"Extreme-Value" Methods Simplified

By Ralph R. Botts

New methods have been devised for fitting skewed statistical distributions that conform to a double exponential model. Such distributions characterize a large class of phenomena of interest to agricultural economists. Distributions of data involving rainfall, temperature, crop yields, crop-hail losses, and others follow this pattern. The new technique, under the name of the theory of extreme values, makes use of graphics and is much less complex and time-consuming than older methods. But most of the available literature is highly mathematical and not easily understood. This paper presents a nonmathematical explanation for the working agricultural economist.

THE THEORY of the statistical distribution of extreme values was developed by E. J. Gumbel.[1] Several others had previously explored the problem.[2] It is particularly appropriate for measuring flood probabilities. For example, a gage height or mean streamflow is available for each day. The highest such reading for the year is an extreme value. So over a period of, say, 36 years, there are 36 such extreme values. Stated another way, the theory applies to a distribution of independent observations, each of which is an extreme value.[3] The control curves (discussed later) are helpful in determining whether or not the theory applies to a particular case.

Such distributions of extreme values tend to be skewed, with the mode or most common value at the left of the mean or average. Probably for that reason, the theory has been applied also to distributions such as those involving rainfall, crop yields, and costs of crop-hail insurance loss, even though the figure for a particular year is a single observation or an average, and not an extreme value.[4]

Most statistical theory applies to normal distributions. Heretofore, it has been necessary to fit Pearsonian curves (particularly type III) to small samples in order to derive the probabilities associated with skewed distributions. The fitting procedure is complex and time-consuming and is not generally understood by economists. But extreme-value methods present a much simpler tool for economists, and they will do well to investigate it.

The purpose of this article is not to review the mathematics involved in the theory of extreme-value distributions; but rather to present this relatively new tool in such a way that it can be applied by those not versed in mathematics. Only a few symbols are used, and no formulas are derived. For illustrative purposes, the method is applied to a distribution of annual "loss costs" of a mutual crop-hail insurance company. Then the necessary steps are summarized and the column headings for a spread sheet are given, so that the calculations may be standardized for mass processing of data by clerks.

Application of the method is simplified immensely by the use of double exponential paper (figs. 1 and 2). On this paper, the vertical scale (y) is evenly spaced. It is used for the observed values. The horizontal scale is used as the probability (m/n+1) scale. It never reaches zero on the left nor 1 on the right. Unfortunately, the

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[1] GUMBEL, E. J. STATISTICAL THEORY OF EXTREME VALUES AND SOME PRACTICAL APPLICATIONS. National Bureau of Standards, Applied Mathematical Series 33, 1954. For sale by Superintendent of Documents, 40 cents. Also PROBABILITY TABLES FOR THE ANALYSIS OF EXTREME-VALUE DATA. National Bureau of Standards, Applied Mathematical Series 22, 1953. For sale by Superintendent of Documents, 25 cents.

[2] For example, DE FINETTI, FISHER, AND TIPPETT. See bibliographies in publications listed in footnote 1.

[3] A chief difficulty in applying the theory has been a lack of independence of data. For tests of randomness in time series, see F. G. FOSTER AND A. STUART. DISTRIBUTION-FREE TESTS IN TIME SERIES BASED ON THE BREAKING OF RECORDS. Jour. Roy. Statis. Soc. Series B, Vol. 16, No. 1 (1954). Pp. 1–22.

[4] BRAKENSIEK, D. L., AND ZINGG, A. W. APPLICATION OF THE EXTREME-VALUE STATISTICAL DISTRIBUTION TO ANNUAL PRECIPITATION AND CROP YIELDS. ARS 41–13, in cooperation with 5 State experiment stations.

probability paper is not for sale.[5] Until it is placed on the market, each person interested in applying the method will have to construct his own paper (a time-consuming job) or obtain it from others.

As an example of application of the method, suppose the cash cost of producing wheat in a Great Plains county is $10 per acre. Using a support price of, say, $2 per bushel, the cash cost represents 5 bushels of wheat. In arid regions, crop yields tend to be skewed, and not normally distributed, over time. Applying extreme-value methods to a distribution of appropriate annual yields, one could determine the probability associated with getting a yield equal to or less than cash costs. He might do this for both continuously cropped and fallowed wheat, from which probability comparisons would be possible. The method therefore has application in the fields of risk reduction and crop-insurance ratemaking.

Or, to take another example, suppose you want to evaluate the chances of having annual rainfall of less than 15 inches (or any minimum level deemed necessary for wheat production) at 2 weather stations. From station records, you could get rainfall data, by years, for each station, prepare probability charts, and read the respective probabilities from them.

The method has special value in the field of crop-hail and windstorm insurance. Using this method, the probabilities associated with various loss levels can be ascertained. But in this field of work, one is usually interested in the other end of the distribution—the higher values of y (loss costs) and their associated values of m/n+1 (probabilities of occurrence).

In the example used, there is an 8-percent, or 1 in 12, chance of having an annual loss cost (aggregate annual losses divided by insurance) that exceeds twice the company's average loss cost (of 56.2 cents per $100). And there is a 3.3 percent, or 1 in 30, chance that annual losses will exceed 250 percent of average. If the company's safety fund or reserve also amounts to 2½ years of average losses, it may want to get aggregate-excess reinsurance, which goes into effect at $1.40 per $100—250 percent of 56.2 cents.

From its own experience, the company can therefore evaluate the chances of having a year of unusually high losses. And, taking the size of its safety fund into consideration, it can arrive at a decision, with respect to the reinsurance protection it needs, that is better than it could otherwise reach.

The basic data for the example used here are shown in table 1. The annual loss costs (losses paid divided by insurance in force) of a crop-hail mutual insurance company are arrayed from lowest to highest, and are ranked in that order. Then, in column 4, the rank for each year is divided by 37, or (n+1) years.

TABLE 1.—*Calculations necessary for plotting data on extreme-value probability paper, based on experience of mutual crop-hail insurance company, 1920–55* [1]

Year	Annual loss cost [2] y	Rank m	$\frac{m}{n+1}$	Z [3]
(1)	(2)	(3)	(4)	(5)
	Cents			
1926	8	1	0. 027	−1. 28424
1938	20	2	. 054	
1932	23	3	. 081	
1942	26	4	. 108	
1927	27	5	. 135	
1933	27	6	. 162	
1941	28	7	. 189	−. 51043
1931	29	8	. 216	
1949	31	9	. 243	
1923	33	10	. 270	
1935	37	11	. 297	
1929	39	12	. 324	
1951	40	13	. 351	−. 04590
1954	40	14	. 378	+. 02751
1924	40	15	. 405	
1936	44	16	. 432	
1945	46	17	. 459	
1955	55	18	. 486	
1930	55	19	. 514	. 40717
1925	56	20	. 541	
1939	57	21	. 568	
1920	58	22	. 595	
1948	58	23	. 622	
1943	61	24	. 649	
1922	67	25	. 676	. 93761
1952	67	26	. 703	
1928	68	27	. 730	
1921	69	28	. 757	
1950	74	29	. 784	
1940	75	30	. 811	

See footnotes at end of table.

[5] The author has obtained a limited supply from the Weather Bureau and from the Climatology Unit, Environmental Protection Section, Research and Development Branch, Military Planning Division, Office of Quartermaster General, U. S. Army.

TABLE 1.—*Calculations necessary for plotting data on extreme-value probability paper, based on experience of mutual crop-hail insurance company, 1920-55* [1]—Continued

Year (1)	Annual loss cost [2] y (2)	Rank m (3)	$\frac{m}{n+1}$ (4)	Z[3] (5)
	Cents			
1946	78	31	0.838	1.73309
1947	84	32	.865	
1944	93	33	.892	
1934	100	34	.919	
1937	128	35	.946	
1953	182	36	.973	
Total	2,023			19.47600
Average	56.2			.5410
Standard deviation	33.2			[4] 1.1313

[1] A few values of Z for selected values of m/n+1 are given in column 5 merely to show that the average and standard deviation of the figures in this column depend only on sample size. For a given sample size (36 in example), the average and standard deviation of the reduced (Z) values always remain the same. The value of these constants for various sample sizes is given in table 2.

[2] Aggregate loss payments divided by amount of insurance in force.

[3] $Z = -\ln[-\ln(m/n+1)]$ . . . where ln stands for natural logarithm.

[4] Square root of [sum of squared deviations from average (0.5410) divided by 36 (not 35)].

The average and standard deviation of the Z-values (see table 1, footnote 3) for a sample consisting of 36 years are shown in table 2. Column 5 of table 1 shows how these values were derived for n=36. The average of 0.5410 and the standard deviation of 1.1313, which appear at the bottom of column 5 of table 1, therefore could have been obtained from table 2 opposite n=36.

The next step is to determine the slope of a regression line through the data and the mode of the distribution.

Slope of regression line=33.2/1.1313=29.35

(See standard deviations shown in table 1.)

Mode=56.2−29.35 (0.5410)=56.2−15.9=40.3

(See averages shown in table 1.)

Therefore

y=point on regression line=mode+ (slope × Z)

and in our problem

(1) . . y=40.3+29.35Z

Three points on the regression line are calculated in table 3, and the coordinates of these points are shown in the first and last columns.

These 3 values or paired observations are then plotted on the extreme-value probability paper (fig. 1). They are joined to form a straight line. Next, the observed values of y and m/n+1 (Cols. 2 and 4 of table 1) are plotted on the probability paper. The 3 coordinates used to fit the line appear as X's while the other coordinates appear as dots on the chart. The latter fall rather close to the line, indicating that the theory applies.

The calculations necessary to fit control curves are shown in table 4.

The vertical distances shown in col. (3) are marked off above and below the respective points on the regression line that are shown in col. (1). Then the points above the line are joined to form the upper boundary or control curve. The points below the regression line are likewise joined to form the lower boundary. Two-thirds of the dots or observations (col. 2 vs. col. 4 in table 1) should fall within the control curves in order for the theory to apply.

Summary of Steps

1. Construct a table like table 1. Only columns 2, 3, and 4 are necessary.

   a. Compute the simple average and the standard deviation of your figures in column 2.

   In the example, the simple average is 56.2 cents and the standard deviation is 33.2 cents.

2. From table 2, find the theoretical average and standard deviation of the "reduced" (Z) values for a sample of the size you have.

   In the example, n=36. For a sample of that size, the corresponding average and standard deviation of the Z values (see column 5, table 1) are, respectively, 0.5410 and 1.1313.

3. Compute the slope of your regression line by dividing the actual by the theoretical standard deviation.

   In the example, the slope is 33.2÷1.1313 or 29.35.

4. Multiply the theoretical average (step 2) by the slope (step 3). Then subtract this product from the sample average. The result is the mode of your sample.

   In the example, the mode is 40.3 as 0.5410×29.35=15.9 and 56.2−15.9=40.3

5. Express your regression line as a straight-line equation which has the mode as a constant

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   In the example, n=36. For a sample of that size, the corresponding average and standard deviation of the Z values (see column 5, table 1) are, respectively, 0.5410 and 1.1313.

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   In the example, the slope is 33.2÷1.1313 or 29.35.

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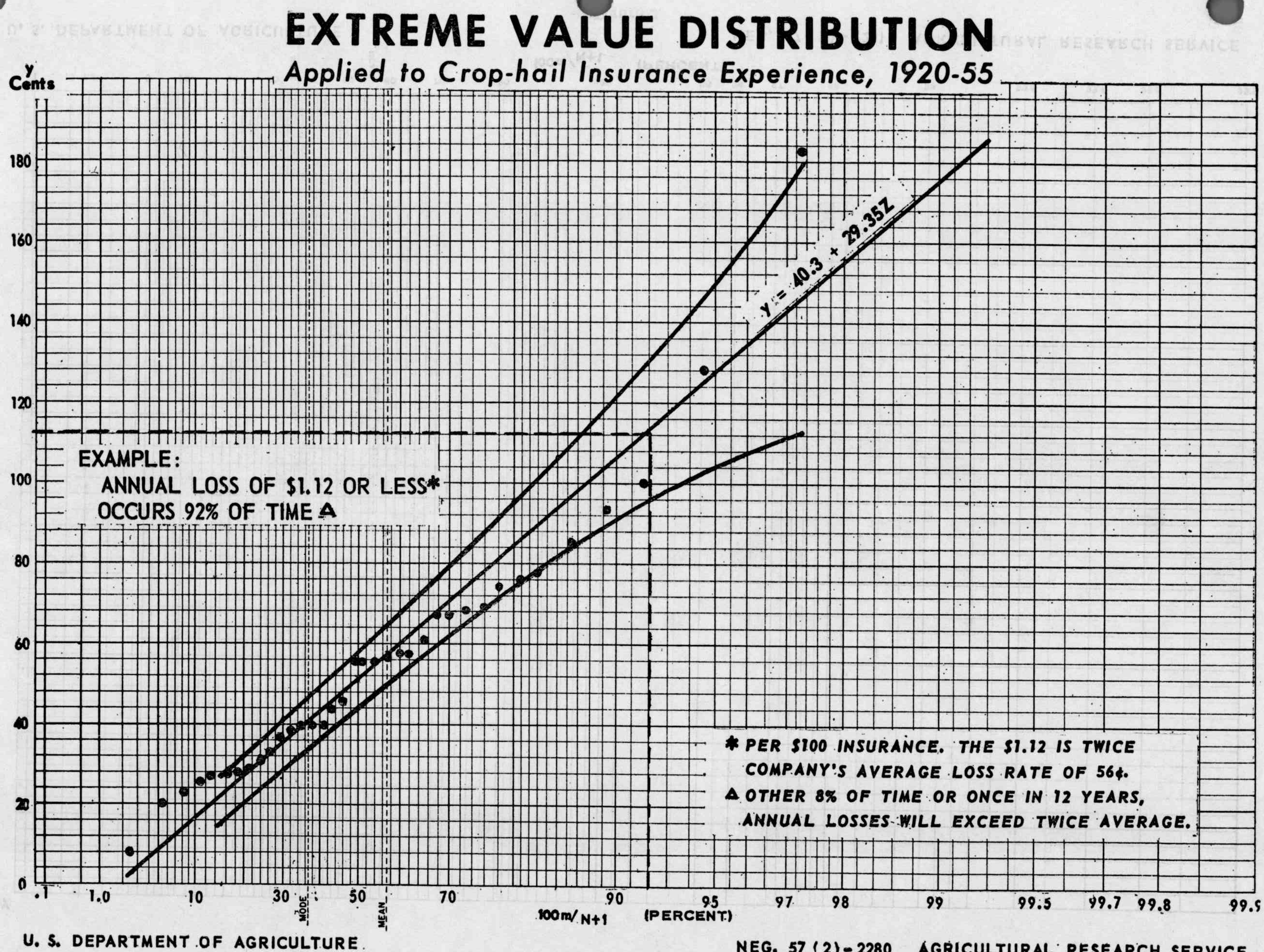

EXTREME VALUE DISTRIBUTION
Applied to Crop-hail Insurance Experience, 1920-55
y Cents
180
160
140
120
100
80
60
40
20
0
.1
1.0
10
30
50
70
90
95
97
98
99
99.5
99.7
99.8
99.9
MODE
MEAN
100m/N+1
(PERCENT)
y = 40.3 + 29.35Z
EXAMPLE:
ANNUAL LOSS OF $1.12 OR LESS*
OCCURS 92% OF TIME △
* PER $100 INSURANCE. THE $1.12 IS TWICE COMPANY'S AVERAGE LOSS RATE OF 56¢.
△ OTHER 8% OF TIME OR ONCE IN 12 YEARS, ANNUAL LOSSES WILL EXCEED TWICE AVERAGE.
U. S. DEPARTMENT OF AGRICULTURE
NEG. 57 (2) - 2280 AGRICULTURAL RESEARCH SERVICE

Figure 1.

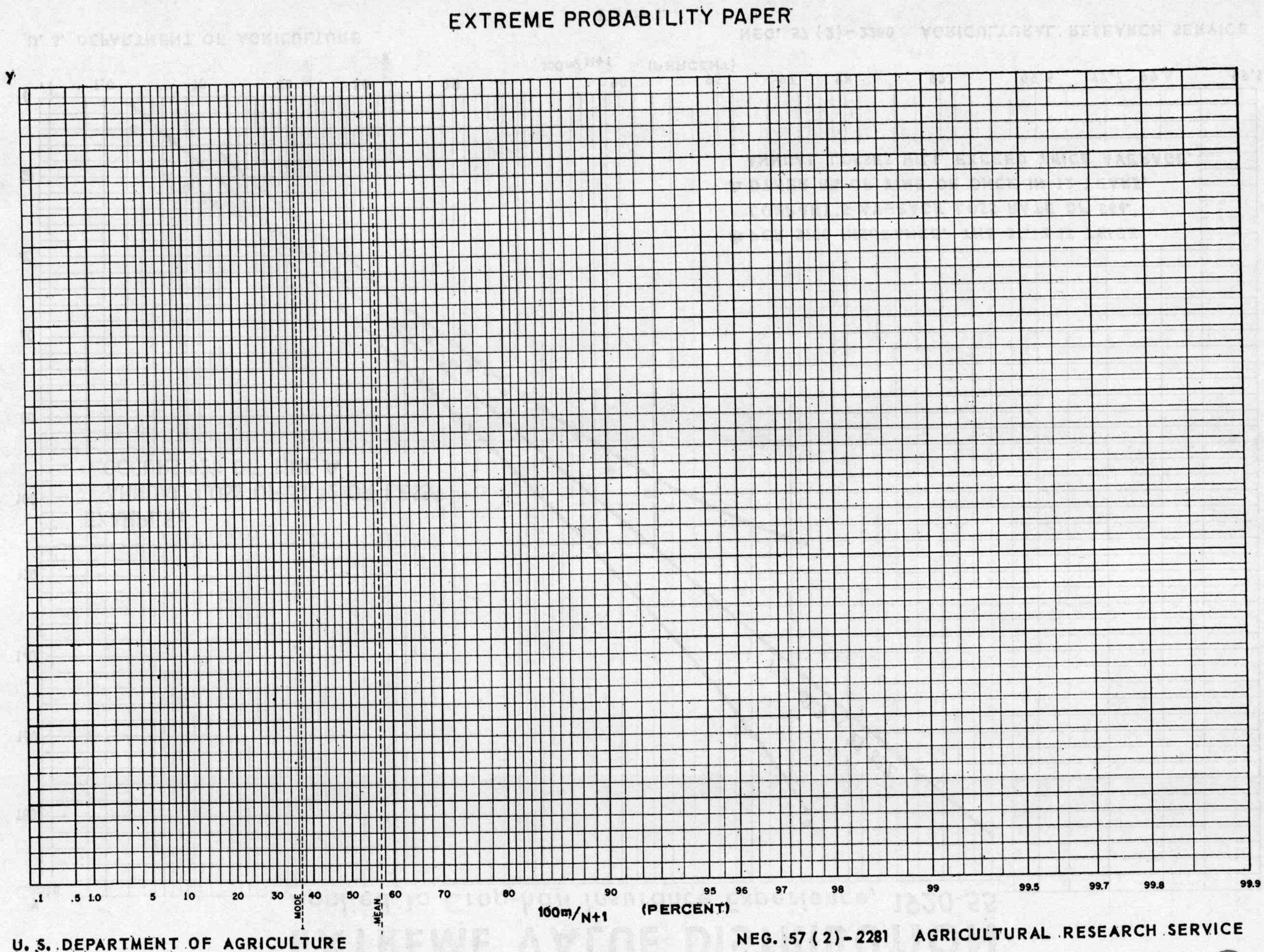
EXTREME PROBABILITY PAPER
Y
.1 .5 1.0 5 10 20 30 40 50 60 70 80 90 95 96 97 98 99 99.5 99.7 99.8 99.9
MODE
MEAN
100m/N+1 (PERCENT)
U. S. DEPARTMENT OF AGRICULTURE
NEG. 57 (2) - 2281
AGRICULTURAL RESEARCH SERVICE

Figure 2.

TABLE 2.—*Averages and standard deviations of reduced (Z) values, by sample size* [1]

Sample size	Average	Standard deviation
15	0. 5128	1. 0206
16	. 5153	1. 0301
17	. 5174	1. 0384
18	. 5196	1. 0471
19	. 5217	1. 0558
20	. 5236	1. 0628
21	. 5252	1. 0696
22	. 5268	1. 0754
23	. 5283	1. 0811
24	. 5296	1. 0864
25	. 5309	1. 0915
26	. 5320	1. 0961
27	. 5332	1. 1004
28	. 5343	1. 1047
29	. 5353	1. 1086
30	. 5362	1. 1124
31	. 5371	1. 1159
32	. 5380	1. 1193
33	. 5388	1. 1226
34	. 5396	1. 1255
35	. 5403	1. 1285
36	. 5410	1, 1313
37	. 5418	1, 1339
38	. 5424	1. 1363
39	. 5430	1. 1388

Sample size	Average	Standard deviation
40	0. 5436	1. 1413
41	. 5442	1. 1436
42	. 5448	1. 1458
43	. 5453	1. 1480
44	. 5458	1. 1499
45	. 5463	1. 1519
46	. 5468	1. 1538
47	. 5473	1. 1557
48	. 5477	1. 1574
49	. 5481	1. 1590
50	. 5485	1. 1607
51	. 5489	1. 1623
52	. 5493	1. 1638
53	. 5497	1. 1658
54	. 5501	1. 1667
55	. 5504	1. 1681
56	. 5508	1. 1696
57	. 5511	1. 1708
58	. 5515	1. 1721
59	. 5518	1. 1734
60	. 5521	1. 1747
61	. 5524	1. 1759
62	. 5527	1. 1770
63	. 5530	1. 1782
64	. 5533	1. 1793

Sample size	Average	Standard deviation
65	. 5535	1. 1803
66	. 5538	1. 1814
67	. 5540	1. 1824
68	. 5543	1. 1834
69	. 5545	1. 1844
70	. 5548	1. 1854
71	. 5550	1. 1863
72	. 5552	1. 1873
73	. 5555	1. 1881
74	. 5557	1. 1890
75	. 5559	1. 1898
76	. 5561	1. 1906
77	. 5565	1. 1915
78	. 5565	1. 1923
79	. 5567	1. 1930
80	. 5569	1. 1938
90	. 5586	1. 2007
100	. 5600	1. 20649
150	. 5646	1. 22534
200	. 5672	1. 23598
Infinity	. 5772	1. 28255

[1] Furnished by Environmental Protection Section, Research and Development Branch, Office of Quartermaster General, U. S. Army (except values for samples of size 16–19).

TABLE 3.—*Determination of 3 points on regression line*

Let m/n+1 equal [1]	Then Z equals [1]	And y equals [2]
0.20	−0. 47588	26. 3
.50	+. 36651	51. 1
.95	+2. 97020	127. 5

[1] These same values of m/n+1 and Z can be used for all problems, thus eliminating the necessity for computing or looking up values of Z corresponding to various values of m/n+1. Z is defined in footnote 3 of table 1.

[2] By substitution in formula (1) above.

and Z as the variable (with a coefficient equal to the slope of the line).

In the example . . . $y=40.3+29.35Z$.

6. Determine 3 points on the regression line. By always using m/n+1 as 0.20, 0.50, and 0.95, respectively, the corresponding values of Z shown in column 2 of table 3 can be used for any example.

   a. Fit these values of Z into your equation and solve for corresponding values of y. (See table 3.) Of course, the equation for each problem will include a different mode and slope; but the 3 values of Z need not change.

TABLE 4.—*Computation of control curves*

When m/n+1 equals [1]	Constant for m/n+1 equals [1]	Vertical distance from point on line to control curve (4.892×column 2)[2]
(1)	(2)	(3)
0.15	1. 255	6. 1
.30	1. 268	6. 2
.50	1. 443	7. 1
.70	1. 835	9. 0
.80	2. 241	11. 0
.85	2. 585	12. 6
Point on line corresponding to:		
Second-highest dot on chart		[3] 22. 3
Highest dot on chart		[4] 33. 5

[1] These values do not need to change from problem to problem.

[2] (1÷square root of n)×slope of regression line. In the illustrative problem, n=36 years, so the square of n=6. The slope of the regression line is 29.35. One-sixth of 29.35=4.892.

[3] 0.7594×slope of regression line. For illustrative problem, 0.7594×29.35=22.3 The figure 0.7594 does not change from problem to problem.

[4] 1.1407×slope of regression line. For illustrative problem, 1.1407×29.35=33.5. The figure 1.1407 does not change from problem to problem.

TABLE 6.—*Spread sheet for extreme-value calculations* [1]

Column number	Column heading	Explanation	Example
1	Identification		Crop-hail mutual
2	n	Number of years	36
3	Sum	From data sheet (table 1)	2, 023
4	Average	(3)÷(2)	56. 2
5	Sum of squared items	From data sheet (table 1)	152, 189
6	Sum squared	Square of (3)	4, 092, 529
7	Correction	(6)÷(2)	113, 681
8	Sum of squares	(5)−(7)	38, 508
9	Variance	(8)÷(figure in column 2 minus 1)	1, 100. 23
10	Standard deviation	Square root of (9)	33. 2
11	Theoretical average	From table 2	0. 5410
12	Theoretical standard deviation	From table 2	1. 1313
13	Slope	(10)÷(22)	29. 35
14	(11)×(13)		15. 9
15	Mode	(4)−(14)	40. 3
	First point on regression line		
16	m/n+1	It can always be 0.20	0. 20
17	Z	If m/n+1 is 0.20, Z does not change	−0. 47588
18	Slope×Z	(13)×(17)	−14. 0
19	y	(15)+(18)	26. 3
	Second point on regression line		
20	m/n+1	It can always be 0.50	0. 50
21	Z	If m/n+1 is 0.50, Z does not change	0. 36651
22	Slope×Z	(13)×(21)	10. 8
23	y	(15)+(22)	51. 1
	Third point on regression line		
24	m/n+1	It can always be 0.95	0. 95
25	Z	If m/n+1 is 0.95, Z does not change	2. 97020
26	Slope×Z	(13)×(25)	87. 2
27	y	(15)+(26)	127. 5
	Control curves		
28	Square root of n	Square root of (2)	6
29	1÷(28)	Reciprocal of (28)	0. 166667
30	(13)×(29)		4. 892
31	(30)×1.255	Vertical distances for m/n+1=0.15	6. 1
32	(30)×1.268	Vertical distances for m/n+1=0.30	6. 2
33	(30)×1.443	Vertical distances for m/n+1=0.50	7. 1
34	(30)×1.835	Vertical distances for m/n+1=0.70	9. 0
35	(30)×2.241	Vertical distances for m/n+1=0.80	11. 0
36	(30)×2.585	Vertical distances for m/n+1=0.85	12. 6
37	0.7594×(13)	Vertical distances for second-highest dot	22. 3
38	1.407×(13)	Vertical distances for highest dot	33. 5

[1] In addition to a spread sheet, for mass production of data, a separate data sheet (like table 1) is needed for *each* set of annual data.

In the *example*,

$\frac{m}{n+1}$	Calculations Mode	slope	Z	Y
0.20	40.3+(29.35×	−.47588)		26. 3
.50	40.3+(29.35×	+.36651)		51. 1
.95	40.3+(29.35×	2.97020)		127. 5

7. Plot the 3 coordinates (of m/n+1 versus y) on extreme-value probability paper.

a. Find the value for m/n+1 on horizontal scale and, from this point, go up to a point opposite y on vertical scale, where an x or dot is placed. Locate 3 such coordinate points.

In the example,

Horizontal scale $\frac{m}{n+1}$:	*Vertical scale y*
0.20	26.2
.50	51.1
.95	127.5

8. Join these 3 points in a straight line. If they do not fall in a straight line, an error has occurred.

9. Now plot your data from table 1 (columns 2 and 4) on the probability paper. The dots should fall quite closely about the line in order for the extreme-value theory to apply.

In the example (from table 1),

Horizontal scale $\frac{m}{n+1}$ (Col. 2):	*Vertical scale y (Col. 4)*
0.027	8
.054	20
.081	23
etc.	etc.

If control curves are desired,

10. Compute vertical distances (above and below) from points on the regression line, as explained in table 4.

11. Join these points to form control curves. (See fig. 1.)

a. If two-thirds of the dots fall within the control curves, the theory applies.

In the example, 30 of 36 dots, or 83 percent, fall within the control curves.

TABLE 5.—*Probabilities that may be read from regression line in figure 1*

Annual loss cost (y)		Cumulative probability of occurrence $\frac{m}{n+1}$	
Cost per $100 of insurance	Percentage of average [1]	Equal to or less than y	More than y
Cents	*Percent*	*Percent*	*Percent*
40.3	72	37	63
56.2	100	57	43
112.4	200	92	8
140.5	250	96.7	3.3
168.6	300	98.7	1.3

[1] Average loss cost 56.2 cents per $100 of insurance.

Interpretation of Data

The annual loss costs in the example are distributed fairly well about the regression line.[6] More than two-thirds of the dots fall within the control curves. The probabilities in table 5 may be read from the regression line in fig. 1:

An extra sheet of the probability paper is included (as fig. 2) in this article for use by those who wish to experiment with the method.

The column headings for a spread sheet, and their explanation, including the data for the example used here, are shown in table 6.

[6] Equation: $y=40.3+29.35Z$.