DYNAMIC DECISIONS UNDER RISK:

Applications of Ito Stochastic Control in Agriculture

Greg Hertzler

Agricultural Economics
School of Agriculture
The University of Western Australia
Nedlands, WA 6009
Australia

Phone: (61) 09 380 2534
Fax: (61) 09 380 1002

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Abstract

In agricultural economics, most studies of dynamic decisions under risk have been empirical with less emphasis on theory. In finance, resource economics and general economics, however, Ito stochastic control is popular for theoretical work. Optimal decisions can be characterized using the powerful Ito stochastic calculus. This paper describes the assumptions and methods of Ito control, constructs a dynamic model of agricultural decisions under risk and illustrates with four examples.

Key words: Agriculture, Risk, Stochastic Calculus, Stochastic Control.

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The art of modeling is the art of striking a bargain. Sometimes tractability must be at the expense of realism. Researchers in finance, natural resource economics and general economics have struck a favorable bargain by accepting the assumptions and gaining the analytical power of Ito stochastic control. Appendix 1 contains a synopsis of this literature. The purposes of this article are to 1) distill the erudite literature on Ito control into a more useful form and 2) adapt Ito control to a dynamic theory of agricultural decisions under risk.

In agricultural economics, most studies of dynamic decisions under risk have been empirical (see Appendix 1). Ito stochastic control is a tool for constructing the theory to complement these empirical results. An analogy might be made to optimal control without risk. Applications are solved in discrete time by mathematical programming or dynamic programming but the theory is usually developed in continuous time using optimal control. With risk, applications must be solved in discrete time by dynamic programming but the theory will almost always be easier to develop in continuous time using Ito control. A more general but less tractable approach to theory integrates directly over transition density functions and differentiates the integrals to find optimality conditions (Taylor; Blume et al.). Ito control simplifies the stochastic structure of the model and finds optimality conditions using a stochastic calculus. It allows expected utility to be maximized over time subject to multiple and correlated risks. Ito control is popular in finance, general economics and natural resource economics because little economic realism is sacrificed for a large gain in analytical power.

In this article the assumptions behind Ito stochastic processes are explained and stochastic calculus and optimal control methods are presented. Then a model is developed which adapts Ito control to the investment, marketing, production and household consumption decisions of a risk-averse farmer. No single model can cover the many topics within the subject of agricultural decisions under risk. But most models will have the basic structure derived and illustrated here. Other researchers can build on this structure, adapt it to their own models and derive theoretical results which complement their empirical results.

Ito Stochastic Processes

State variables, sometimes called quasi-fixed factors, distinguish a dynamic model from a static one. State variables cannot adjust instantaneously, as can variable factors, but they do change, unlike fixed factors. Wealth is a state variable common to all farmers. It changes over time with the retention of earnings from net income. Other state variables may include inventories of machinery and livestock, stocks of soil moisture and soil fertility, water tables, minerals or prices.

A farmer makes decisions at the beginning of each discrete time interval. A decision is risky if a state variable changes unexpectedly once the decision is made. Thus the change in the state variable is described by a stochastic difference equation.

$$S_\tau - S_t = f(t, S_t, c_t, \xi_t)[\tau - t].$$

S is a state variable; c is a vector of control variables; ξ is a vector stochastic process of dimensions nx1; t and τ are current and future times; and τ-t is the length of each time interval. Three assumptions convert the difference equation into an Ito differential equation.

Assumption 1: Markov Property. The current state summarizes all relevant information. The probability of making a transition conditioned upon past and present states of the system or past and present stochastic events equals the probability of making a transition conditioned upon just the current state. The past and the future are statistically independent.

Assumption 2: Continuous Time. The difference equation is approximated by a differential equation in continuous time.

Assumption 3: Rapid Events. Stochastic events occur "rapidly". For example, weather and prices can be difficult to predict from one growing season to the next.[1]

These three assumptions have several implications. The Markov property and continuous time together imply that the approximating differential equation is linear in the stochastic process (Horsthemke and Lefever, p. 97; Arnold, Chpt 9).

$$S_\tau - S_t = [\delta(t,S_t,c_t) + \sigma(t,S_t,c_t)\xi_t][\tau - t] + o(\tau - t).$$

The expected change in the state per unit of time is δ. The standard deviation, σ, is a vector function of dimension 1xn. It can be a function of the control and state variables or it can be heteroscedastic as a function of time. Special cases include Kalman filtering, and multiplicative and additive risks (Mangel, p. 33; Arnold, p. 206; Fleming and Rishel, p. 135; Kendrick, Chpts. 5 and 6). The order function, $o(\tau - t)$, is a remainder which goes to zero faster than time interval τ-t approaches dt in the limit of continuous time.

Continuous time is a simple assumption in a deterministic model. By the fundamental theorem of calculus, a difference equation converges in the limit to a unique differential equation. There is only one calculus for deterministic equations. But the limit of a stochastic difference equation is not unique. If the stochastic difference equation is evaluated at the midpoint of each time interval it converges to a Stratonovich differential equation (Wong and Zakai). If it is evaluated at the beginning of each time interval, it converges to an Ito differential equation (Maruyama). Two different stochastic calculi result from the two approximations (Sethi and Lehoczky; Horsthemke and Lefever, p. 101; Schuss, p. 93). The Stratonovich calculus has the same rules as ordinary calculus (Stratonovich). Unfortunately the Markov property is destroyed. The state of the system at the beginning of a time interval does not summarize all the information necessary for approximating at the midpoint. The Ito calculus has different rules of integration and differentiation, but it preserves the Markov property by approximating at the beginning of each time interval (Ito, 1944, 1951a, 1951b).

The stochastic processes must be identically and independently distributed at each point in time. If stochastic processes were not independent but autocorrelated, past events would contain information not summarized by the current state and the Markov property would be violated. For rapid events, the first two moments become sufficient statistics (Feller, p. 335; Arnold, pp. 39 and 156; Mailliaris and Brock, p. 99; Horsthemke and Lefever, p. 72). The first, second and higher moments of the change in the state variable are

$$E\{S_\tau - S_t \mid S_t = s_t\} = \delta(t,S_t,c_t)[\tau - t] + o(\tau - t);$$

$$\text{(1)} \qquad E\{[S_\tau - S_t]^2 \mid S_t = s_t\} = \sigma(t,S_t,c_t)\Omega\sigma'(t,S_t,c_t)[\tau - t] + o(\tau - t);$$

$$E\{[S_\tau - S_t]^{2+\alpha} \mid S_t = s_t\} = o(\tau - t).$$

Ω is the contemporaneous correlation matrix and $\sigma\Omega\sigma'$ is the covariance; α is a positive parameter and expectations are conditioned on a known current stock, s_t. As τ-t approaches dt in the limit, the order terms vanish, the first moment becomes the drift term, δdt, the second moment becomes the diffusion term, $\sigma\Omega\sigma'$dt, and higher moments go to zero, regardless of the underlying probability distribution.

Finally, the ordinary stochastic process, ξ, must be replaced by white noise, ϵ (Horsthemke and Lefever, p. 59). The ordinary stochastic process has a finite variance which prevails over each discrete time

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[1] Merton (1982) also describes "rare" events. For example, machines fail and agricultural commodity programs are revised only rarely. The theory of rapid events is more developed and is used in the analysis to follow.

interval. As time is divided into shorter and shorter intervals in the passage to continuous time, this finite variance would be parcelled out and the variance in each short interval would go to zero. For the differential equation to remain stochastic, the variance must increase to infinity as time becomes continuous. An infinite variance, an expected value of zero and no autocorrelation is a description of white noise[2].

$$E\{\epsilon_t\} = 0;$$

$$E\{\epsilon_t \epsilon'_\tau\} = \begin{cases} \Omega / [\tau-t] \text{ goes to } \infty \text{ as } \tau-t \text{ goes to dt;} \\ 0 \text{ for } \tau \neq t. \end{cases}$$

To see how white noise replaces an ordinary stochastic process in the passage to continuous time, consider a special kind of state variable, Z, which has a drift of zero, a standard deviation of unity and a normally distributed stochastic process.

$$Z_\tau - Z_t = \xi_t[\tau-t] + o(\tau-t).$$

By equation (1), the first moment is $o(\tau-t)$ and the second moment is $\Omega[\tau-t] + o(\tau-t)$. Thus, $\xi\xi'$ must equal $\Omega / [\tau-t]$ and the change in Z converges in distribution to white noise as time becomes continuous.

(2) $$dZ = \epsilon dt.$$

Z is called Brownian motion or a Weiner process. Its differential over time is normally-distributed white noise.

Finally, an Ito differential equation is defined by taking the limit of the difference equation for state variable S as τ-t becomes dt and ordinary stochastic process, ξ, becomes white noise, ϵ, then replacing white noise over time by the differential of the Weiner process[3].

(3) $$dS = \delta(t,S_t,c_t)dt + \sigma(t,S_t,c_t)dZ.$$

The Ito differential equation retains the essential character of the original difference equation. Time is asymmetric. Decisions are made at the beginning of each infinitely-short time interval based on information summarized by the current state. The state variable is not differentiable with respect to time in the ordinary sense. It evolves stochastically and is differentiable only by stochastic calculus.

Ito Stochastic Calculus and Control

Like ordinary calculus, Ito stochastic calculus includes integration and differentiation. Integration is useful (Arnold, Chpts 7 and 8) but differentiation is essential. Let variable X be a function of both time and the stochastic state variable, or X = f(t,S), and take the Taylor expansion.

$$X_\tau - X_t = f_t[\tau-t] + f_S[S_\tau-S_t] + \tfrac{1}{2}[f_{tt}[\tau-t]^2 + 2f_{tS}[\tau-t][S_\tau-S_t] + f_{SS}[S_\tau-S_t]^2]$$
$$+ O([\tau-t]^3) + O([S_\tau-S_t]^3).$$

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[2]White noise can have different probability distributions including normal and Poisson (Gel'fand and Vilenkin). Normal distributions result in Weiner processes and Poisson distributions result in Poisson processes and model rapid and rare events, respectively.

[3]Rare events could be included by adding the term σdP where σ is a standard deviation and dP is the differential of a Poisson process (Merton, 1971; Malliaris and Brock, p. 121; Mangel, p. 22).

f_t, f_S, f_{tt}, f_{tS} and f_{SS} are partial derivatives and the terms $O([\bullet]^3)$ are order functions which converge to nonzero constants.

Ito's lemma (Ito, 1951a, 1951b) simplifies the Taylor expansion. Like differentiation by ordinary calculus, only first-order terms must be retained in the passage to continuous time. Terms with $[\tau-t]^2$ and $[\tau-t][S_\tau-S_t]$ are of second order and vanish in the limit. However, the second moment, $[S_\tau-S_t]^2$, is actually of first order in τ-t according to equation (1). The limit of the Taylor expansion is the stochastic differential equation for X which, with the aid of equation (3), can be written in either of two forms.

$$
(4) \quad \begin{aligned} dX &= f_t dt + f_S dS + \tfrac{1}{2} f_{SS} dS^2 \\ &= [f_t + f_S \delta(t,S,c) + \tfrac{1}{2} f_{SS} \sigma(t,S,c)\Omega\sigma'(t,S,c)]dt + f_S \sigma(t,S,c)dZ. \end{aligned}
$$

Three simple rules convert the second moment of the state variable, dS^2, into the diffusion term, $\sigma\Omega\sigma'dt$.

$$
(5) \qquad dt^2 = 0; \quad dtdZ = 0; \quad dZdZ' = \Omega dt.
$$

When differential equation (3) is squared, all terms with dt^2 go to zero, including dtdZ which equals ϵdt^2. Although dZdZ' equals $\epsilon\epsilon' dt^2$, it is not zero because $\epsilon\epsilon'$ equals Ω / dt.

The trademark of an Ito derivative is the diffusion for S inserted into the drift for X. For example, if X = f(t,S) were concave, variance in S would be expected to decrease X. This trademark results from the Markov property in which expectations are formed at the beginning of each small time interval. Examples of Ito differentiation can be found in Kamien and Schwartz (Chpt 21) and Malliaris and Brock (pp. 89 and 220). Appendix 2 contains the differentiation formula for when S is a vector of state equations.

Like deterministic optimal control, Ito control maximizes an objective function subject to differential equations for the state variables. The objective function of a farmer who wishes to maximize his expected utility is

$$
(6) \qquad J(S_0) = \underset{c}{\text{Max}}\, E\{ \int_0^T e^{-\rho t} U(S_t,c_t)dt + e^{-\rho T} V(S_T) \mid S_0 = s_0 \}.
$$

Maximization is subject to the stochastic evolution of the state variable in equation (3). The farmer's control and state variables are c and S; his direct utility functions are U and V; his indirect utility after optimization is J; his rate of time preference is ρ; and his planning horizon is T. In a typical model, the farmer would maximize the expected utility of current consumption and terminal wealth but many other control and state variables are possible.

Either dynamic programming or a stochastic maximum principle can optimize equation (6) (Malliaris and Brock, pp. 108, 112 and 118). Dynamic programming is more common and optimality conditions are derived from the Ito version of the Hamilton-Jacobi-Bellman (HJB) equation.

$$
(7) \qquad 0 = J_t + \underset{c}{\text{Max}} \{ e^{-\rho t} U(S,c) + J_S \delta(t,S,c) + \tfrac{1}{2} J_{SS} \sigma(t,S,c)\Omega\sigma'(t,S,c) \}.
$$

J_t, J_S and J_{SS} are partial derivatives and the HJB equation is a partial differential equation in t and S. It has end condition $J(T,S_T) = e^{-\rho T} V(S_T)$. The expression in brackets to be maximized is a dynamic certainty equivalent denominated in utils. Its first term is current direct utility; its second term is the dynamic cost of an expected change in the state; and its third term is the dynamic risk premium.

Because the HJB equation is derived by Ito differentiation it contains the covariance of the state variable in its risk premium. This gives Ito control all the power of mean-variance analysis without the restrictive assumptions. The only assumptions required are rapid uncertainty and the Markov property in continuous time. The first two moments become sufficient statistics and the utility functions can be of any form, allowing all types of risk preferences. The second partial derivative, J_{SS}, is zero or negative for a risk-neutral or a risk-averse farmer and changes endogenously with the level of the state.

Control variables, c, are chosen by differentiating the HJB equation.

$$(8) \qquad 0 = e^{-\rho t}U_c + J_S\delta_c + J_{SS}\sigma\Omega\sigma'_c.$$

U_c is marginal direct utility, δ_c is the partial derivative of the expected change and $\sigma\Omega\sigma'_c$ is one-half the partial derivative of the covariance where σ'_c is an n×1 vector of partial derivatives of the standard deviation vector. To make his decision, a farmer compares discounted marginal utility to the marginal dynamic costs of an expected change in the state and a marginal risk premium.

Another optimality condition describes the shadow price, or costate, J_S. Just as the state variable changes stochastically over time, so does the costate. Its differential equation is presented in Appendix 2.

A Model for Agriculture

A farmer faces many risks from production, from prices and from investments. No single model can include them all. But every model must consider the farmer's bottom line, risky income which makes the accumulation of wealth stochastic. A risk-averse farmer may wish to maximize his expected utility subject to his stochastic change in wealth.

$$(9) \qquad J(W_0) = \underset{q,c}{\text{Max}}\, E\{ \int_0^T e^{-\rho t}U(q)dt + e^{-\rho T}V(W_T) \mid W_0 = w_0\};$$

subject to:

$$(10) \qquad dW = [\delta_w W + \sum_i(\delta_i - \delta_w)p_iA_i - \sum_i p_ig_i(A,c) + \pi - p_qq]dt + \sum_i p_iA_i\sigma_idz_i - \sum_i p_is_i(A,c)dZ_i + d\pi.$$

U and V are direct utility of consumption and terminal wealth; J is indirect utility of wealth; q is consumption at price p_q; c is a vector of production decisions to be defined in later examples; ρ is the farmer's rate of time preference; W is wealth which can be invested off-farm at the risk-free rate δ_w; A_i is the inventory of an agricultural asset valued at price p_i and expected to receive premium δ_i-δ_w above the risk-free rate; g_i is the physical rate of degradation of an inventory; π is the revenue from production above variable costs, or gross margin; σ_i and s_i are standard deviations of returns to assets and of physical degradation; dz_i, and dZ_i are Weiner processes; and $d\pi$ is the stochastic change in the gross margin.

The stochastic change in wealth, equation (10), is derived in Appendix 3. It is a general equation which must be specialized for a particular topic in risk. Investment in financial assets will treat the term $(\delta_i$-$\delta_w)p_iA_i$ as real capital gains. Investment in depreciating assets such as machinery may treat this same term as fixed costs. Negative δ_i is a rate of depreciation and δ_w is the rate of interest on investment. The term $p_iA_i\sigma_idz_i$ is risk from capital gains or depreciation. Investment in degradable assets, such as soil which erodes and livestock which dies, will treat the term p_ig_i as a user-cost or quasi-fixed cost. The amount of the asset used, g_i, will cost p_i to replace. Degradation is risky because of the term $p_is_idZ_i$. Most importantly, however, gross margin, π, and its stochastic change, $d\pi$, must be specified. Gross margin could include many combinations of stochastic yields and prices. A few combinations are illustrated in the following examples.

Example 1: Farmland Investment

The capitalization approach to farmland valuation has been investigated by many authors including Burt, Alston, and Featherstone and Baker. Using Ito control, this approach can be generalized to include risk and risk preferences.

Assume the farmer's only asset is land, L, valued at price p_ℓ and expected to appreciate at rate δ_ℓ. Assume the land does not degrade physically. Also assume the gross margin is nonstochastic and equals the gross margin per hectare times the number of hectares of land.

$$\pi = (p_yY(x) - p_xx)L.$$

Y is yield per hectare as a function of variable input x; p_y and p_x are prices for yield and the variable input. With these assumptions, the change in wealth, equation (10), becomes

$$dW = [\delta_wW + (\delta_\ell - \delta_w)p_\ell L + (p_yY - p_xx)L - p_qq]dt + p_\ell L\sigma_\ell dz_\ell.$$

Optimal farmland investment will be derived from the Hamilton-Jacobi-Bellman (HJB) equation (7). The state variable denoted by S in the HJB equation corresponds to wealth. The control vector, c, contains the variable input and hectares of land. The drift term, δdt, is the first moment, E(dW), and the diffusion term, $\sigma\Omega\sigma'dt$, equals the second moment, E(dW^2).

$$\delta dt = [\delta_wW + (\delta_\ell - \delta_w)p_\ell L + (p_yY - p_xx)L - p_qq]dt;$$

$$\sigma\Omega\sigma'dt = \delta^2dt^2 + 2\delta p_\ell L\sigma_\ell dtdz_\ell + (p_\ell L\sigma_\ell)^2dz_\ell^2$$
$$= (p_\ell L\sigma_\ell)^2dt.$$

According to the rules in (5), dz_ℓ^2 is a first-order term equal to time increment dt. But dt^2 and $dtdz_\ell$ are second-order terms which vanish in the limit of continuous time.

From equation (8), the optimality condition for land equals J_W times the derivative of drift plus J_{WW} times one-half the derivative of diffusion.

$$0 = J_W[(\delta_\ell - \delta_w)p_\ell + p_yY - p_xx] + J_{WW}L(p_\ell\sigma_\ell)^2.$$

This can be solved for the demand for farmland.

(11) $$L = [p_yY - p_xx - (\delta_w - \delta_\ell)p_\ell] / [-J_{WW} / J_W](p_\ell\sigma_\ell)^2.$$

The numerator is the gross margin per hectare minus the opportunity cost of investment after expected capital gains. The denominator is the variance of land prices weighted by the farmer's coefficient of absolute risk aversion, $-J_{WW} / J_W$. A risk-averse farmer has negative J_{WW} and invests in farmland if the gross margin exceeds the expected opportunity cost. The greater his risk aversion the lower a farmer's demand for land.

Further insight comes from solving for the price of farmland.

$$p_\ell = [p_yY - p_xx - L[-J_{WW} / J_W](p_\ell\sigma_\ell)^2] / (\delta_w - \delta_\ell)$$

To calculate the maximum price he would be willing to bid, a risk-neutral farmer divides the gross margin by the real rate of interest after expected capital gains. An aspiring farmer who doesn't yet own farmland has no risk exposure. He may be risk averse but would initially bid as though he were risk neutral. As he accumulates farmland his risk exposure increases and he lowers his bid. If the gross margin were risky he would be even more conservative.

This stochastic price equation implies that farmers who are very averse to risk might be willing to purchase some land but farmers who are less averse will purchase more. In addition to economies-of-scale, degrees of risk aversion may influence farm size.

Example 2: Forward Selling

Many authors have investigated forward selling through forward contracts or futures markets. Kahl, Bond et al., Thompson and Bond, and Robison and Barry (Chpt 10) used mean-variance analysis to determine the optimal proportion of yields to hedge. Using Stein's theorem, Grant extended the results to an expected utility model with bivariate, normally distributed risks. Using Ito control, the results can be extended even further to a dynamic expected utility model with multiple sources of correlated risks.

Assume the farmer has a fixed land area which does not appreciate in value. All his financial investments are risk-free. The gross margin, however, is now stochastic and has a term for hedging income.

$$\pi = p_y Y(x) - p_x x - (p_f - p_c)F.$$

The farmer hedges quantity F by either a forward or futures contract and receives contract price p_c. With a forward contract he must deliver at harvest time or, equivalently, sell all his yield and purchase enough at future price p_f to make the promised delivery. With a futures contract, he purchases an offsetting contract before the original contract matures. Future price, p_f, equals spot price, p_y, for a forward contract. The future price should converge to the spot price for a futures contract but may be subject to a different stochastic influence leading to basis risk. Because of spot and futures price risks, the gross margin has an Ito derivative as defined by equation (A4) in Appendix 2.

$$d\pi = dp_y Y - dp_f F.$$

In the finance literature, it is typical to assume price expectations that are log-normal. In other words, a farmer expects prices to be nonnegative and their percentage change to be normally distributed.

$$dp_y/p_y = [\delta_y + \sigma_y \epsilon_y]dt = \delta_y dt + \sigma_y dz_y;$$

$$dp_f/p_f = [\delta_f + \sigma_f \epsilon_f]dt = \delta_f dt + \sigma_f dz_f.$$

Expected rates of change, δ_y and δ_f are forecast with errors $\sigma_y \epsilon_y$ and $\sigma_f \epsilon_f$, where the σ's are standard deviations and the ϵ's are normally distributed white noise. By equation (2), ϵdt is the change in a Weiner process, dz. The Weiner processes for the spot and futures prices may be highly correlated.

After substituting in the gross margin, its Ito derivative and price expectations, the change in wealth, equation (10), becomes

$$dW = [\delta_w W + p_y(1 + \delta_y)Y - p_x x - (p_f(1 + \delta_f) - p_c)F - p_q q]dt + p_y Y \sigma_y dz_y - p_f F \sigma_f dz_f.$$

As in the previous example, the drift term, δdt, is the expected change in wealth. In this example, however, there are two sources of risk and the diffusion term, $\sigma\Omega\sigma' dt$, is the product of matrices.

$$\sigma\Omega\sigma' dt = [p_y Y \sigma_y; -p_f F \sigma_f] \begin{bmatrix} dz_y \\ dz_f \end{bmatrix} [dz_y; dz_f] \begin{bmatrix} p_y Y \sigma_y \\ -p_f F \sigma_f \end{bmatrix}$$

$$= [p_y Y \sigma_y; -p_f F \sigma_f] \begin{bmatrix} 1 & \omega_{yf} \\ \omega_{yf} & 1 \end{bmatrix} \begin{bmatrix} p_y Y \sigma_y \\ -p_f F \sigma_f \end{bmatrix} dt.$$

Using the rules in (5), the product of Weiner increments, $dz_y dz_f$, equals $\omega_{yf} dt$ where ω_{yf} is the correlation coefficient.

Optimality condition (8) is applied by differentiating drift with respect to F and multiplying by J_W. To this is added the derivative of σ' premultiplied by $J_{WW}\sigma\Omega$.

$$0 = -J_W[p_f(1 + \delta_f) - p_c] + J_{WW}[F(p_f\sigma_f)^2 - Yp_y p_f \sigma_y \sigma_f \omega_{yf}].$$

The demand for futures is found by solving for F.

$$F = Y(p_y \sigma_y \omega_{yf}) / p_f \sigma_f + [p_c - p_f(1 + \delta_f)] / [-J_{WW} / J_W](p_f\sigma_f)^2. \tag{12}$$

The first term on the right-hand side is the demand for futures as a hedge. The second term is speculative demand. Suppose the futures contract is for a completely different commodity than the one produced and the spot and futures prices are uncorrelated. A risk-averse farmer may still speculate if the contract price exceeds the expected futures price. The more risk-averse a farmer the less will be his speculative demand. Positively correlated spot and futures prices increase the demand for futures as a hedge. Prices may never be perfectly correlated because of basis risk and the farmer will hedge only a portion of his yield. A forward contract, however, is a special case in which spot and futures prices are identical. Price risk is eliminated. The farmer hedges all his yield and uses contract price p_c to calculate the marginal value product in making his production decisions. If yields were stochastic, the decisions of a risk-averse farmer would be more complex and he might not hedge all of his yield.

Example 3: Production from Nutrient Stocks

Typically, studies of production under risk consider only variable inputs. This is true for studies of fertilizer applications, as an example (Rosegrant and Roumasset; SriRamaratnam et al.). Most nutrients, however, are quasi-fixed stocks which carryover from one year to the next (Lanzer and Paris). The rate of carryover is stochastic, depending upon rainfall. Yields are a stochastic function of these nutrient stocks and other variables beyond the farmer's control such as soil moisture, temperature and pest attacks. Thus the gross margin is stochastic because of yield.

$$\pi = p_y Y(N,x) - p_x x;$$

$$d\pi = p_y dY.$$

Yield has many stochastic influences and evolves according to a complex differential equation. But the drift and deviation terms of this equation can be summarized as functions of nutrient stocks, N.

$$dY = \delta_Y(N)dt + \sigma_Y(N)dZ_Y.$$

The change in yields is linear in the stochastic process. This imparts the same desirable properties postulated by Just and Pope (1978, 1979) for stochastic yields in a static model.

The assumptions about nutrient degradation, gross margins and yield transform the change in wealth, equation (10), into

$$dW = [\delta_w W + (\delta_n - \delta_w)p_n N - p_n g(N) + p_y(Y(N,x) + \delta_Y(N)) - p_x x - p_q q]dt$$

$$+ p_n N \sigma_n dz_n - p_n s_N dZ_N + p_y \sigma_Y(N)dZ_Y.$$

The drift term now contains investment and degradation costs for nutrients. Expectations must be formed about the price of nutrients, the degradation of nutrients and about yield. Degradation and yield both depend on the weather and may be highly correlated in the diffusion term.

$$\sigma\Omega\sigma' dt = [p_n N\sigma_n;\ -p_n s_N;\ p_y\sigma_Y(N)] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \omega_{NY} \\ 0 & \omega_{NY} & 1 \end{bmatrix} \begin{bmatrix} p_n N\sigma_n \\ -p_n s_N \\ p_y\sigma_Y(N) \end{bmatrix} dt.$$

The derivatives of drift and diffusion are substituted into equation (8) to find the optimality condition for nutrients which is then rearranged.

$$(13)\qquad p_y(\partial Y/\partial N + \partial\delta_Y/\partial N) = p_n(\delta_w + \partial g/\partial N - \delta_n)$$

$$+ N[-J_{ww}/J_w](p_n\sigma_n)^2 + [-J_{ww}/J_w][(p_y\sigma_Y)^2 - p_n p_y s_N \sigma_Y \omega_{NY}](\partial\sigma_Y/\partial N)/\sigma_Y.$$

The farmer compares the expected marginal-value product on the left-hand side to the marginal investment cost and marginal risk premiums on the right-hand side. The marginal investment cost depends on an effective rate of return on investment. This rate equals the interest rate plus the marginal rate of degradation minus the expected rate of change in the nutrient price. An easily leached nutrient such as nitrogen is almost a variable input because the marginal rate of degradation is nearly one. A slowly leached nutrient is almost a fixed cost because the marginal rate of degradation is nearly zero. If he expects its price to increase, the farmer will apply more nutrient to carry into the future.

A risk-neutral farmer sets the marginal risk premiums to zero but his decisions may appear to be risk averse (Just). Stochastic nutrients modify the expected marginal value product through the term δ_Y. With no risk, the farmer would expect the marginal value product to be $p_y\partial Y/\partial N$. But with risky nutrients the expected marginal value product may increase if $\partial\delta_Y/\partial N$ is positive. The farmer would demand more. A risk-averse farmer will behave the same as a risk-neutral farmer if fertilizer prices are certain and the nutrient neither increases nor decreases yield risk. He will demand less than a risk-neutral farmer if the nutrient price is risky and he will demand more if the nutrient decreases yield risk with $(\partial\sigma_Y/\partial N)/\sigma_Y$ negative. Nutrient availability and yields may have a negative covariance in the marginal risk premium. Unexpected rainfall may cause rapid degradation and lower yields. Costs would be unexpectedly high and returns unexpectedly low, increasing the variance of income and the incentive of a risk-averse farmer to apply extra nutrients.

Example 4: Household Demand

In Example 1, the farmer chose whether to invest his wealth at a risk-free rate or buy farmland. But there is a third choice, consume rather than invest. The farmer equates the discounted marginal utility of consumption with the marginal utility of wealth multiplied by the price of the consumption good.

$$e^{-\rho t}\partial U/\partial q = J_w p_q.$$

Using equation (A9) from Appendix 2, it can be shown that the farmer expects the marginal utility of wealth to decline at the risk-free rate of interest or, in other words, to be constant in real terms. Suppose the farmer's marginal utility of wealth is constant and his rate of time preference equals the rate of interest. Then, perhaps, demand could be estimated as if it were static.

A more rigourous but less flexible approach would estimate a closed-form demand equation. A closed-form equation can be integrated analytically to find indirect utility. Merton (1971) proved that a closed-form is linear if and only if the direct utility function belongs to the hyperbolic absolute risk aversion (HARA) class of functions. This class includes many popular utility functions such as the Stone-Geary function.

$$U(q) = \beta(q - \gamma)^{\alpha}.$$

To find closed-form demand, first U is differentiated and substituted into the optimality condition. Then the optimality is solved for q.

$$q = [p_q e^{\rho t} J_W / \alpha\beta]^{1/(\alpha-1)} + \gamma.$$

Next, q and the demand for land, L, from Equation (11) are substituted into the HJB equation which is simplified.

$$0 = J_t + (1-\alpha)\beta e^{-\rho t}[p_q e^{\rho t} J_W / \alpha\beta]^{\alpha/(\alpha-1)} - \tfrac{1}{2}[J^2_W / J_{WW}]m^2 + J_W[\delta_w W - p_q\gamma].$$

The coefficient m is the gross margin per hectare of land above interest on investment, standardized for risk.

$$m = [p_y Y - p_x x - (\delta_w - \delta_\ell)p_\ell] / p_\ell \sigma_\ell.$$

The HJB equation is a partial differential equation in wealth and time. Assuming a zero utility of terminal wealth, Merton (1973b) integrated to find indirect utility.

$$J(W,t) = \beta e^{-\rho t}[(1-\alpha)(1-e^{-r(T-t)/(1-\alpha)}) / r]^{(1-\alpha)}[W / p_q - \gamma(1-e^{-\delta_w(T-t)}) / \delta_w]^{\alpha}.$$

Rate r is the farmer's rate of time preference above a risk-adjusted interest rate.

$$r = \rho - \alpha[\delta_w + \tfrac{1}{2}m^2 / (1-\alpha)].$$

Finally indirect utility, J, is differentiated with respect to wealth and substituted into the optimality condition for q to get a dynamic linear-expenditure equation.

$$(14) \qquad p_q q = r[W + p_q\gamma(1-e^{-\delta_w(T-t)}) / \delta_w] / [(1-\alpha)(1-e^{-r(T-t)/(1-\alpha)})] + p_q\gamma.$$

This dynamic demand equation differs from its static counterparts in having wealth as an argument instead of income and depending explicitly on the farmer's time horizon, rate of time preference, rate of interest and exposure to risk. Because demand is a closed-form solution, movements in commodity prices or reductions in risk translate exactly into the change in utility. There is no need to calculate changes in consumer's surplus as a proxy. Finally, more general utility functions including utility of terminal wealth, more sources of risk and multiple commodities are possible in a dynamic linear-expenditure system.

Conclusions

In agricultural economics, most studies of dynamic decisions under risk are empirical. Fewer studies are theoretical, perhaps because the models are difficult to analyze. Ito stochastic control is an ideal tool for theoretical analysis. The crucial assumptions are the Markov property and continuous time. The Markov property is natural for economic models because decisions are made at the beginning of each time interval. The limit of continuous time is no more restrictive than in deterministic models. With these and an added assumption of rapid stochastic events, the first and second moments of a stochastic process become sufficient statistics. Ito control has the power of mean-variance analysis without objectionable assumptions about probability distributions or utility functions.

A model for agricultural decisions under risk was derived and proposed as a basic model. Then the model was adapted for four topics: a farmer's investment in risky assets, marketing with risky prices, production from stock inputs that degrade stochastically and dynamic household demand. A few results from the literature were strengthened and new results were obtained. The capitalization approach to farmland valuation was generalized to include risk and risk preferences. Mean-variance and variance decomposition results for optimal hedging decisions were strengthened by the less restrictive assumptions of Ito control. New results for the optimal carryover of risky nutrients were derived. And a dynamic linear expenditure system under risk was presented. These topics were chosen to demonstrate the methods of Ito calculus and Ito control.

Further topics might include correlated prices and yields, options markets, share-rent versus cash-rent leases, crop insurance, soil erosion, salinization, machinery repair and replacement, livestock replacement and breeding, weed and pest control, commodity programs, buffer stocks, international trade under risk, flexible demand systems, and stochastic dynamic duality.

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Appendix 1: Synopsis of the Literature

Ito stochastic processes are the foundation of most modern mathematics literature on stochastic control. Mathematics texts are usually inaccessible to economists. Exceptions are Arnold, Horsthemke and Lefever, Fleming and Rishel, Schuss, Karlin and Taylor and Feller (1968, 1971). In the economics literature, Ito processes were first used to study finance. Famous examples are Black and Scholes option pricing formula and Merton's (1971, 1973a) portfolio rules and capital-asset pricing model. Merton (1975) was among the first to use Ito control in the study of growth theory under risk and was followed by others, including Chang. Pindyck (1980, 1981, 1984), in particular, introduced Ito control into the natural resources literature. Pindyck (1982) also introduced Ito control into the adjustment cost literature and was followed by Abel and by Stefanou. Review articles by Smith (1976, 1984) and Malliaris give an overview of Ito control in finance. Anderson and Sutinen review the literature for marine economics. Chow reviews the literature for finance and natural resources. The book by Malliaris and Brock is an in-depth survey of stochastic control in finance and general economics. The book by Mangel contains original work in natural resources. And the book by Merton (1990) summarizes the work of pioneers in finance.

In agricultural economics, dynamic decisions under risk have been studied by various methods. Burt et al., Taylor and Burt, Zacharias and Grube and McGuckin et al. either solved stochastic dynamic programming problems in discrete time or approximated the solutions. Karp and Pope solved a stochastic dynamic programming problem by linear programming. Taylor and Talpaz applied certainty-equivalence rules. Karp as well as Dixon and Howitt solved linear-quadratic-gaussian control problems. So-called dual or adaptive control in which estimates of the model are updated as decisions are made was introduced into the agricultural economics literature by Rausser and applied by Taylor and Chavas and by Zavaleta et al.. Antle and Hatchett estimated a sequential decision process. And the book by Kennedy applies dynamic programming to agricultural and natural resource economics.

Appendix 2: Vector Differentiation and Control

Ito differentiation and control with a vector of state variables are no different in interpretation but much more detailed.

Vector Differentiation. Let the dimensions of S be m×1, δ be m×1, σ be m×n and dZ be n×1. The vector formula corresponding to equation (4) differentiates the scalar X with respect to the vector S, where X = f(t,S).

$$dX = f_t dt + \sum_{i=1}^{m} f_{S_i} dS_i + \tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} f_{S_iS_j} dS_i dS_j$$

(A4)

$$= [f_t + \sum_{i=1}^{m} f_{S_i}(\delta)_i + \tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} f_{S_iS_j}(\sigma\Omega\sigma')_{ij}]dt + \sum_{i=1}^{m} f_{S_i}(\sigma dZ)_i.$$

S_i is the i^{th} element of the S vector and f_{S_i} and $f_{S_iS_j}$ are first and second partial derivatives of f with respect to elements of the S vector. The notation $(\delta)_i$ denotes the i^{th} element of the δ vector; $(\sigma\Omega\sigma')_{ij}$ denotes the ij^{th} element of the m×m covariance matrix; and $(\sigma dZ)_i$ denotes the i^{th} element of the m×1 stochastic vector.

Vector Control. The Hamilton-Jacobi-Bellman equation for a vector of state variables corresponds to equation (7).

$$\text{(A7)} \quad 0 = J_t + \underset{c}{\text{Max}}\,\{e^{-\gamma t}U + \sum_{i=1}^{m} J_{S_i}(\delta)_i + \tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} J_{S_iS_j}(\sigma\Omega\sigma')_{ij}\}.$$

J_{S_i} and $J_{S_iS_j}$ are partial derivatives of J with respect to elements of the S vector.

Optimality conditions when there is a vector of state variables are not widely available in the literature. The proof is tedious but it can be shown that the optimality condition for the controls is

$$\text{(A8)} \quad 0 = e^{-\gamma t}U_c + \sum_{i=1}^{m} J_{S_i}(\delta_c)_i + \sum_{i=1}^{m}[\sum_{j=1}^{m} J_{S_iS_j}(\sigma\Omega)_{j1}; \ldots ;\sum_{j=1}^{m} J_{S_iS_j}(\sigma\Omega)_{jn}](\sigma'_c)_i.$$

U_c is the partial derivative of utility with respect to control, c, and $(\delta_c)_i$ is the partial derivative of the i^{th} element of δ. The term in square brackets is a 1×n vector written out explicitly. The notations $(\sigma\Omega)_{j1}$ and $(\sigma\Omega)_{jn}$ denote the $j1^{st}$ and the jn^{th} elements of the m×n $\sigma\Omega$ matrix. $(\sigma'_c)_i$ is the partial derivative of the i^{th} column of the n×m matrix, σ'.

The marginal utility of a state is the shadow price or costate variable. Costates can be important in analyzing a theory, particularly when there is a vector of states and, hence, a vector of costate variables. It can be shown that the Ito differential equation for costate dJ_{S_k}, corresponding to state S_k, k = 1,...,m, is

$$\text{(A9)} \quad dJ_{S_k} = -[e^{-\gamma t}U_{S_k} + \sum_{i=1}^{m} J_{S_i}(\delta_{S_k})_i + \sum_{i=1}^{m}[\sum_{j=1}^{m} J_{S_iS_j}(\sigma\Omega)_{j1}; \ldots ;\sum_{j=1}^{m} J_{S_iS_j}(\sigma\Omega)_{jn}](\sigma'_{S_k})_i]dt + \sum_{i=1}^{m} J_{S_kS_i}(\sigma dZ)_i.$$

U_{S_k}, δ_{S_k} and σ_{S_k} are derivatives with respect to S_k.

Setting m equal to one in equations (A7), (A8) and (A9) gives the scalar case discussed in equations (7) and (8).

Appendix 3: Stochastic Wealth

A farmer will have inventories of risky assets, A_i, valued at prices, p_i, and, perhaps, a risk-free bond, B, valued at p_w. Liabilities are negative assets. A farmer's wealth is the value of assets and liabilities summed.

$$W = \sum_i p_i A_i + p_w B.$$

Wealth is stochastic because assets and prices are. Decisions are made at time t and the outcomes are revealed at time t+dt. Ito differentiate wealth using equation (A4) in Appendix 2.

$$dW = \sum_i dp_iA_i + dp_wB + \sum_i [p_i + dp_i]dA_i + [p_w + dp_w]dB.$$

The first two terms are capital gains on beginning inventories. The last two terms are additions to inventories valued at ending prices. At the beginning of each time period, the farmer forms expectations about ending prices. It is typical in the finance literature to assume price expectations are log-normally distributed.

$$dp_i/p_i = [\delta_i + \sigma_i\epsilon_i]dt = \delta_i dt + \sigma_i dz_i;$$

$$dp_w/p_w = \delta_w dt.$$

δ denotes an expected rate of appreciation or depreciation, σ is the standard deviation of the forecast, ϵ is normally distributed white noise and dz is a Weiner process. Price expectations may be correlated across risky assets. Given these expectations, the farmer chooses a portfolio. Once the A_i's are known, B is determined by the wealth constraint.

$$B = [W - \sum_i p_iA_i]/p_w.$$

In addition to appreciating or depreciating in value, inventories may be acquired or be degraded.

$$dA_i = [a_i - g_i(A,c) - s_i(A,c)e_i]dt = [a_i - g_i(A,c)]dt - s_i(A,c)dZ_i.$$

Acquisitions are a_i. The rate of degradation, g_i, is predicted with error s_ie_i, where s is a standard deviation, e is white noise and dZ is a Weiner process. Stochastic degradation may be correlated across inventories but assume there is no correlation with prices. By definition, acquisitions of risky assets and purchases of risk-free bonds must be financed from production income above variable costs, called gross margin, π, after consumption expenditures, p_qq, have been subtracted. The portion, w, of wealth accumulated from the gross margin above consumption expenditures is a stochastic integral over time.

$$w(t,\pi) = \int_0^t (\pi - p_qq)ds.$$

Ito differentiate this contribution to wealth using equation (4) and equate the result to the value of risky asset acquisitions plus risk-free bond purchases.

$$\sum_i [p_i + dp_i]a_idt + [p_w + dp_w]dB = \pi dt + d\pi - p_qqdt.$$

Finally, substitute price expectations, the wealth constraint for risk-free bonds, inventory expectations and the financing of assets and bonds into the change in wealth.

$$dW = [\delta_wW + \sum_i(\delta_i - \delta_w)p_iA_i - \sum_i p_ig_i(A,c) + \pi - p_qq]dt + \sum_i p_iA_i\sigma_idz_i - \sum_i p_is_i(A,c)dZ_i + d\pi.$$

This is Equation (10). Maximizing expected utility subject to this single equation is equivalent to maximizing expected utility subject to multiple equations for wealth, assets and prices. In a model with multiple equations, the asset and price equations are as given previously and the wealth equation would be

$$dW = [\delta_w[W - \sum_i p_iA_i] - \sum_i p_ia_i + \pi - p_qq]dt + d\pi.$$

Verifying that the two models give the same answer requires patience, vector control techniques from Appendix 2 and the identities, $J_{Ai} = J_W[\partial W/\partial A_i] = J_W p_i$ and $J_{pi} = J_W[\partial W/\partial p_i] = J_W A_i$.