (ABARE 1990). Thus all four types of firms are likely to be present in this population.

IV. Elasticity of demand for credit

For simplicity of notation, it will be assumed throughout this section that σ, π and λ always take their optimal values $\sigma^*(i)$, $\pi^*(i)$ and $\lambda^*(i)$. In any particular case, given a utility function and a specific family of probability distributions, equations (1), (3) and (4) can be solved either explicitly or numerically to give a demand curve for credit. For example, it is easy to show the following result if returns are normally distributed and the firm has constant absolute risk aversion k (in this case $S_\sigma = k\sigma\lambda^2$).

PROPOSITION 4: *Assume that* r *is normally distributed and that the coefficient of local absolute risk aversion* k *is constant. Then*

$$\lambda = \frac{\pi_\sigma}{2k\sigma}.$$

and the demand curve is downward sloping.

Proposition 4 is illustrated in Figure 5, which shows typical constant absolute risk aversion demand curves for the risk constraint $\pi = 1 - 1/\sigma$. In the general case the sign of the slope is not so clear cut.

INSERT FIGURE 5 ABOUT HERE

PROPOSITION 5: *The slope of the demand curve for credit is*

$$\lambda_i = \frac{-\lambda}{\sigma^2}\left[\frac{1}{\varepsilon\theta} + \frac{1}{S_\sigma}\right].$$

The demand curve is downward sloping unless

$$-\varepsilon\theta \leq S_\sigma \leq 0.$$

If absolute risk aversion is decreasing and relative risk aversion is constant or increasing then the demand curve slopes down.

PROOF: Differentiating Equation (4) with respect to i gives

$$(\pi_\sigma)_i = \sigma_i S_\sigma + \lambda_i S_\lambda.$$

The equation then follows using Proposition 1 and Lemma 1. The sign of λ_i under constant or increasing relative risk aversion is a consequence of Lemma 1 and the convexity of the risk constraint.

Can the demand curve for credit slope backward? From Lemma 1, S_σ can be negative if relative risk aversion is decreasing at the right rate. What would this mean in economic terms? To say that risk aversion is a decreasing function of income is equivalent to saying that as income falls risk aversion rapidly increases. Consider what happens as interest rates fall. As the price of borrowing decreases one might expect, by a normal substitution effect, that demand would increase. However as interest rates fall the firm moves down the risk frontier in Figure 1, choosing a more conservative technique, and gross expected income decreases. Interest costs will also fall and the effect on net expected income is indeterminate. If however $\lambda < 1$, as may well be the case for a very risk averse firm, then interest payments are revenue not a cost. In this case expected income falls unambiguously as interest rates fall. If risk aversion increases rapidly as a consequence of falling income, then desired leverage may fall since reducing leverage always reduces income variance. Thus it is conceivable that the demand curve may bend backward.

I can report, after some days spent doing numerical integrations, that it is not easy to explore this issue empirically. Utility functions which display decreasing relative risk aversion have a singularity at zero, where they approach negative infinity. Expected utility is not defined for many probability distributions, such as the normal distribution, which have an unbounded range. If utility functions such as lognormal or rectangular distributions are used then the problem re-emerges whenever the bottom of the range is translated past the singularity when leverage or the variance change. To make matters more difficult, the phenomenon in question can be expected to occur at very low incomes, close to the singularity.

The assumption of constant relative risk aversion is analytically intractable, but there is an alternative assumption of constant relative global risk aversion which is easier to handle (Bardsley ,1991). In situations where risk is large relative to income Bardsley has argued that a global measure of risk aversion $-E[U'']/2E[U']$ may be more appropriate than the Arrow Pratt local measure $-U''/2U'$. There appears to be reasonable evidence, summarized by Newbery and Stiglitz (1981, Chapter 7) for constant partial risk aversion (that is, constant relative risk aversion conditioned on income not wealth) with a coefficient of about 1. If Bardsley's argument is accepted, then this evidence would also be consistent with a hypothesis of constant global relative income risk aversion, particularly if the data were collected in an environment where risk was not vanishingly small. In any case, this hypothesis provides quite a tractable framework in which to further explore the slope of the demand curve under assumptions which may be less unrealistic than constant absolute risk aversion.

The hypothesis of constant relative global risk aversion is consistent with the following preference functional (or any monotone transformation of it) in the mean variance plane:

$$V(\sigma_Y, \bar{Y}) \quad = \quad \bar{Y}^2/2 - R\sigma_Y.$$

If σ_Y is small relative to $\bar{Y}$ then the indifference curves are like those associated with constant relative risk aversion. It can be shown, using the test in Bardsley (1991) that this choice rule is not consistent with the expected utility hypothesis. It is however consistent with Machina's generalized expected utility framework (Machina 1982) provided that all probability distributions have a bounded range.

PROPOSITION 6: *Assume that the relative global risk aversion coefficient* R *is constant. Then*

(i) $$S \quad = \quad \frac{-R\sigma\lambda}{\pi\lambda - I};$$

(ii) $$\lambda \quad = \quad \frac{i}{\sigma\left[\frac{2R}{\pi_\sigma} - \pi_\sigma\right]}.$$

Let $\varphi = \sigma/\pi$ *be the coefficient of variation when* $i = 0$. *Assume* *that* $R > \frac{1}{2\varphi^2}$. *Then*

(iii) $$\lambda_i(0) = \frac{1}{\sigma\left[\frac{2R}{\pi_\sigma} - \pi_\sigma\right]};$$

(iv) *the demand curve has a backward bending section whenever* $R > \frac{1}{2\varphi^2}$.

PROOF:(i), (ii) and (iii) can be derived by straight forward algebra from the first order conditions. If $R < \frac{1}{2\varphi^2}$ then the demand curve goes asymptotically to infinity at a positive interest rate, so the behaviour of the curve in the vicinity of $i = 0$ has no economic meaning (the first order condition is for utility minimization rather than maximization). If $R > \frac{1}{2\varphi^2}$ then $\left[\frac{2R}{\pi_\sigma} - \pi_\sigma\right] > 0$ at $i = 0$. Thus $\lambda(0) = 0$ and $\lambda_i(0) > 0$. Thus the demand curve passes through the origin and has positive slope in this vicinity. Figure 6 illustrates a typical family of demand curves under the hypothesis of constant global relative risk aversion for the risk constraint $\pi = 1 - 1/\sigma$.

INSERT FIGURE 6 ABOUT HERE

Is this just a theoretical curiosity? Proposition 6 allows some insight into the conditions where the demand curve can bend backwards. Newbery and Stiglitz (1981) suggest, after reviewing the literature, that a typical coefficient of variation in agriculture is .33, with a range from .2 to .5. Given these values of φ, Proposition 6 (iv) indicates that the demand curve will slope backwards if R exceeds 4.5, 12.5, and 2 respectively. Newbery and Stiglitz suggest that R is typically in the range 0 to 2, with few individuals having a value of R greater than 2. Thus it seems unlikely that a backward bending curve would be observed in practice, and then only for the most risk averse individuals in highly risky situations.

V. Summary and Conclusion

This paper discusses the production and financing decision of a risk averse utility maximizing firm which has access to financial markets in which it may lend or borrow. It is assumed that the cost of borrowing may increase with the quantity borrowed and that credit may be rationed beyond a certain point.

Sandmo (1971) describes a similar model of production under price risk but in which there is no access to credit markets. The main interest of this paper is in how this access changes the behaviour described by Sandmo. Once there is access to credit markets, firms face two decisions. The first is the production decision: choice of crop or livestock mix, intensity of input use and so on. This choice is referred to in this paper as the choice of technique. The second decision is whether to borrow to gain access to more capital for the production process. Increasing the capital base may lead to a greater gross income but this must be balanced against borrowing costs and the greater risk exposure created by the need to service debt. This second choice is referred to here as the choice of scale or the choice of leverage.

It is shown that these choices are sequential. In contrast to Sandmo's model, risk attitudes do not directly affect the production decision or the choice of technique. Given the same marginal borrowing cost, all firms will make identical production decisions irrespective of their attitudes to risk.

An increase in the marginal interest rate leads to the use of a more profitable but also more risky technique.

Risk attitudes do however directly influence the financing decision. For a given production technology the amount which will be borrowed depends on both the marginal and the average interest rate. Only under special assumptions on the shape of the supply curve does there exist a conventional demand curve as a function of a single interest rate. Whether it is treated as a function of one interest rate or of two, greater aversion to risk shifts the demand curve to the left. Depending on the shape of the credit supply curve, this shift may lead to a reduction in the amount borrowed, a fall in the marginal interest rate, or both. Any fall in the interest rate will in turn cause an adjustment to the production choice, leading to a less profitable but safer technique. Thus risk attitudes indirectly affect the choice of technique.

Sandmo's model is recovered when credit is rationed or constrained. In this case the supply curve is vertical, and the marginal interest rate should be interpreted as a shadow price on this constraint. Greater aversion to risk reduces the demand for credit, reducing the shadow interest rate, leading in turn to a more conservative choice of technique. In Sandmo's model this is achieved by reducing the output level.

Risk response thus depends on the shape of both the supply and demand curves for credit. The shape of the supply curve should be derived endogenously within the model, but this difficult problem is not tackled in this paper. It leads to interesting problems of nonlinear pricing, signaling equilibria and market structure. A not implausible simplification of the supply side is studied, though, in which credit is supplied perfectly elastically up to an exogenously imposed credit limit. It is shown that the interaction of supply and demand may segment the population into a number of subgroups who respond quite differently to risk. Some adjust their production technique while others make no production change but adjust in stead their financial structure. Because of this heterogeneity policy interventions may have quite different effects on different subgroups.

The shape of the demand curve is also quite important. It is shown that the credit demand elasticity can be calculated from a specification of the production technology and of attitudes to risk. The shape of the demand curve is quite sensitive to the specification of risk attitudes. In some cases demand reaches a finite maximum when the interest rate approaches zero; in other cases it goes asymptotically to infinity at some positive interest rate. In other cases, demand may become inelastic at low interest rates and may even be backward bending (although this is unlikely to be observed at realistic parameter values).

The model presented here is fairly simple, but it reveals patterns of behaviour which are potentially quite complex. More work is clearly needed to model the supply side more satisfactorily, but in the mean time the model may be of some use in considering such issues as the supply response to risk and the distributional effects of commodity price stabilization.

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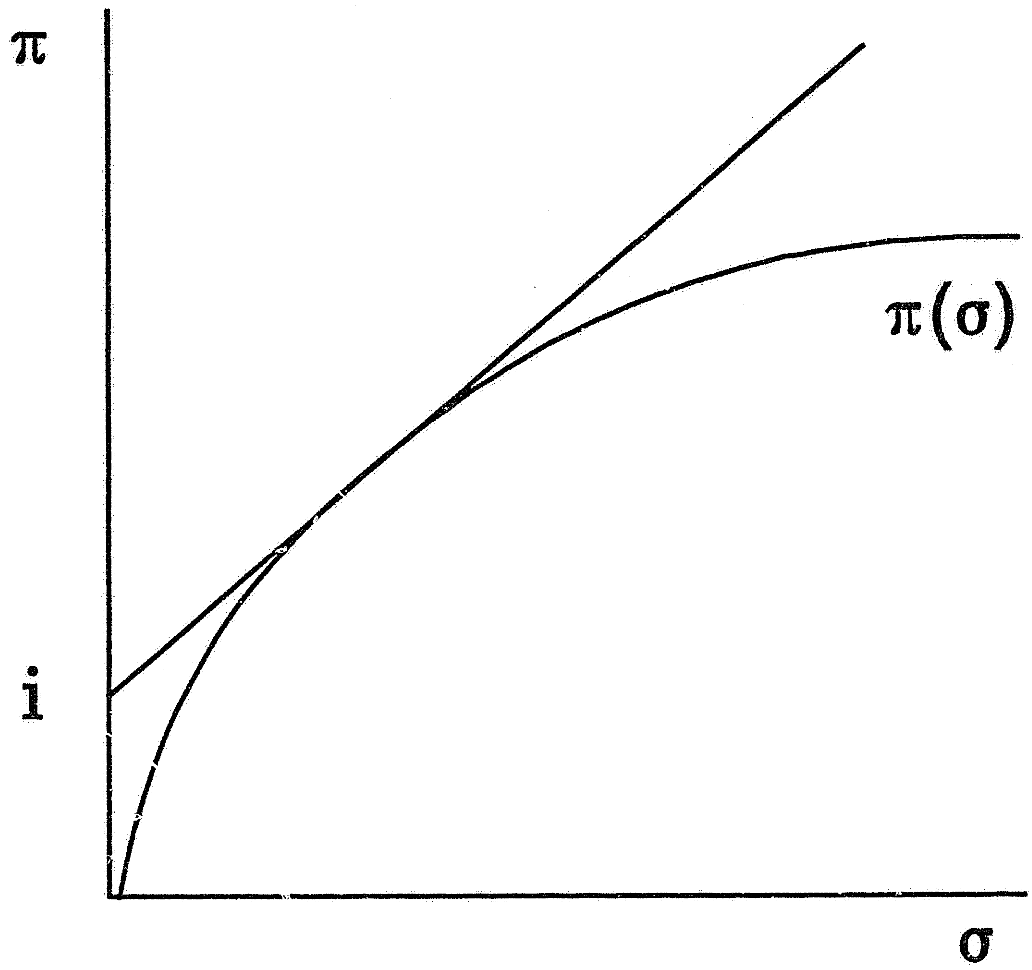

Figure 1. Choice of technique.
Production risk and return.

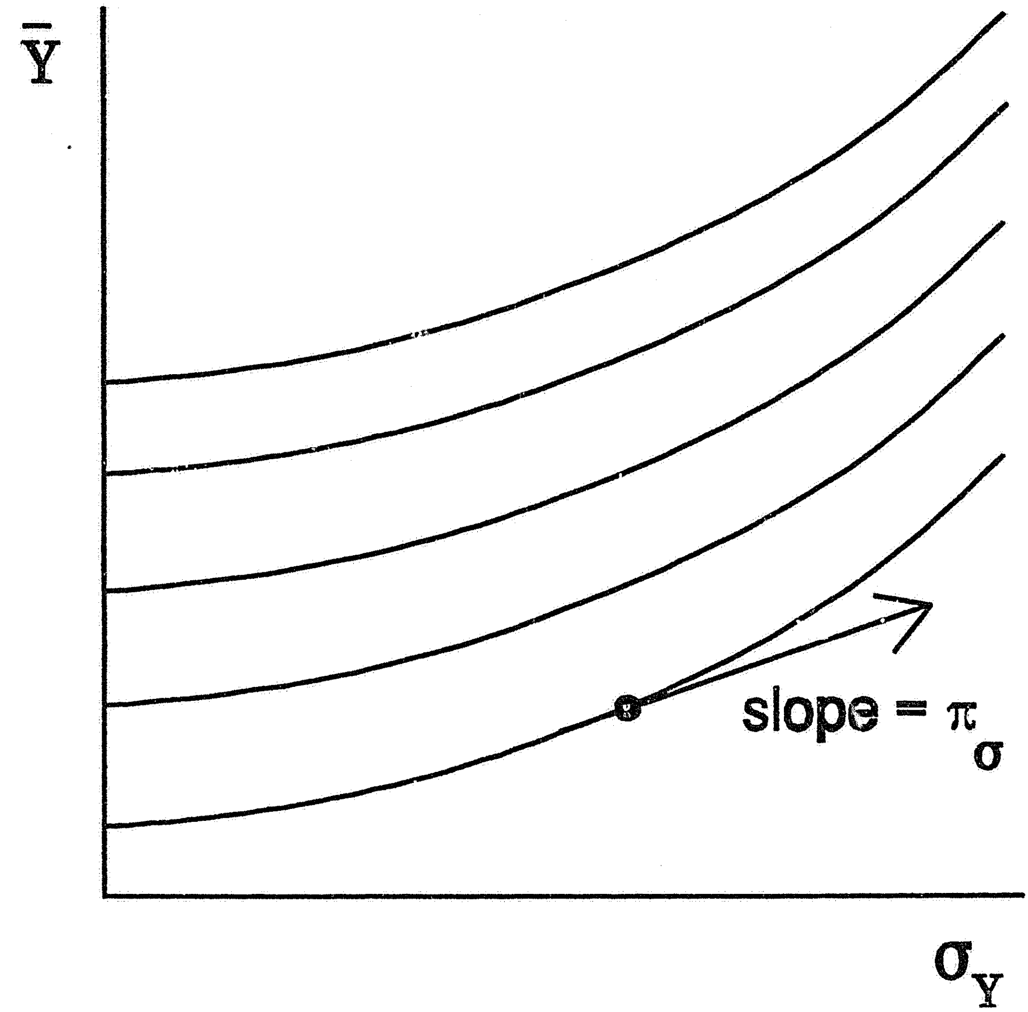

Figure 2. Choice of scale and financial leverage.

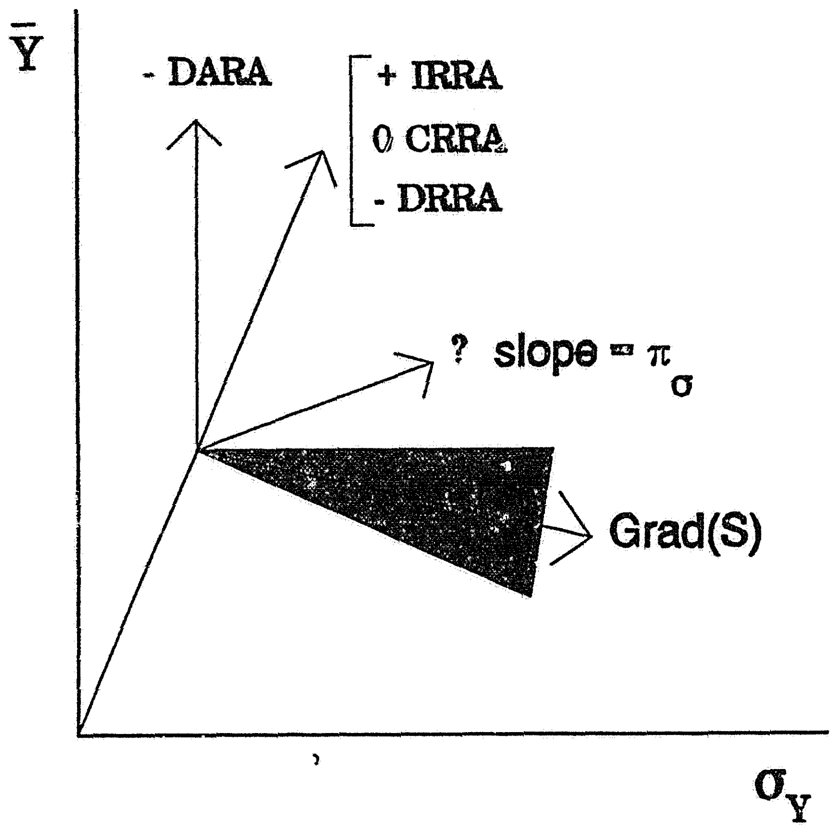

Figure 3. Signing the gradient of S

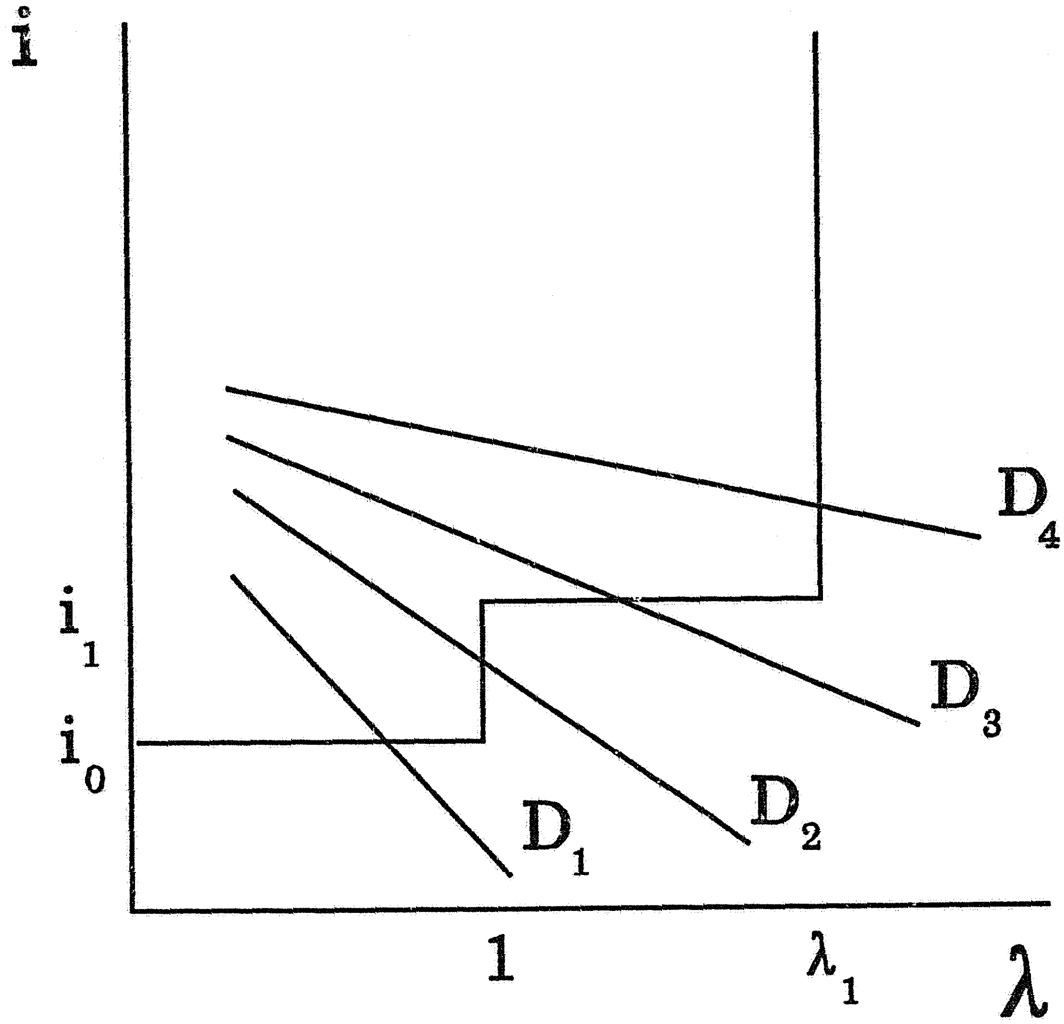

Figure 4. Supply of and demand for credit.

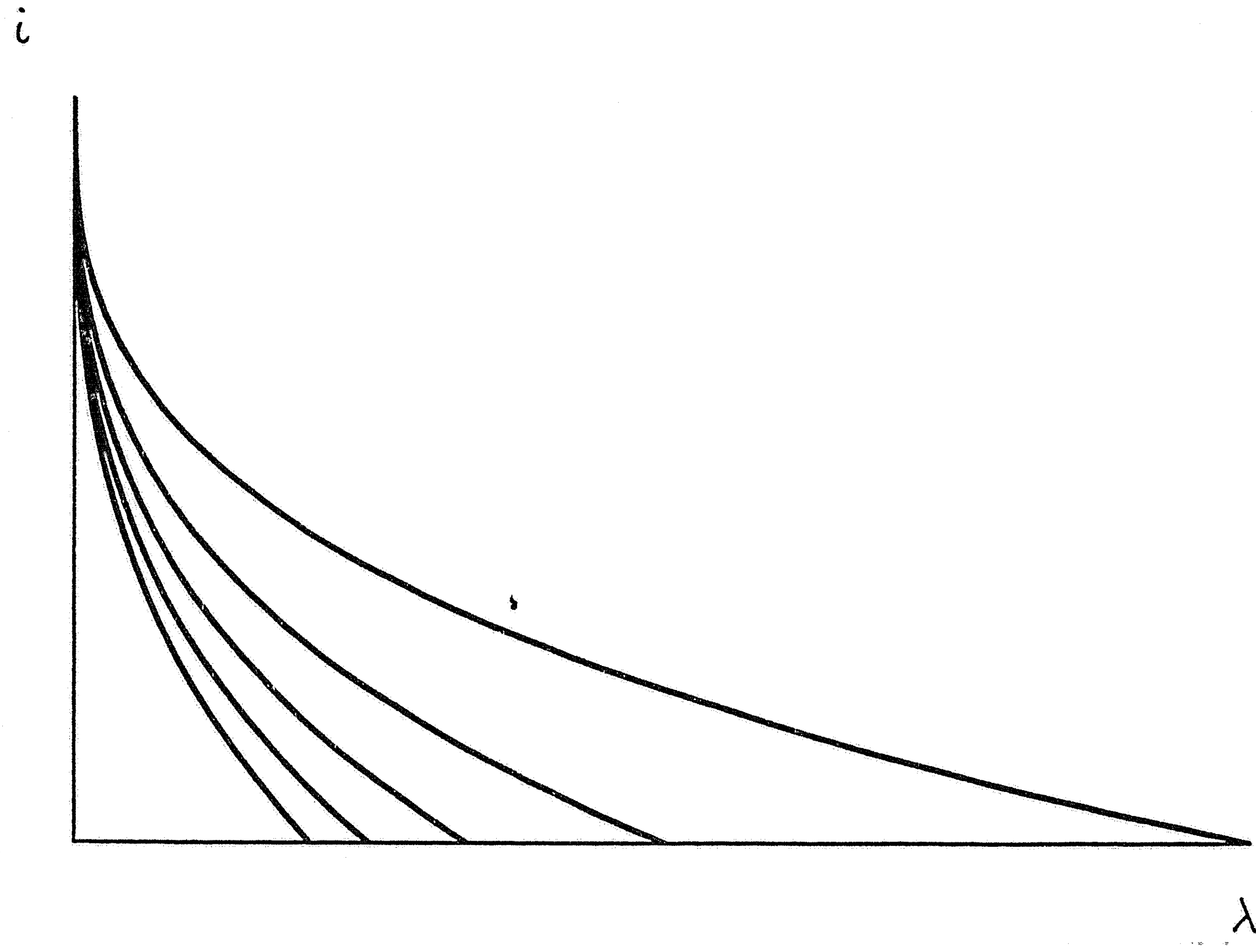

Figure 5. The demand for credit: constant absolute risk aversion.

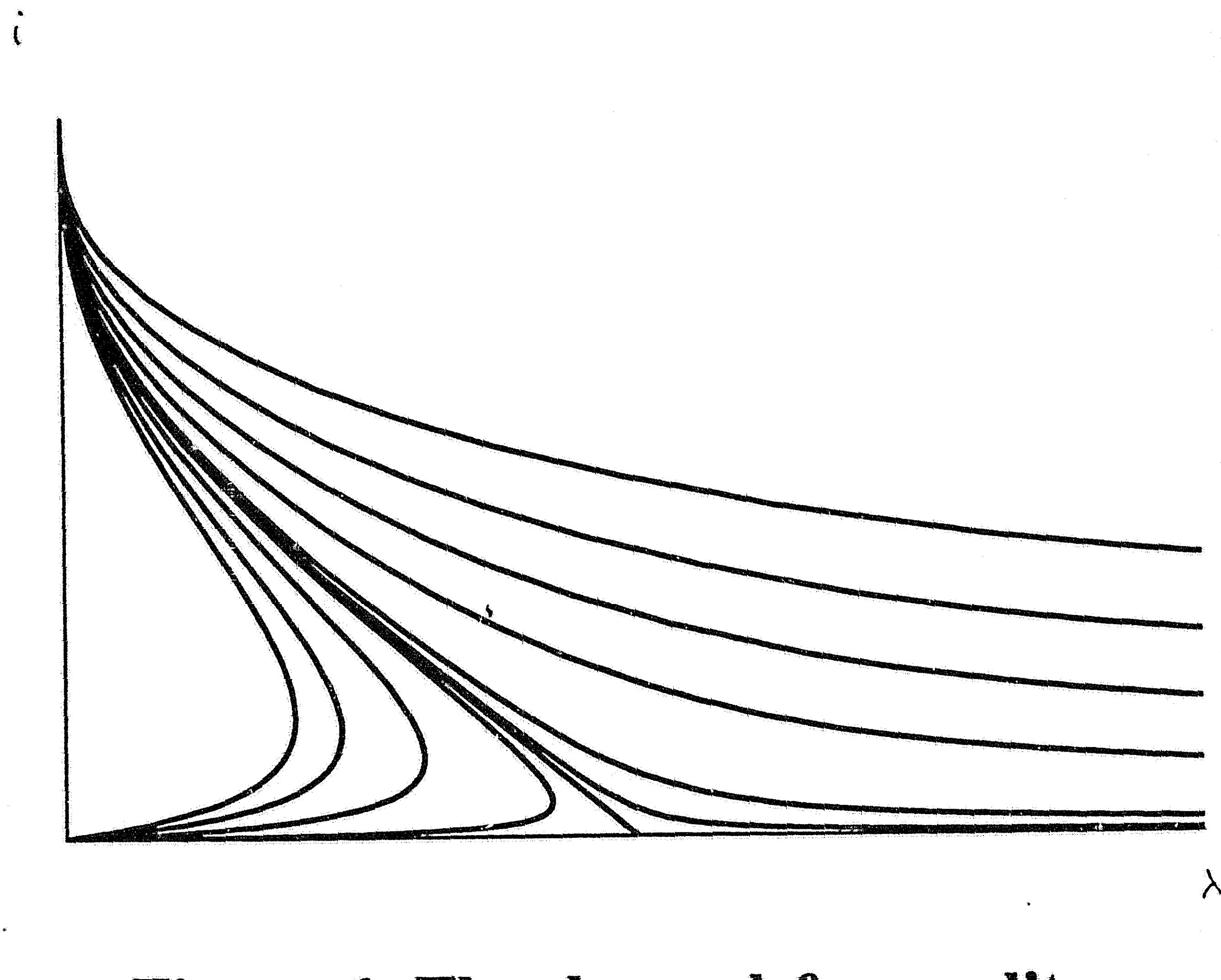

Figure 6. The demand for credit: constant relative global risk aversion.