Exact Aggregation--A Discussion of Miller's Theorem

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TOM MILLER, in an article in this journal,[1] has shown that the optimal responses of different farms to a given set of relative product prices will be proportional if two conditions are met. The conditions are that the farms have homogeneous activity vectors and that the same activities appear in the linear programming solution vector for each farm.

Miller's paper and this discussion of it are concerned with the aggregation problem, which is essentially one of grouping farms that respond alike in linear programming models of agricultural supply. Miller starts by observing that levels of management and practices are assumed given or are specified in most research projects so the input-output matrices for large groups of farms are identical. He further observes that all farm managers can be assumed to behave in a consistent, objective manner; thus, an acceptable procedure is to assume all farmers have the same net return expectations. These two observations allow Miller to focus his concern on Richard Day's condition of proportionality of resource vectors among all farms in the aggregate.[2] Miller sets forth the following theorem:

> "Sufficient conditions for exact aggregation are (1) that all farms have identical coefficient matrices, that is, that $B^* = B_g$ for all g, and (2) that all farms have qualitatively homogeneous output vectors,"

where the asterisk (*) denotes the aggregate set of farms and the subscript g represents the individual farms. "Qualitatively homogeneous output vectors" means that all farms in the aggregate have the same activities included in the solution vector.

Several points can be made about Miller's aggregation theorem. One weakness is that the groups of farms having qualitatively homogeneous output vectors are unique for each set of relative product prices. This is because, ceteris paribus, the nonzero activities in the solution vector depend on relative activity net returns. These, in turn, depend on relative product prices. For each additional set of prices considered, all farms have to be reprogrammed to determine which farms are common to a given group over the whole range of prices. This does not invalidate Miller's theorem. It does imply that in a model designed to estimate supply response to wide price changes, a burdensome amount of computation may be required.

Richard Day's proportionality conditions are general with respect to price changes; that is, the same farms lend themselves to exact aggregation at all price combinations. (However, one could point out that Day's farm groupings are not general if one considers, say, the disproportionate effects on the resource structure of farms resulting from changes in Government allotment programs.) A group of farms aggregated under Day's more restrictive proportionality conditions are a subset of a group aggregated under Miller's less restrictive conditions for a specified set of product prices. As the product price ratios are varied over an infinite range, Miller's sets of farms approach but do not reduce to Day's subsets. The import of this will be pointed out later.

Another shortcoming of Miller's aggregation theorem centers around its practical applicability. The fact that his conditions are defined as a requirement of the solutions to the individual farm problems, rather than a requirement of the farms themselves, provides a less than ideal approach to the problem of delineating representative farms. Miller recognizes this. Day's conditions could be used to group farms simply by observing the farm characteristics.

Miller's work, as it stands, represents progress. In a fairly homogeneous farming area, large groups of farmers employ similar

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[1] Thomas A. Miller, "Sufficient Conditions for Exact Aggregation in Linear Programming Models," this journal, this issue, p. 52.

[2] Richard H. Day, "On Aggregating Linear Programming Models of Production," Jour. Farm Econ., Vol. 45, Nov. 1963, pp. 797-813.

production practices, and view essentially the same alternatives. Thus, they have similar coefficient matrices and similar sets of activities in their "subjective solution vectors." In addition, one may be concerned with supply response to a relatively narrow range of price ratios such that the subset of farms contained in the unbiased aggregate could be easily determined for that range of prices.

However, these practical observations may not be the most valuable results of Miller's work. Miller hinted at but did not exploit an extension of his analysis, which could potentially lead to translation of the farm solution vector conditions into observable characteristics of the farms themselves. This potential is revealed in the dual to Miller's primal problem.

In the primal problem, Miller proved that for a group of farms having qualitatively homogeneous solution vectors and identical B matrices, if

$$C^* = \sum_{g=1}^{n} C_g, \text{ then } X^* = \sum_{g=1}^{n} X_g.$$

For the dual solutions, Miller observes that a parallel argument could be developed to show that if

$$Z^* = \frac{1}{n} \sum_{g=1}^{n} Z_g, \text{ then } r^* = \frac{1}{n} \sum_{g=1}^{n} r_g.$$

Note that the marginal revenue products of the n farms are not weighted by the relative share of aggregate resources belonging to each farm (as was the case in Day's paper). The shadow prices of the individual farms are simply added and divided by n, and they turn out to be exactly the same as the shadow prices of the aggregate problem. It follows then that if the net returns expectations are identical for all farms in the group,

$$Z_g = \frac{1}{n} \sum_{g=1}^{n} Z_g.$$

Thus

$$Z^* = \frac{1}{n} \sum_{g=1}^{n} Z_g = Z_g$$

for any and all farms. From this, it is obvious that

$$r^* = \frac{1}{n} \sum_{g=1}^{n} r_g = r_g$$

that is, the marginal revenue products are the same for all farms and are constant over the range of resource ratios represented by the aggregated farms. It now becomes clear that the observed ranges of resource ratios represented can be used as criteria for grouping farms on the basis of observable characteristics. In effect we have developed a new aggregation theorem which can be stated as follows:

> "Sufficient conditions for exact aggregation are (1) that all farms have identical coefficient matrices, (2) that all farms have the same net returns expectations, and (3) that the range of resource ratios be such that the dual solution vector is the same for all farms."

This is simply the dual counterpart of Miller's theorem. It would delineate sets of farms identical to those delineated by the original theorem. However, it may be more useful since it lends itself to interpretation in terms of observable characteristics. The link between theorem and application is the empirical task of determining the exact ranges of resource ratios over which the marginal revenue product is constant.

The potential of this approach may be demonstrated graphically. Suppose there exists a group of farms each possessing some combination of two resources C and L, each viewing the same three production processes, A_1, A_2, and A_3, with technical coefficients common to all farms, and each having identical net returns expectations. The situation is depicted in figure 1.[3]

[3] The graphic exposition was first suggested in private correspondence to Lee M. Day by John Stovall of the University of Kentucky. Stovall reported the attempts of a graduate student at the University of Kentucky to group farms on which capital and labor resources were fixed and all other resources variable. The student held capital constant and obtained primal and dual programming solutions with varying amounts of labor. He then determined the points at which the marginal value product (MVP) of labor changed and drew lines from the origin through these points. He called these lines "MVP boundaries" but did not indicate awareness that these boundaries were in fact activity vectors or that they represented the resource ratios at which the optimum activity mix changed.

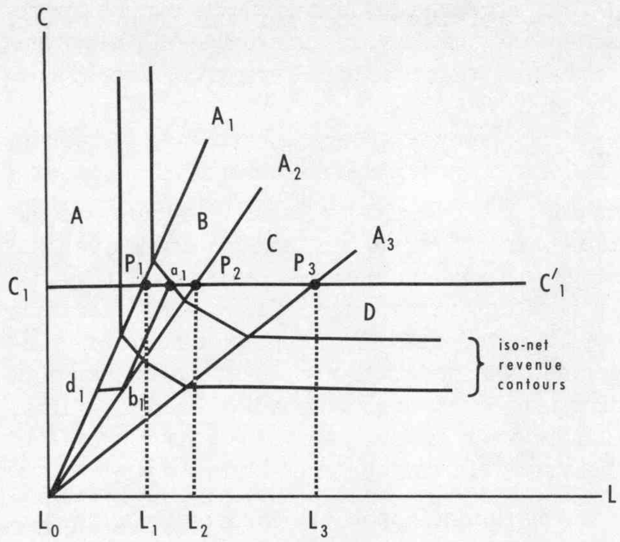

Figure 1

All possible resource ratios are depicted by points on the horizontal bar C_1 C_1' derived by adding varying amounts of resource L to a fixed amount (C_1) of resource C. The net returns expectations for an initial set of product prices (or input prices) create a field of iso-revenue curves exemplified by the solid iso-net revenue curves shown.

Given the situation described above and portrayed in figure 1, the output and net revenue for farms with C_1 of resource C and none of resource L will be zero. As the level of resource L increases in small increments from L_0 to L_1, L remains the limiting resource and net revenue increases in proportion to increases in L.[4] In other words, as resource L is increased from L_0 to L_1, net revenue is maximized by moving up the iso-revenue field along the A_1 activity vector (since the A_1 vector represents the most efficient utilization of resources as long as resource L is limiting).

Since net revenue changes in proportion to changes in the limiting resource, L, between L_0 and L_1, the marginal value product (shadow price) of L is constant over the same range. Under the conditions of the "dual aggregation theorem" and under the assumptions applied to figure 1, all farms with resources C and L in ratios ranging between C_1/L_0 and C_1/L_1 can be aggregated without bias so that

$$X^* = \sum_{g=1}^{n} X_g.$$

With resource combination $C_1 L_1$ (denoted by point P_1), both resources are exactly used by activity vector A_1. With further increases in resource L (beyond L_1) both resources C and L are limiting. However, the full amount of both resources can be utilized and net revenue maximized by combinations of activities A_1 and A_2 (for example, L_0 b_1 of A_2 and b_1 a_1 = d_1 P_1 of A_1 in figure 1). The locus of resource combinations, P_1 P_2, is also the path of net revenue expansion as resource L is increased. This expansion path intersects the iso-net revenue field at constant angles (i.e., as L increases, the net revenue from A_2 substitutes for net revenue from A_1 at constant rates). Thus, between L_1 and L_2, the marginal revenue product of L is constant and the conditions of the dual to Miller's theorem are again met. Note that the MVP of L between L_1 and L_2, while constant, is less than the constant MVP of L between L_0 and L_1. The reason is that as L is increased it becomes less scarce relative to resource C. This is reflected in the flatter slope of the iso-revenue curve. Obviously, farms with resource ratios between C_1/L_1 and C_1/L_2 can be aggregated without bias.

As resource L is increased from L_2 to L_3, its MVP is again constant though lower than previously. Farms with resource ratios between C_1/L_2 and C_1/L_3 meet the conditions for exact aggregation. Beyond L_3 amounts of resource L, C becomes the only limiting resource; A_3 is the only activity in the solution and the MVP of L is constant at zero. Thus, all farms with resource ratios of C_1/L_3 or less can be aggregated without bias.

With resource C fixed at C_1, the line L_0 P_1 C_1' represents the maximum efficiency net revenue expansion path as L is increased from L_0 to infinity. The angle at which this path cuts the field of iso-net revenue curves determines the marginal revenue product (shadow price) of L.

Thus, in figure 1, at the set of prices reflected in the iso-revenue contours, the number of

[4] Assuming, of course, no internal or external economies or diseconomies of scale.

groups of farms needed to eliminate aggregation bias is four. In effect, the activity vectors together with the axis of figure 1 represent MVP "borders." All farms with combinations of C and L falling between two adjacent "borders" have the same shadow prices in the dual, have the same activities in the primal solution vector, and can be aggregated without bias.

If the activities A_1, A_2, and A_3 are the only alternatives available to the farms, the four groups of farms, A, B, C, and D, represent the maximum number of groupings needed for zero-bias aggregation purposes, regardless of the relative product or input prices. For a specific set of prices, a smaller number of groups may suffice for zero-bias but will not be general.

The preceding exposition can be used to demonstrate that Miller's sufficient conditions for exact aggregation are indeed less restrictive than Day's. In figure 1, every point on the locus of resource ratios, $C_1 C_1'$, represents a separate group of farms under Day's condition that each farm in a group have resources in the same proportion. One of Miller's groups contains all such points--and all of Day's groups--that lie between two MVP "boundaries." Miller's groups (of farms which can be aggregated without bias) reduce to Day's groups only when the number of alternative activities is infinite.

It is apparent from the preceding discussion that the key to determination of the resource ratios relevant to bias-free grouping of farms is the relationship between the ratios in which resources are required by alternative activities and the ratios in which resources are available to farms. In the two-dimensional example, the marginal value product borders are determined from the technical coefficients themselves without having to solve the linear program. A generalization of the grouping procedure illustrated in that example is now being developed for extension to multiproduct-multiresource farm populations.

In summary, Miller's theorem does not represent a final resolution of the problem of aggregation bias in linear programming models. However, an extension of the dual to that theorem indicates the range over which resource ratios may vary without introducing bias. The ideas presented, when combined with those of Day, Stovall, and others, provide the pieces from which a general, practical aggregation procedure will eventually be developed.