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**Fifth Joint Conference on
Agriculture, Food and the Environment**

Proceedings of a Conference Sponsored by
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SESSION VIII: LAND USE AND RURAL DEVELOPMENT

**PAPER 2: AN ARBITRAGE-FREE APPROACH TO
QUASI-OPTION VALUE**

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AN ARBITRAGE-FREE APPROACH TO QUASI-OPTION VALUE*

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ABSTRACT

In the presence of uncertainty and irreversibility, dynamic decision problems should not be solved using expected net present value analysis. The right to delay a decision can be valuable. We show that the value of this right equals Arrow and Fisher's (1974) quasi-option value. In a discrete model we show how to derive quasi-option value using methods from finance, methods that remove altogether the need to take expected values of future stochastic variables. Two main findings are presented. First, if the stochastic dynamic process underlying the problem is known, the Arrow and Fisher and Henry (1974) result, that improper use of net present value leads to too much early development, is correct. Second, if the process is not known perfectly, their result can be incorrect in the sense that net present value methods lead to the correct outcome while the dynamic rule does not.

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AN ARBITRAGE-FREE APPROACH TO QUASI-OPTION VALUE

*If you choose not to decide
You still have made a choice.*

—Geddy Lee of Rush, *Freewill*

1. Introduction

Decision problems with stochastic, dynamic, and irreversible elements should not be treated using expected net present value (ENPV) analysis. Replacing future stochastic variables with their expected values, discounting costs and returns back to the present, and investing whenever ENPV is positive—as the rule prescribes—can lead to costly mistakes. It is better to solve the correct dynamic problem, accounting properly for the irreversibility, than to take advantage of the shortcut that is ENPV. This is the advice handed down by Arrow and Fisher (1974) and by Henry (1974)—hereafter AFH—in two remarkably similar articles addressing natural resource development problems specifically. Both papers also claim that use of ENPV will systematically favor too much early development, or development too early.

Interest in this matter has not waned in the past two decades, and for good reason. The idea contains, after all, a provocative ideological element. If it is true that the use of ENPV leads to too much development, then those who wish to preserve wild places can legitimately employ the AFH finding to defend against many development projects. What is more, ENPV is ubiquitous in environmental policy. Much of the environmental planning carried out by and on behalf of U.S. regulatory agencies from the EPA to the U.S. Army Corps of Engineers rests upon cost-benefit exercises that use ENPV analysis.¹ If the method employed by regulatory agencies is biased systematically in a certain direction, there is a clear need for a thorough understanding of the method, the claims against its use, and the potential consequences when it is used.

The literature on this matter in resource economics is extensive. The breakthrough early piece is by Weisbrod (1964), who asserted that preservation of a natural resource confers “option value” over and above the value of development. His presentation of the idea was not formal, and in the ensuing decade two separate strands of literature arose from it. The first, due primarily to

¹For a relatively broad introduction to environmental policy from a practitioner’s viewpoint, see Arnold (1995), who provides numerous examples supporting our point.

Bohm (1975), Graham (1981), and Schmalensee (1972), seized the phrase option value to describe a premium accruing to preservation that can be attributed to risk aversion.² The second, due to AFH, is expressly independent of risk preferences. Arrow and Fisher (1974) introduced the phrase “quasi-option value” (QOV) to describe the extra value that can be gained if one eschews ENPV analysis in favor of the fully dynamic alternative. The literature on quasi-option value is now vast and, in certain respects, less than transparent. Agreement has yet to be achieved concerning basic definitions and nomenclature. More substantive disagreements have reared up from time to time. Some have been resolved; evidently others have not. In recent years the problem has been attacked using sophisticated tools from financial economics. While powerful, by their nature these tools have sometimes contributed to the apparent confusion to which we allude.

In this paper we mean to lessen the confusion. A crucial ingredient for our story is the distinction between uncertainty generated by a known stochastic dynamic process and uncertainty about the parameters of the process itself. In the former, the AFH insight is impeccable: ENPV is biased toward immediate development. In the latter, if the fundamental stochastic process is imperfectly understood, the AFH insight can be wrong: ENPV might yield the correct outcome while the AFH rule does not.

The first observation gives us the first theme of the paper, which is that techniques from financial economics can often be used to place a value upon the right to delay an irreversible development decision. It will be seen that these techniques make expected value itself obsolete by removing all dependence of present values upon the probability of alternative future outcomes. We develop an arbitrage-free approach to net present value. The methods themselves can be traced to the discovery of option-valuation equations in 1973—at about the same time AFH appeared. They require thinking of an investment opportunity as a call option. By comparison to recent work the approaches of AFH and their early followers were relatively straightforward. The trend has been toward ever more sophisticated techniques, with the result that seeing how the elegant and profound AFH insight matters in applied work is sometimes obscured. There can be no doubt, however, that the powerful tools are both exceedingly useful and capable of addressing precisely the concern raised by AFH.

The second observation gives us the second theme of the paper, which is that the AFH insight might be overturned. If the level of volatility is misestimated, following the AFH recipe can lead

²We will have nothing to say about this notion or the connected literature. The interested reader may consult Bishop (1982) and Cory and Saliba (1987) for masterful treatments of option value.

to an incorrect development decision when a decision maker using ENPV would have got it right. We shall argue that this result is not just a technical curiosity. Indeed, it is most likely to occur under conditions which hold in resource development problems.

Though most recent work is based on continuous-time models, ours is in discrete time. The reader may wonder why this choice was made, and whether it limits the usefulness of our treatment. Regarding the latter, we argue emphatically that it does not. The discrete model can be extended in a number of directions, making it more realistic. Most of the data one could use in implementing our method empirically are discrete, which means that for some purposes one must adopt a discrete approach. In any case one should not think of a discrete model as a short cut compared to a continuous-time model. For most purposes these matters are dealt with more easily in continuous time than in discrete time. (See Cox and Rubinstein (1985) or Dixit and Pindyck (1994) for arguments along these lines.) As to why we chose a discrete approach, the main reason is that we feel it permits a clearer exposition of the fundamental ideas. One of the ways that this helps is that it is much more difficult to obscure important but unsavory assumptions in a discrete model, and we want to be scrupulously honest about the assumptions we maintain and their implications.

2. Background

The AFH admonition against using ENPV in a dynamic and stochastic setting was not, in itself, new in 1974. It is, after all, akin to the suggestion—familiar from dynamic programming—that a dynamic problem should be solved using the appropriate dynamic techniques. Even the effects of the presence of an irreversibility, a twist that AFH included, was studied before 1974.³ By assuming that the development project is all or nothing, AFH added another twist, which together with irreversibility yields a scenario quite different from many dynamic programming problems in resource economics. Non-renewable resource extraction models, for example, have as solution a path of control decisions. With its one-time, all-or-nothing nature, the solution in the AFH model amounts to an optimal stopping rule, where the intuition is relatively simple.

For resource economists, the special appeal of the AFH papers is of course their focus on resource development problems. Bellwether papers in resource economics following the AFH tradition include, among others, Conrad (1980), Fisher and Hanemann (1987), and Hanemann (1989). For a 2-period model, assuming that nothing after the second period matters, Conrad showed that quasi-option value is equivalent to the value of information. Also working with a 2-period model,

³Arrow and Fisher (1974) themselves cite Arrow (1968) and Arrow and Kurz (1970).

Fisher and Hanemann and Hanemann developed a definition of QOV that they called the value of information conditional upon no development in the first period. Emphasis in their work is on whether, in the second period, the decision maker knows which state has occurred. The connection to AFH is plain: one should not rely upon ENPV analysis.

In section 3 we explain why the mistake one makes by inappropriately employing ENPV is the same in every respect as the mistake one makes by ignoring the right to delay the decision. The value of the mistake *is* quasi-option value, and it can be expressed

$$\text{Quasi-option value} =$$

$$\text{Value of the investment opportunity including delay option} - \text{Expected net present value.}$$

Alternatively, this could be written so that the value of the investment opportunity consists of two parts: its expected value if investment occurs today, and the value of the right to delay the decision. In some situations $\text{QOV} = 0$, in which case the right to delay the development decision has no value. This can happen for two diametrically opposed reasons, a fact that is sometimes unclear. On one hand, it might be true that a given project should never, under any circumstances, be developed. Waiting to see what happens tomorrow is valueless here. On the other hand, it might be true that a given project should be developed immediately. Again, waiting to see what happens tomorrow is valueless.

At about the time the AFH papers appeared, Black and Sholes (1973) and Merton (1973) solved the problem of pricing a European option. Their work provided the analytical machinery upon which the subsequent explosive growth of the financial derivatives industry was based. A European call option gives its holder the right—but not the obligation—to purchase an underlying asset at a pre-specified price (the “strike price”) on a specified date (the “expiration date”). (The right to *sell* an asset at a pre-specified price is a put option.) The owner of a call gains if at expiration the price of the underlying asset is above the strike price. An American call is equivalent to a European call, except that it can be exercised at any time up to the expiration date.

In the 1980's, the usefulness of these methods for solving general investment problems was first appreciated. Many investment opportunities, it turns out, are akin to an American call option. The right to build a manufacturing plant before a zoning permit expires is one example. The strike price in this case would be the fixed cost of building the plant, the expiration date would be the deadline for construction, and the underlying asset would be the plant itself. Just as the price

of the financial asset fluctuates through time, so does the value of the output of the plant. Just as the owner of a financial option may delay exercising it in the hope that the price rises, so the decision to construct the plant might be delayed. Prominent among the early papers exploiting the relationship between a real investment and a financial option are Bernanke (1983), McDonald and Siegel (1985, 1986), and Majd and Pindyck (1987). Dixit and Pindyck's (1994) recent book contains a thorough and authoritative survey of the literature and of the techniques employed there.

While resource economists were developing the literature based upon AFH, a parallel literature sprang up in which the option-pricing methods were applied to resource valuation problems. Tourhino (1979) appears to have originated this application. The papers by Brennan and Schwartz (1985) and Paddock, Siegel, and Smith (1988) are important and much-cited pieces. It is noteworthy that in the main these papers do not cite Arrow and Fisher (1974). More recently, the financial methods have been applied to natural resource problems by authors who also trace their roots to AFH. The two literatures, once largely separate, are now coming together.⁴ The reader should be aware, though, that the terminology is not yet unified. Whether or not this quantity is emphasized, and whatever it is called, these continuous-time models all contain a quantity that is analogous to QOV. Its role is summed up nicely by Brennan and Schwartz, who write, "[O]nce the possibility of postponing an investment decision is recognized, it is clear that it is not in general optimal to proceed with construction simply because the net present value of construction is positive: there is a 'timing option' and it may pay to wait in the expectation that the net present value of construction will increase," (1985, p. 150).

The contributions of the present paper are threefold. The first, pedagogical in nature, is a clear description of the portfolio approach to valuation. Second, it is shown that if arbitrage-free methods can be used, the probabilities of alternative future price realizations become irrelevant. We will see that this fact has the fundamental implication that expectation in the usual sense no longer plays a role. Third, as we have noted, if the level of volatility of the underlying stochastic process is misestimated, the AFH insight can be incorrect.

3. ENPV, the Investment Decision, and Quasi-Option Value

Suppose a certain tract of wilderness land can be put to two alternative uses. It can either be preserved in its present natural, undeveloped state or it can be developed to extract some

⁴Cochrane (1990), Zinkhan (1991), Reed (1993), and Conrad (1995) are just a few members of this synthetic literature.

marketable resource. In its undeveloped state, in each year t ($t = 0, 1, \dots$) the land yields a constant amenity value A from recreation uses.⁵ Letting r denote the constant risk-free interest rate and $R = (1 + r)$ the risk-free compounding factor, the value of the land if preserved is $\tilde{A} = \sum_{i=0}^{\infty} A/R^i = AR/(R - 1)$. If the land is developed at t , the stream of amenity returns is immediately and irretrievably sacrificed in return for a stream of random net revenues from the project. We assume, then, that the development decision is completely irreversible in that the amenity value is gone forever and the project, once in operation, cannot be shut down. The entire value of the land is due to the potential for amenity revenues and for returns to the developed project.

This basic framework can accommodate any number of resource development problems. Returns from preservation might stem from hiking, hunting, or from a variety of recreation activities. The project might be construction of a dam, logging of a forest, or the construction of a mine; returns from development will correspond to the project. For purposes of discussion we will sometimes refer to the project as a mine, with output of copper.

The developed project exhibits the following physical properties. It has a known, finite life of n years ($1 \leq n < \infty$). If it is undertaken in any period t , the project ceases to function after period $t + n - 1$. The project is assumed to be normalized so that in each period it yields one unit of the marketable output. There is no uncertainty about output quantity. We assume that the mine is not reclaimed at the end of its life, neglecting the possibility of restoring some portion of the amenity benefits. We also rule out temporary shutdown. With the exception of a known output level, relaxing these assumptions changes things very little.

Development requires payment of a stream of costs, and yields a stream of revenues. Construction at period t (which we assume is completed immediately) requires a known fixed, irreversible investment of $I > 0$. Annual per-unit operating costs $c > 0$ are assumed to be constant. With this assumption the present value of the n -period stream of operating costs is given by $\sum_{i=0}^{n-1} c/R^i = c[(R^n - 1)/R^{n-1}(R - 1)] = c\Theta$.

Output price is the lone source of uncertainty. Price at t is denoted P_t . We assume P_t follows a stationary multiplicative random walk. That is, between t and $t + 1$ the price either rises by U percent to $P_{t+1}^+ = uP_t$ or falls by D percent to $P_{t+1}^- = dP_t$, where “+” and “-” denote the up and

⁵This amenity value could also include non-use values so long as these values can be counted legitimately as returns to the owner of the land. The land could be owned privately or publicly.

down states respectively and where $u = 1 + U$ and $d = 1 + D$.⁶ The objective probability of an up state (respectively a down state) at $t + 1$ is $q \in (0, 1)$ (respectively $1 - q$). We assume that q is fixed. These conditions imply that the (P_t/P_{t-1}) are independently and identically distributed for all $t \geq 1$. Leaving aside for now the question of estimation, suppose the parameters u , d , and q are known with certainty. Formally, we have the following definition.

DEFINITION. A **price process starting at $t = 0$** , denoted $PP = \{P_0, u, d, q\}$, is a binomial tree consisting of P_0 , together with the up and down increments u and d and a scalar $q \in (0, 1)$ describing the probability that the up state occurs at each t .

Given that the developed project yields one unit of output in each period, price and gross revenues coincide. Given P_t , for $i \geq 0$ the expected price at $t+i$ is simply the weighted average of uP_t and dP_t , using q for the weights and raising the result to the i th power: $E_t(P_{t+i}) = (qu + (1-q)d)^i P_t$. At time 0 the expected value of the stream of revenues from immediate development is $G_0 = \sum_{i=0}^{n-1} E_0(P_i)/R^i$. Let $\mu = (qu + (1-q)d)$ and note that the expectation at time 0 of the price i years hence is $E_0(P_i) = \mu^i P_0$. Rearranging and using the expression $\sum_{i=0}^{n-1} \mu^i = (1 - \mu^n)/(1 - \mu)$, we have

$$G_0(q) = P_0 \left(\frac{(R^n - \mu^n)}{R^{n-1}(R - \mu)} \right) = P_0 \Omega(q),$$

where $\Omega(q)$ denotes the ratio in parentheses.⁷

A decision maker employing the ENPV rule would behave as follows:

$$\begin{aligned} &\text{if } P_0 \Omega(q) - (c\Theta + I) \geq \tilde{A}, \quad \text{develop;} \\ &\text{if } P_0 \Omega(q) - (c\Theta + I) < \tilde{A}, \quad \text{do not develop.} \end{aligned}$$

The value of the investment opportunity under the ENPV rule is

$$W_0(q) = \max[\tilde{A}, P_0 \Omega(q) - (c\Theta + I)]. \quad (1)$$

Note that if $P_0 \Omega(q) - (c\Theta + I) < \tilde{A}$, the opportunity itself has value \tilde{A} . Because it refers to the ENPV rule, all of this assumes the decision must be made today. If this is not true, in the presence of an irreversibility and if the decision can be delayed, the ENPV rule should simply not

⁶To prevent riskless arbitrage, u and d must satisfy the inequality $u > R > d$.

⁷Note that G and Ω depend on q here. Soon it will be seen that when the valuation calculation is done properly, q will disappear.

be employed. A decision maker using it will sometimes choose to develop the project when it would be better to delay the decision.

The problem with ENPV is precisely this: it ignores the possibility of delaying the decision. At times the right to delay a decision is valuable, so any decision rule—including ENPV—that ignores it will fail to account for a valuable asset. To see this, suppose the development decision can be delayed one period, but no longer. To calculate the value of the investment opportunity in this situation, we begin at $t = 1$ (after which further delay is impossible so ignoring delay causes no problem) and work backwards. The value of the opportunity at in the up and down states is given respectively by

$$W_1^+(q) = \max[\tilde{A}, P_1^+ \Omega(q) - (c\Theta + I)] \quad \text{and} \quad (2a)$$

$$W_1^-(q) = \max[\tilde{A}, P_1^- \Omega(q) - (c\Theta + I)]. \quad (2b)$$

These expressions, like (1), convey the fact that development should occur—this time at $t = 1$ —only if discounted expected net revenues exceed the discounted returns to preservation. Our interest is in the value of the investment opportunity at $t = 0$. *If the development decision is delayed*, recalling that A is collected in period 0, the value of the opportunity is

$$W_0^*(q) = A + \frac{qW_1^+(q) + (1 - q)W_1^-(q)}{R}. \quad (3)$$

The quantity $(qW_1^+(q) + (1 - q)W_1^-(q))$, discounted one period, gives the current expected value of the optimal decision at $t = 1$. The value of W_0^* , as we shall see, can exceed W_0 by a significant amount.

The decision whether to develop at $t = 0$ is based upon a comparison between $W_0(q)$ and $W_0^*(q)$. If $W_0(q) > W_0^*(q)$ then development should take place immediately; otherwise the optimal choice is to delay the decision. At $t = 0$, the value of the investment opportunity itself is the larger of these:

$$W_0^{**}(q) = \max[W_0(q), W_0^*(q)].$$

Finally, the value of the right to delay the decision is the difference between $W_0^*(q)$ and $W_0(q)$ if this difference is non-negative, and 0 otherwise. This quantity is quasi-option value. It is here defined as

$$O^*(q) = \max[0, W_0^*(q) - W_0(q)].$$

A brief discussion of an example will be helpful. Suppose the parameters in the model take the values given in the following table.

Table 1. Example parameter values.

Parameter	P_0	R	n	u	d	q	I	A	c
Value	280	1.1	10	1.3	0.7	0.60	1000	75	50

Using these values, one can readily calculate $W_0(q)$, $W_0^*(q)$, and $O^*(q)$:

$$W_0(q) = 1045.6, \quad W_0^*(q) = 1335.4, \quad \text{and} \quad O^*(q) = 289.8.$$

Because $W_0(q) > \tilde{A} = 825$, the ENPV rule would lead to immediate development. If delay were impossible, this would be the correct decision. But delay is possible here, and because $W_0^*(q) > W_0(q)$, the optimal decision is to wait. The value today of the right to delay is the increment of present value accruing to the decision maker who chooses to delay and then behaves optimally in the future. In some situations, however, we might have $W_0(q) > W_0^*(q)$, in which case $O^* = 0$ and immediate development is optimal.

The value of the right to delay the development decision equals the cost of the mistake one commits by using ENPV in this setting. Our framework can be seen to agree with that of Fisher and Hanemann (1987), though their emphasis is somewhat different. This will be clearer in the following section, but our approach to QOV emphasizes whether development should take place immediately in a way that theirs, with its explicit conditioning of QOV on delay, does not.⁸ A more important difference emerges in the next section, where q disappears.

⁸Fisher and Hanemann define V^* and \hat{V} to be the expected value at $t = 0$ of the development opportunity at $t = 1$ if at $t = 1$ the decision maker does not (in the case of V^*) or does (in the case of \hat{V}) know P_1 —conditional on no development at $t = 0$. In our notation, V^* and \hat{V} correspond to the expected value at $t = 0$ of

$$\begin{aligned} W_1(q) &= \max[\tilde{A}, (qP_1^+ + (1-q)P_1^-)\Omega(q) - (c\Theta + I)] \quad \text{and} \\ W_1^*(q) &= \max[\tilde{A}, qW_1^+(q) + (1-q)W_1^-(q)], \end{aligned}$$

respectively. In $W_1(q)$ the development decision at $t = 1$ cannot depend on P_1 , while in $W_1^*(q)$ it can. The mapping between our setup and Fisher and Hanemann's setup is given by

$$\begin{aligned} V^* &= E_0(W_1(q)) = W_1(q)/R = W_0(q) \\ \hat{V} &= E_0(W_1^*(q)) = W_1^*(q)/R = W_0^*(q). \end{aligned}$$

So long as $W_0^* \geq W_0$, our O^* corresponds to their quasi-option value, V_q .

4. Arbitrage-Free Quasi-Option Value: Known Stochastic Process

In its treatment of quasi-option value, the preceding section is in principle perfectly sound. It separates the value of the project into two parts: its value if development occurs immediately and the value of the right to delay: $W_0^{**} = W_0 + O^*$. The treatment of the dynamic and uncertain problem itself is flawed, however. The flaw inheres in the mismatch between the riskiness of the project and the rate used to discount future costs and returns. Though R is riskless, the project is not. It is—or should be—a cardinal rule in such problems to ensure that the discount rate reflects the project's riskiness. If there were no market price for the project's risk, we would be forced to employ some equilibrium method such as CAPM to adjust R and so obtain a risk-adjusted discount rate appropriate for this particular project. This discount rate could then be used to compute O^* as above.⁹

But we have developed our problem in such a way that yet another method can be used. The first goal of this section is to present the method. Free of the inconsistency between R and the project's riskiness, the method draws upon financial economics and exploits the kinship between the natural resource development opportunity under study and an American call option. The formal technique upon which we rely goes by many names, including risk-free or arbitrage-free valuation. It is based on an elegant but exceedingly powerful insight, due to Black and Sholes (1973).

The problem at hand is to discover the value of the investment opportunity in such a way that the value of the delay option is included. For this asset there may be no reliable market price. The model is set up so that a spanning condition is satisfied. Put simply, this means that there are exactly as many states as there are assets whose prices can be discovered in the market. In the present problem the "priced" assets are the output of the project and a riskless bond. These can be used to price any asset whose value is a function of them, as the "unpriced" asset—the investment opportunity—is. For some problems in natural resources the spanning condition will not be satisfied, in which case our approach could not be used in exactly this form. The key is whether the risk in question (for us, this is P_t) is priced in the market.¹⁰ This is especially true if there is no market for the output of a given developed project. Likewise, if the stream of amenity benefits were stochastic and unpriced, as is often true, the approach we use would not work. When

⁹This approach usually follows Samuelson (1965). Adjusting the discount rate in this manner is never easy.

¹⁰Though the precise form we use is out of reach if the spanning condition is not satisfied, it is still possible to draw upon financial methods. One would then have to estimate the price of the risk of interest, and use dynamic programming methods to value the investment opportunity. See Cox and Rubinstein (1985) or Dixit and Pindyck (1994). Our aim is to describe the method in the most favorable setting, where it is most easily understood.

the arbitrage-free approach can be used, though, it should be viewed as a compelling choice.

In addition to the spanning condition, for the remainder of the paper we shall maintain three assumptions. First, we assume the absence of riskless arbitrage opportunities. Second, we assume no limit upon short sales. Third, market positions may be taken without cost: there are no brokerage fees or other transactions costs. These assumptions are fairly ubiquitous in financial models. Relaxing them will seldom alter the results materially.

Though the method employed here is quite different, there are parallels to the previous section. Partly to highlight the parallels and partly to preserve order in notation, we will continue to use W_0 , W_0^* , and W_0^{**} respectively to denote the value of the investment opportunity if delay is impossible, the value of the investment opportunity if development is delayed, and the maximum of the two. Note that each has been shorn of q . Similarly, we will once again use the symbol Ω but without the argument q . This is more than just a notational tic. The innovation of this section is the fact that everything is done without relying upon q .

Assume as before that the decision can be delayed one period and that the stochastic price process is known. Rather than starting with W_0 , we begin here by deriving W_0^* . We reason backward from $t = 1$, when delay is impossible. Let W_1 denote the value of the opportunity at $t = 1$. We have

$$W_1^+ = \max[\tilde{A}, P_1^+ \Omega - (c\Theta + I)] \quad \text{and} \quad (4a)$$

$$W_1^- = \max[\tilde{A}, P_1^- \Omega - (c\Theta + I)] \quad (4b)$$

in the up and down states respectively. We will return presently to derive Ω , which we note does not in general equal $\Omega(q)$.

Now consider forming a portfolio at time $t = 0$ consisting of the investment opportunity itself (which is already owned by the decision-maker) and a short position in the quantity y of the underlying asset, copper. This is called a *hedge portfolio*.¹¹ It is built so as to be riskless: the future value of the portfolio is independent of P_1 . It is a hedge in the sense that if P rises, the opportunity to build a mine increases in value while the short position creates a loss. The opposite is true if P falls. Given y , the value of the portfolio at $t = 0$ is $\Pi_0 = W_0^* - yP_0$. Its value at $t = 1$ is

$$\Pi_1^+ = W_1^+ - yuP_0 + RA \quad \text{and} \quad (5a)$$

$$\Pi_1^- = W_1^- - ydP_0 + RA \quad (5b)$$

¹¹We could alternatively have formed a *replicating portfolio*, consisting of the riskless bond and the underlying asset, to replicate the payoffs to the investment opportunity. The results would be identical.

in the up and down state respectively. Note that at $t = 1$ the portfolio includes the compounded value of A , the amenity benefits from period 0. This is because by assumption development is delayed, so A is collected at $t = 0$. The key to risklessness is to choose y so that the portfolio has the same value in each state at $t = 1$. Setting (5a) and (5b) equal, we can solve for y as

$$y = \frac{W_1^+ - W_1^-}{P_0(u - d)}, \quad (6)$$

from which we find $\Pi_1 = RA + (uW_1^- - dW_1^+)/ (u - d)$.

To solve for W_0^* , we invoke the requirement that the portfolio, being riskless, must yield precisely the riskless rate.¹² It is in this requirement that the price of the riskless bond—the other spanning asset—is put to use. Letting $S = (R - 1)yP_0$ denote the cost of holding the short position (the trader on the other side of the transaction requires this payment), the capital gain to holding the portfolio is $\Pi_1 - \Pi_0 - S$. This quantity must equal $(R - 1)\Pi_0$, the riskless return on holding Π_0 for one period: $\Pi_1 - \Pi_0 - S = (R - 1)\Pi_0$. Using the expression for y , with a bit of straightforward algebra this can be manipulated to yield

$$RW_0^* = RA + \frac{uW_1^- - dW_1^+}{u - d} + \frac{W_1^+ - W_1^-}{u - d}.$$

For later reference, note that $\Pi_0 = (\Pi_1 - S)/R$. Finally, we can write

$$\begin{aligned} W_0^* &= A + \frac{1}{R} \left[\frac{(1 - d)W_1^+ - (1 - u)W_1^-}{u - d} \right] \\ &= A + \frac{pW_1^+ + (1 - p)W_1^-}{R}, \end{aligned} \quad (7)$$

where $p = (1 - d)/(u - d)$.

Note the similarity between $W_0^*(q)$ and W_0^* in equations (3) and (7). In (3), the expected net return at $t = 1$ was calculated using the $W_1^\pm(q)$ (which depended on q and on the expected price, $E_0(P_1) = \mu P_0$) and R . These are mismatched: the $W_1^\pm(q)$ reflects a risky project while R is riskless. In (7), the use of R is appropriate. By forming the hedge and squeezing the risk out of the investment problem, we have earned the right to discount future returns using R . The reader should note, in fact, that q is entirely absent from this section. Remarkably the probability q of an up state, and by extension the expected price, is irrelevant to the value of the investment

¹²If it did not, then our decision maker would be the source of arbitrage profits.

opportunity.¹³

What is one to make of p ? It appears now to play the role of q , which suggests that p is also a probability. It does not describe the likelihood of any actual event, though. Cox and Ross (1976) called p a “risk-neutral probability,” because when it is teamed with R as in (7) we obtain the value W_0^* that a risk-neutral decision maker would place on the portfolio. The phrase “risk-neutral” is unfortunate insofar as it suggests the result is valid only if the decision maker is risk neutral. On the contrary, the replication method works regardless of risk preferences. The discipline of the market, and the information in the price process it generates, force risk-averse investors and risk-loving investors to agree on the correct value of W_0^* .

One gap—the definition of Ω —must be closed before moving on. Recall that $P_t\Omega$ gives the value at t of the stream of revenues if the project is developed at t . It turns out that Ω depends on p , which we have only now derived. Yet the derivation leading to p used Ω in several places. Which comes first? The answer is that p comes first, and from here on we can think of p and R , the primary ingredients in Ω , as a matched pair.

To see that it is appropriate to use p in defining Ω , suppose the project has been in operation for $n - 1$ years, so that one year of production remains. How much is the project worth in this penultimate period? We can derive this value by forming a hedge portfolio just as before, removing the risk due to price uncertainty in the last year, and discounting it one period using R and assigning “probability” p to an up state. The parameters u and d are unchanged, which means p is unchanged as well. At time $t + n - 2$, then, with one period of its life remaining the project has value $[(pu + (1 - p)d)P_{t+n-2}]/R$. Following this algorithm back, we see that the arbitrage-free value at period t of the stochastic returns to the project developed at t is

$$G_t = \sum_{i=1}^{n-1} \frac{\varphi^i}{R^i} P_t = P_t \Omega,$$

where $\varphi = pu + (1 - p)d$ and $\Omega = (R^n - \varphi^n)/(R^{n-1}(R - \varphi))$. But it is easy to check that given our definition of p , $\varphi \equiv 1$. Thus, our approach to valuing a stochastic stream of returns amounts

¹³In a most wonderful understatement, when describing the differential equation at the heart of Black and Sholes’s (1973) continuous-time option pricing model, to which our treatment can be traced, Hull (1989, p. 96, emphasis added) writes, “The Black-Sholes differential equation would not be independent of risk preferences if it involved the expected return on the stock, μ . This is because the value of μ does depend on risk preferences. The higher the level of risk aversion by investors, the higher μ will be for any given stock. *It is fortunate that μ happens to drop out in the derivation of the equation.*” Fortunate indeed. The result is the linchpin—and the genius—of the method.

to the same thing as assuming the price remains fixed at P_0 in the future.¹⁴

Now that we have Ω , we are nearly finished. The counterpart to $W_0(q)$ from the previous section is

$$W_0 = \max[\tilde{A}, P_0\Omega - (c\Theta + I)].$$

This is not an expected value, so rather than ENPV we will call it ANPV: “arbitrage-free” net present value. Everything else follows naturally. The value of the investment opportunity is¹⁵

$$W_0^{**} = \max[W_0, W_0^*].$$

Quasi-option value, which is the value of the right to delay the decision, is given by

$$O^* = \max[0, W_0^* - W_0].$$

The reader may wonder why ENPV has been forsaken altogether. Would it not be more interesting to compare the arbitrage-free W_0^* to $W_0(q)$, which is the value an ENPV maximizer would place upon the investment opportunity? This might be interesting, but it would not be appropriate. Because of the way they use information in the market, none of the values calculated in the previous section are comparable to the values derived here. One can compare W_0 to W_0^* , or one can compare $W_0(q)$ to $W_0^*(q)$, but any comparison that mixes them is illegitimate.

This is a good time to return to the example. Figure 1 contains the four curves of interest— W_0 , W_0^* , W_0^{**} , and O^* —all depending on P_0 . The other parameters are fixed at the values given in Table 1, so $\tilde{A} = 825$. To begin, observe that there are two kinks in W_0^* and one in W_0 and denote them P'_0 , P''_0 , and P'''_0 from left to right as labelled in the figure. The interpretations to be placed on them are as follows.

P'_0 : This is the price corresponding to the first kink in the W_0^* curve. If $P_0 < P'_0$, a decision maker who defers the decision *would not invest* at $t = 1$ even in the up state. Thus, if $P_0 < P'_0$ the decision maker should never invest. Regardless of the state at $t = 1$ it is more profitable simply to collect A each period in perpetuity.

¹⁴This is a subtle but important point. Rather than presenting this argument for deriving the arbitrage-free G_t , we could simply have assumed that at the moment of development the owner could contract with a buyer to sell the entire stream of production at price P_0 . The results would all have been identical to what we obtain. This is because assuming we can create the riskless hedge is equivalent to assuming that the riskiness of the project is completely diversifiable—which it must be if such a forward contract is available.

¹⁵The counterpart to W_0^{**} in financial option-pricing models is called the value of the option or the option value. Note that our QOV is a portion of W_0^{**} .

P_0'' : This is the price corresponding to the kink in the W_0 curve. If $P_0 > P_0''$, a decision maker employing the ANPV rule would invest immediately. Otherwise, he or she would not invest.

P_0''' : This is the price corresponding to the second kink in the W_0^* curve. If $P_0 > P_0'''$, a decision maker who defers the decision *would invest* at $t = 1$ even in the down state.

One other price is important. Indeed, it turns out to be the most important for our story.

P_0^* : This is the price at which W_0 and W_0^* intersect. If $P_0 > P_0^*$, the optimizing decision maker would choose to develop the project immediately.

It turns out that P_0''' is never interesting. If P_0 is this large, investment should take place immediately. To see that this is true in the example, note that $P_0''' > P_0^*$. It is a straightforward exercise to show that this relation will hold in general. If the current price is sufficiently large that development should occur at $t = 1$ even in the down state, development should not be delayed.

Turn now to the O^* curve. We see that $O^* = 0$ in two regions: where $P_0 < P_0'$ and where $P_0 > P_0^*$. In the first, development should never take place. The right to wait until tomorrow to decide whether to invest has no value. In the second, development should take place immediately. Once again the right to wait until tomorrow has no value. This possibility appears to be what Arrow and Fisher (1974, p. 319) meant when they wrote, "Just because an action is irreversible does not mean that it should not be undertaken." Although one must contemplate forming the portfolio here, and indeed the calculation of the thresholds requires contemplating doing so, if $O^* = 0$ there will be no short sale, no hedge, no portfolio.

For $P_0 \in (P_0', P_0^*)$, though, the story is very different. Here the right to wait is valuable. The hedge is actually carried out. In some situations W_0^* can exceed W_0 by a sizable amount. Note that O^* achieves its maximum at P_0'' , where W_0 first rises above 0. The most important range of initial prices, though, at least in the context of the AFH insight, is (P_0'', P_0^*) . For prices in this interval, ANPV analysis leads one to invest immediately though the best decision is to wait and invest at $t = 1$ only in the up state. It is here that the ANPV rule leads to too much development, as AFH warned.

The price P_0^* , then, can be said to play an important role. Given a collection of values for the other parameters, if the current price exceeds P_0^* development should occur immediately. Here, ignoring the delay option and employing ANPV is not costly. Otherwise, it might be. Of course,

in an empirical setting one must calculate the hedge to derive P_0^* , and so it is never safe to use the ANPV rule alone. Given its function as a threshold or cut-off value, we wish to discover how P_0^* responds to changes in these other parameters.

Consider first the effect of a change in n , A , or I . As either the amenity value of the undeveloped resource (A) or the fixed investment (I) grows the threshold development price rises, making immediate development less likely. This is not surprising, for in either case the net returns to the developed project fall relative to the preservation alternative. As the lifetime of the project increases, the threshold development price falls. This is not unexpected either, since the length of the project's life can be considered a proxy for its size. These intuitive relationships may be formalized as follows.

PROPOSITION 1. *For any given parameter vector (R, n, u, d, q, I, A, c) , P_0^* is (i) increasing in A , (ii) increasing in I , and (iii) decreasing in n .*

PROOF: We begin by deriving an analytical expression for P_0^* . Note first that, since $P_0''' > P_0^*$, if development is delayed it must be that in the relevant region $W_1^- = \tilde{A}$. The expression for W_0 thus becomes

$$W_0^* = \frac{p(uP_0\Omega - (c\Theta + I)) + (1-p)\tilde{A}}{R} + A. \quad (8)$$

Using the definition of W_0 , we can solve for P_0^* by setting $W_0 = W_0^*$, which yields

$$P_0^* = \frac{1}{\Omega} \left(\frac{(R-p)(c\Theta + I) + (1-p)\tilde{A} + RA}{R - pu} \right). \quad (9)$$

The denominator of the rightmost term must be positive. To see this, note that $pu = (u - ud)/(u - d) < 1$, while $R > 1$ by assumption. Thus, the proof of (i) and (ii) follows without difficulty from partial differentiation of P_0^* with respect to A and I .

Result (iii) is a bit more subtle. The only place where n enters (8) is in the Ω and Θ terms. It suffices to show that $P_0^*(n+1)/P_0^*(n) < 1$, where the n argument has the natural interpretation. Given that Ω is a finite sum of strictly positive numbers, it must be true that $\Omega(n+1) > \Omega(n)$ for any $n \geq 1$. Because $\varphi = 1$, it is also true that $\Omega = \Theta$. Using this substitution we may write the ratio of P_0^* 's as

$$\frac{P_0^*(n)}{P_0^*(n+1)} = \frac{\Omega(n)}{\Omega(n+1)} \left(\frac{((R-p)(c\Omega(n+1) + I) + (1-p)\tilde{A} + RA)/(R - pu)}{((R-p)(c\Omega(n) + I) + (1-p)\tilde{A} + RA)/(R - pu)} \right) \quad (10)$$

Combining constant terms common to the numerator and denominator, let $b = ((R - p)I + (1 - p)\tilde{A} + RA)$. The ratio in (10) may now be written

$$\frac{P_0^*(n)}{P_0^*(n+1)} = \frac{\Omega(n)}{\Omega(n+1)} \left(\frac{c(R-p)\Omega(n+1) + b}{c(R-p)\Omega(n) + b} \right) = \frac{c(R-p)\Omega(n)\Omega(n+1) + b\Omega(n)}{c(R-p)\Omega(n)\Omega(n+1) + b\Omega(n+1)} < 1.$$

The last inequality holds because $\Omega(n+1) > \Omega(n)$. This completes the proof of the proposition. ■

The effect on P_0^* of a change in R is also interesting, but is less straightforward. Indeed, the graph of P_0^* as a function of R is U-shaped. One can show that if R is very small, P_0^* declines as R increases. As R grows, eventually P_0^* begins to increase along with R .

Together with the irreversibility of the investment decision, everything to this point has been driven by uncertainty about the future returns to an investment opportunity. To complete this section and to set the stage for the next, we now turn to a deeper question: what if the *level* of uncertainty increases?¹⁶ We examine the effect upon P_0^* and O^* of an increase in the volatility of the price process, which in this binomial setting takes the form of an increase in the spread of the price realizations in each period. Of course the precise form that an increase in variance takes is important. Hence, we consider a mean-preserving spread, which has the virtue that it leaves $E_t(P_{t+i})$ unchanged.

Up to this point, the relationship between u , d , and q has been quite general. In practice these parameters can be chosen in such way that the binomial process approximates any time series process. If the data are discrete realizations of an underlying continuous-time process, and if that process follows geometric brownian motion, the data can be used to estimate our u , d , and q . Let μ and σ denote estimates of the trend and volatility (or standard deviation) of the unobserved continuous process.¹⁷ Letting Δt denote the time increment between observations in the data, the mapping between μ and σ (which are estimated directly from the discrete data) and u , d , and q (which describe the binomial process) is as follows.

$$u = e^{\sigma \sqrt{\Delta t}}, \quad (11a)$$

$$d = 1/u, \quad (11b)$$

$$q = (\mu \Delta t - d)/(u - d). \quad (11c)$$

¹⁶This question was investigated by Hanemann (1989), who found that his quasi-option value increases in response to an increase in the uncertainty of future realizations.

¹⁷That is, $dP = \mu P dt + \sigma P dz$, where z is a standard Weiner process. Furthermore, daily or more frequent data can be used to turn σ into an annual measure of volatility.

In our case, because the model calls for annual decisions, we are interested in an annual specification of the process. Thus, $\Delta t = 1$ and σ is an annual standard deviation.¹⁸

This formulation, linking u and d to a single parameter that captures the level of variability of the process, will prove useful in the sequel. So long as u , d , and q are computed according to equations (11), an increase in σ with μ fixed is a mean-preserving spread in the process. It leaves the expected future price unchanged, as the reader can easily confirm by checking to see that $(qu + (1 - q)d) = \mu$, which means that $E_0(P_1) = \mu P_0$ is unchanged. Formally, a mean-preserving spread is defined as follows.

DEFINITION. *Given a price process PP , where u , d , and q are defined by equations (11), together with a pair (μ, σ) , a **mean-preserving spread** is an increase in σ .*

Note that since φ remains constant, a mean-preserving spread leaves Ω and W_0 unchanged. We are now prepared to show that P_0^* , the threshold price, increases in σ . Thus, the greater the uncertainty about the future profitability of a developed project, the greater the inclination to delay the investment.

PROPOSITION 2. *A mean-preserving spread in the price process PP causes an increase in P_0^* . That is, $\partial P_0^* / \partial \sigma > 0$.*

PROOF: Given the definition of d , we know that $p = (u + 1)^{-1}$. Thus, $dp/du = -1/(u + 1)^2 < 0$. Let $\delta = (R - p)(c\Theta + I) + (1 - p)\tilde{A} + RA > 0$ denote the numerator of the term in parentheses in equation (9). Then

$$\begin{aligned} \frac{\partial P^*}{\partial \sigma} &= \frac{\partial P^*}{\partial u} \frac{du}{d\sigma} \\ &= \frac{1}{\Omega} \left(\frac{\delta(p + u(dp/du)) - (R - pu)(c\Theta + I + \tilde{A})(dp/du)}{(R - pu)^2} \right) e^\sigma. \end{aligned}$$

The term multiplying δ , $p + u(dp/du)$, can be reduced to $1/(1 + u)^2 > 0$. Because $dp/du < 0$, then, the term in parentheses is positive. Thus, $\partial P^* / \partial \sigma > 0$, which was to be proved. ■

We now show that an increase in σ leads unambiguously to an increase in quasi-option value. Intuitively, this is due to the ability to avoid an ever less attractive outcome by waiting. As the price in the up state (and hence the payoff to the developed project) increases, and the price in the down state falls, the value of waiting to discover which state will occur increases.

¹⁸See Cox and Rubinstein (1985) for a discussion of this estimation procedure and of the close connection between the discrete- and continuous-time versions of the problem. In particular, the reader should note that as $\Delta t \rightarrow 0$, the discrete-time version approaches the continuous-time version.

PROPOSITION 3. *Quasi-option value O^* is nondecreasing in response to a mean-preserving spread in the price process. If $O^* > 0$, it is strictly increasing in response to a mean-preserving spread.*

PROOF: Consider first the case in which $O^* = 0$, so that $P_0 < P_0^*$. Obviously, by its definition, O^* cannot decrease. Now suppose that $O^* > 0$, which means that $P_0' < P_0 < P_0^*$. Because φ always equals 1, Ω is constant as σ changes and $W_0 = \max[\tilde{A}, \Omega P_0 - (c\Theta + I)]$ is invariant to changes in σ . Thus, O^* increases precisely when W_0^* increases. But in the relevant region W_0^* is given by equation (8). Thus, the response of O^* to an increase in σ is given by the following derivative.

$$\begin{aligned} \frac{\partial O^*}{\partial \sigma} &= \frac{\partial W_0^*}{\partial \sigma} = \frac{\partial W_0^*}{\partial u} \frac{du}{d\sigma} \\ &= \frac{1}{R} \left(P_0 \Omega \left(p + u \frac{dp}{du} \right) - \frac{dp}{du} (c\Theta + I + \tilde{A}) \right) \frac{du}{d\sigma}. \end{aligned}$$

But from the proof of Proposition 2, we know that $(p + u(dp/du)) > 0$ and that $dp/du < 0$. Thus, the derivative $\partial O^*/\partial \sigma$ must be strictly positive, and the proof is complete. ■

Figure 2 illustrates the effect of changes in σ on W_0 , W_0^* , W_0^{**} , and O^* . The four panels of the figure differ only in the value of P_0 in each. In 2a, where $P_0 = 150$, for σ smaller than about 0.75, $O^* = 0$. This is because at such a low price, if σ is small development should never occur. If σ is above 0.75, though, waiting becomes valuable as P_1 becomes large enough in the up state to drive W_1^+ above \tilde{A} . The decision maker should create the hedge. Once O^* rises above 0, it increases monotonically in σ . Figure 2b is similar, except there is no value of σ for which $O^* = 0$. Notice that O^* is also everywhere greater than it was in Figure 2a.

In Figure 2c, where $P_0 = 450$, we once again find that $O^* = 0$ for small σ . It is essential for an understanding of our argument to see why. In this case $O^* = 0$ means not that development should never occur but that it should occur immediately which, as is evident from the diagram, means that $W_0 > W_0^*$. The difference between $O^* = 0$ in 2a and in 2c could not be greater. In Figure 2c, as σ increases W_0^* eventually grows while W_0 remains fixed and we find that O^* becomes positive. Thus, an increase in the spread of future prices can cause our decision maker to delay a decision when development would have occurred immediately with the smaller σ .

This result linking σ to the value of the right to delay seems important in many real-world resource development problems. The volatility of perceived future returns is often very large. And the larger it is, the more valuable waiting becomes. Note also that we cannot say that the right to delay the decision increases as P_0 increases. It does so as P_0 rises from 150 to 300, but as it

increases further we see that the immediate value of the developed project rises more quickly than the value of the project if it is developed tomorrow. Thus, O^* falls as P_0 increases. (This outcome is also evident in Figure 1.)

Figure 3 contains a 3-dimensional diagram of O^* as it depends on both σ and P_0 . The figure nicely summarizes what we have learnt so far. Proposition 2 is apparent in the curve in the (σ, P_0) -plane along the “upper” edge of the O^* wedge jutting out of the plane: P_0^* is increasing in σ . Proposition 3 is apparent in the upward slope of O^* along any P_0 slice: O^* is increasing in σ .

Quite a lot has happened in this lengthy section. Let us recap briefly before moving on. First, we showed how to compute quasi-option value using replication arguments from finance. This involved creating a riskless hedge portfolio and using it to place a value upon the investment opportunity. Then we saw how the threshold price P_0^* changes as various parameters change. Finally, we defined a mean-preserving spread and we saw how both P_0^* and O^* change as σ changes.

In all of this, it was assumed that the underlying stochastic process was known perfectly. There was uncertainty, but no uncertainty about the nature and form of this uncertainty. And in this setting there is nothing more that one can know. The stochastic process is all. It yields a realization each period. One can observe these, but having observed more realizations tells one nothing new about the process. It is now time to relax the assumption of a perfectly known process, and introduce uncertainty on the part of our decision maker about the process itself.

5. Arbitrage-Free Quasi-Option Value: Unknown Stochastic Process

Suppose the price process is described by equations (11), and that our decision maker knows μ but does not know σ . What does this assumption mean for our replication argument, for W_0^{**} and O^* , and for the investment decision itself? The decision maker must make a decision, and this decision will be guided by the machinery of Section 4. The only thing to do, then, is to choose some value for σ and proceed based upon it. We say nothing about how σ is estimated. It may be obtained according to a Bayesian procedure based upon previous realizations (see Graham-Tomasi 1995). It may be an estimate purchased from an outside agency.

Is this scenario—in which μ is known but σ is not—far-fetched? We say it is not. Estimating the volatility of a stochastic process is sometimes exceedingly difficult. Volatility estimation is often sensitive to the length of the data series used to estimate it. Though it might be argued that if σ is not known, surely μ would not be known either, this may not be true. In any event, whether μ is actually known is in the end irrelevant to the problem: the replication method ignores it. The

hardest complaint for us to answer is this: if the process for P is generated by an asset that is traded in a visible market, how could there be information about the process that is not available in, say, the daily newspaper listings? Our answer lies in the recent behavior of copper prices. A scandal involving a market-maker caused a severe shock to the price series, which can only be interpreted in our context as a dramatic shift in σ . Thus, even the best estimate of σ based upon realizations up to t may turn out to be quite wrong tomorrow. This is the scenario we wish to posit. To what does it lead?

Following this reasoning, imagine that the true underlying volatility is denoted σ , and that our decision maker's belief about this parameter is denoted σ_e . If $\sigma \neq \sigma_e$, we wish to understand the effect upon the valuation problem and the investment decision. The section possesses two main thrusts. First, we show that the effect of a mistake in specifying the stochastic process can be calculated using the portfolio methods described above. Second and more important, we show that it is possible in this case that following the AFH program (invest immediately only if $W_0^* > W_0$) can lead to a *worse* outcome than following the simple ANPV rule (invest immediately only if $W_0 > \tilde{A}$).

If the decision maker follows the method of section 4 using an incorrect value for σ (that is, if $\sigma \neq \sigma_e$), the effect is felt in two ways. First, because σ_e is wrong our decision maker will choose an incorrect short position and the hedge will be imperfect. The *financial loss* due to being wrong about σ is the value today of the difference in the portfolio value that would be achieved if σ were known, and the portfolio value that is achieved by acting as if σ_e were correct. In both cases, we evaluate the portfolio using the values of P_1^\pm that obtain under the correct σ . Second, because σ_e is wrong there may be an error in the development decision itself. This mistake could go either way: the project may be developed immediately when it should be delayed, or it may be delayed when it should be developed immediately. We will call this the *real loss* due to being wrong about σ . The cost of being wrong is the sum of the two.

Recall that the hedge portfolio consists of the development opportunity (the land) and a short position in copper. We have seen that this portfolio's value at $t = 0$ equals the value of the portfolio at $t = 1$ less the cost of the short position, discounted 1 period: $\Pi_0 = (\Pi_1 - S)/R$. We wish to compare this to the value of the portfolio if an incorrect belief about σ leads to an incorrect portfolio. Formally, let $u_e = e^{\sigma_e}$ and $d_e = 1/u_e$ be the up and down increments as a function of σ_e . Let y_e be the short position in the hedge portfolio, calculated as in equation (6), but using σ_e .

This portfolio differs from the correct one only in the size of the short position in copper. Though y_e is chosen based upon σ_e and the erroneous (anticipated) future prices based upon σ_e , the value of the portfolio at time $t = 1$ will be determined by the true σ and the actual future prices. At $t = 1$, then, the portfolio's value in the up and down states will be given by equations (5) using y_e and the correct P_1^\pm . Denote these values $\Pi_{1,e}^\pm$. This portfolio is not riskless: $\Pi_{1,e}^+ \neq \Pi_{1,e}^-$. Thus, to obtain a current value of the portfolio we discount using R and assign “probabilities” p and $1 - p$: $\Pi_{1,e} = p\Pi_{1,e}^+ + (1 - p)\Pi_{1,e}^-$. Thus, using $S_e = (R - 1)y_e P_0$,

$$\Pi_{0,e} = \frac{\Pi_{1,e} - S_e}{R}.$$

If the portfolio is formed, the financial loss due to a mistaken σ_e is the difference

$$\begin{aligned} L_f(\sigma, \sigma_e) &= \Pi_0 - \Pi_{0,e} \\ &= \frac{p(y_e - y)P_1^+ + (1 - p)(y_e - y)P_1^- - (R - 1)(y - y_e)P_0}{R}. \end{aligned}$$

Note that if this quantity is positive, the mistake leads to a gain; if it is negative, the mistake leads to a loss. In some situations—if $P_0 < P'_0$ or if $P_0 > P_0^*$ —there will be no portfolio. In this case $L_f = 0$.

The financial loss is only part of the story. Potentially more important is the possibility that a mistake in estimating σ might lead to an incorrect development decision. For a given set of parameter values, we can define the threshold value σ^* at which O^* becomes positive:

$$\sigma^* = \inf\{\sigma : O^* > 0\}.$$

This threshold value plays a crucial role in determining the real loss due to a mistake in estimating σ . We observe without proof that so long as P_0 is finite, σ^* is finite, and for intuition we direct the reader to Figure 2. There are two possibilities: either $\text{ANPV} = \tilde{A}$ (as in Figure 2a) or $\text{ANPV} > \tilde{A}$ (as in Figures 2c and 2d). If $\text{ANPV} = \tilde{A}$, $\sigma \leq \sigma^*$ means that investment should never take place, while $\sigma > \sigma^*$ means that investment should be delayed and the portfolio should be formed. In the latter case, development will occur at $t = 1$ in the up state, and the hedge is designed to remove the riskiness of the investment opportunity. If $\text{ANPV} > \tilde{A}$, $\sigma \leq \sigma^*$ means that investment should take place immediately, while $\sigma > \sigma^*$ once again means that investment should be delayed and the portfolio should be formed.

Based upon σ^* , for each of the two situations just described ($\text{ANPV} = \tilde{A}$ or $\text{ANPV} > \tilde{A}$) we can divide (σ, σ_e) -space into 4 regions. The following table describes the possible relationships between the decision made using σ and the decision made using σ_e if $\text{ANPV} = \tilde{A}$, when an ANPV decision maker *would not* invest at $t = 0$.

		If $\text{ANPV} = \tilde{A}$:	
		$\sigma \leq \sigma^*$	$\sigma > \sigma^*$
$\sigma_e \leq \sigma^*$	Region I: Correctly walk away; form no portfolio	Region II: Incorrectly walk away; should form a portfolio	
$\sigma_e > \sigma^*$	Region III: Incorrectly form a portfolio; should walk away	Region IV: Correctly form a portfolio	

Note that here our person will never develop the project immediately. No matter what σ_e happens to be, this is the correct decision and thus there is no real loss. If $\text{ANPV} > \tilde{A}$, though, this is no longer true. The next table describes the relationships between the decision made using σ and the decision made using σ_e if $\text{ANPV} > \tilde{A}$, when an ANPV decision maker *would* invest at $t = 0$.

		If $\text{ANPV} > \tilde{A}$:	
		$\sigma \leq \sigma^*$	$\sigma > \sigma^*$
$\sigma_e \leq \sigma^*$	Region I: Correctly invest immediately; form no portfolio	Region II: Incorrectly invest immediately; should delay, form portfolio	
$\sigma_e > \sigma^*$	Region III: Incorrectly delay; should invest immediately	Region IV: Correctly delay, form portfolio	

Now it is possible that a mistaken σ_e can lead to an incorrect development decision. In Region II, development occurs immediately when it should be delayed. The real cost of this error is simply foregone quasi-option value, O^* . Because no portfolio is formed, there is no financial loss. In Region III, development is delayed when it should occur immediately. The real cost of this error is $W_0^* - W_0$, a negative number since in this region $W_0 > W_0^*$.

The real loss suffered due to a mistaken σ is 0 if the correct development decision (either delay or develop) is made. As a function of σ and σ_e , real loss can be expressed as follows:

$$L_r(\sigma, \sigma_e) = \begin{cases} 0 & \text{if } \text{ANPV} = \tilde{A}; \\ 0 & \text{if } \text{ANPV} > \tilde{A} \text{ and Region I or IV;} \\ -O^* & \text{if } \text{ANPV} > \tilde{A} \text{ and Region II;} \\ W_0^* - W_0 & \text{if } \text{ANPV} > \tilde{A} \text{ and Region III.} \end{cases}$$

Total loss due to a mistaken σ is the sum of L_f and L_r , which we may express:

$$L(\sigma, \sigma_e) = L_f + L_r = \begin{cases} L_f & \text{if ANPV} = \tilde{A}; \\ L_f & \text{if ANPV} > \tilde{A} \text{ and Region I or IV}; \\ L_f - O^* & \text{if ANPV} > \tilde{A} \text{ and Region II}; \\ L_f + W_0^* - W_0 & \text{if ANPV} > \tilde{A} \text{ and Region III}. \end{cases}$$

The 3-dimensional surface of $L(\sigma, \sigma_e)$ is depicted in Figure 4 for four different values of P_0 , the initial price.

From the figure, note that the financial loss, L_f , can be positive. In Figure 4a, for example, where P_0 is such that $\text{ANPV} = \tilde{A}$, $L_f > 0$ in what we have called Region II. The reason is simple enough, though it may be unexpected. One might expect that being wrong in an arbitrage-free setting is always costly. Not so. In Figure 4a, if $\sigma = 1.2$ and $\sigma_e = 0.5$, our decision maker has formed a portfolio which, by pure luck, yields a financial gain. Thinking that volatility is less than it actually is, this person takes too small a short position in copper: $y_e < y$. Because of this, in the down state at $t = 1$ the short sale loss is smaller than it would have been using y . The value of the investment opportunity in the up state, meanwhile, is greater than anticipated. The development decision is correctly delayed (as it always is when $\text{ANPV} = \tilde{A}$), so this financial gain is not offset by a real loss.

In Figure 4c, $L < 0$ in region II, while $L > 0$ in Region III. In both cases the wrong development decision is made. In Region II, σ_e is small enough that it leads incorrectly to immediate development. The realized loss here is O^* . This is the same mistake an ANPV decision maker would make by ignoring the value of the delay option. In Region III, because $\sigma < \sigma^*$, the correct decision would have been to develop the project immediately. Our decision maker delays, which yields a real loss of $W_0^* - W_0 < 0$. The financial position, however, yields a gain that more than offsets this loss. The end result is that $L > 0$, representing mistakes that do not cause an incorrect development decision but are profitable financially.

Before turning to our main and concluding point, it is worth noting a few instructive points. First, it is always true that $L = 0$ in Region I. It should be clear why this is true; reasoning it out is a useful exercise. Second, $L = 0$ along the diagonal represented by $\sigma = \sigma_e$. There is no loss associated with being correct about σ . Third, in all cases there is a portion of Region IV in which $L > 0$.

Finally, all that remains is to support the claim that a mistake about σ can lead to a situation in which the ANPV rule would yield the correct development decision, but the more sophisticated

AFH rule yields the incorrect development decision. The illustration of this claim is found in Figure 4d, in which $P_0 = 650$. Here, we have $\sigma^* = 0.85$. The result we seek is achieved if we have $\sigma = 0.5$ and $\sigma_e = .87$. Because $ANPV > \bar{A}$, the wrong development decision is made in Regions II and III. In Region III, development is delayed when it should not be *and the ANPV rule would have led correctly to immediate development*. In many cases this cost of the real mistake is offset by a gain in the financial side, but in a few cases we find both parts of the claim supported: the development decision is wrong and the value achieved is lower under this rule than it would have been under ANPV.

PROPOSITION 4. *There exist situations in which the AFH rule leads to delay when development should occur immediately. In some of these, the period-0 value of the returns to the investment opportunity—including the outcome of the financial position—is negative.*

PROOF: The first statement is true if $ANPV > \bar{A}$ and $\sigma < \sigma^* < \sigma_e$. In order to prove the second, we need only present one situation in which it is true. Let the parameters P_0 , R , n , I , A , and c take values 280, 1.1, 10, 1000, 75, and 50 respectively. Let $\sigma = 0.5$ and let $\sigma_e = 0.87$. In this situation we have (after rounding)

$$\begin{aligned} W_0 &= 3055.4 & W_0^* &= 2852.6 \\ L_f &= 148.9 & L_r &= -202.8 & L &= -53.9. \end{aligned}$$

From these numbers it is evident that the correct decision would be to develop the project immediately. The decision maker using the ANPV rule would correctly develop immediately. The decision maker using the AFH rule, however, would incorrectly delay development. This would lead to a real loss of $L_r = -202.8$. The financial position taken to remove the perceived riskiness of this project yields a positive return of $L_f(\sigma_e) = 148.9$. The sum of these, however, is negative. This completes the proof of the proposition. ■

Figures 4c and 4d suggest the following caution. If the ANPV calculation recommends immediate development, it appears that less harm is done by overestimating than by underestimating σ . Of course, which of the different possibilities is most common is an empirical matter. But while this section has shown that sometimes the dynamic AFH rule (invest if $W_0^* > W_0$) is faulty, we may also extract a final recommendation that is certainly in the same spirit as AFH: be conservative. The most costly mistakes tend to be in the direction of premature development.

6. Concluding Remarks

The option to delay an irreversible decision can be valuable. This is true in natural resources as in general investment problems. Ignoring this option, which one does when net present value rules are used, can be costly. How costly? We have shown how to calculate the cost, QOV, for a simple model using arbitrage-free methods. If the stochastic process is known, in our setting the AFH result holds true. One should not use a net present value rule. If the process is not known with certainty, however, things are not so clear. In some situations the net present value rule yields the correct decision (which is the decision a person who knows the process exactly would make) while the dynamic AFH rule yields the incorrect decision.

We also showed that QOV grows as the volatility of the process grows. For a development problem with a source of uncertainty whose volatility is low, then, the potential cost of early improper development is low. At the same time, as Figures 4c and 4d indicate, the worst mistake one can make is to underestimate volatility and invest too early.

Our simple, discrete model could be extended in many directions. The delay option could be extended to an arbitrary number of periods. The amenity benefits, A , could be made stochastic. A number of additional options could be introduced. These might include the option to shut down the project, to expand it, to speed up production (thereby shortening its life), or to reclaim the land after the project is over. If the delay option lives several period, the hedge becomes dynamic, and the short position in the portfolio must be adjusted each period.

These extensions are all feasible in a discrete model. They would, without doubt, be more easily accomplished in a continuous-time model, because the discrete version becomes unwieldy very quickly. We used a discrete model, as we have said, primarily because we feel it is a better vehicle for describing how methods from financial economics can be employed to value natural resources for which markets do not exist.

For purposes of creating realistic models of development problems, particularly if these models are to be employed empirically using discrete data, the subtlety and difficulty of continuous-time models should not be underestimated. One appeal of the Black-Sholes model is that it has an analytical solution. Seemingly innocuous complications can erase this advantage. Only very seldom can a real-world problem be squeezed into the pure Black-Sholes framework. The most exciting current work on resource development decision making, it seems to us, consists in stretching the replication methods to cover more and more realistic cases.

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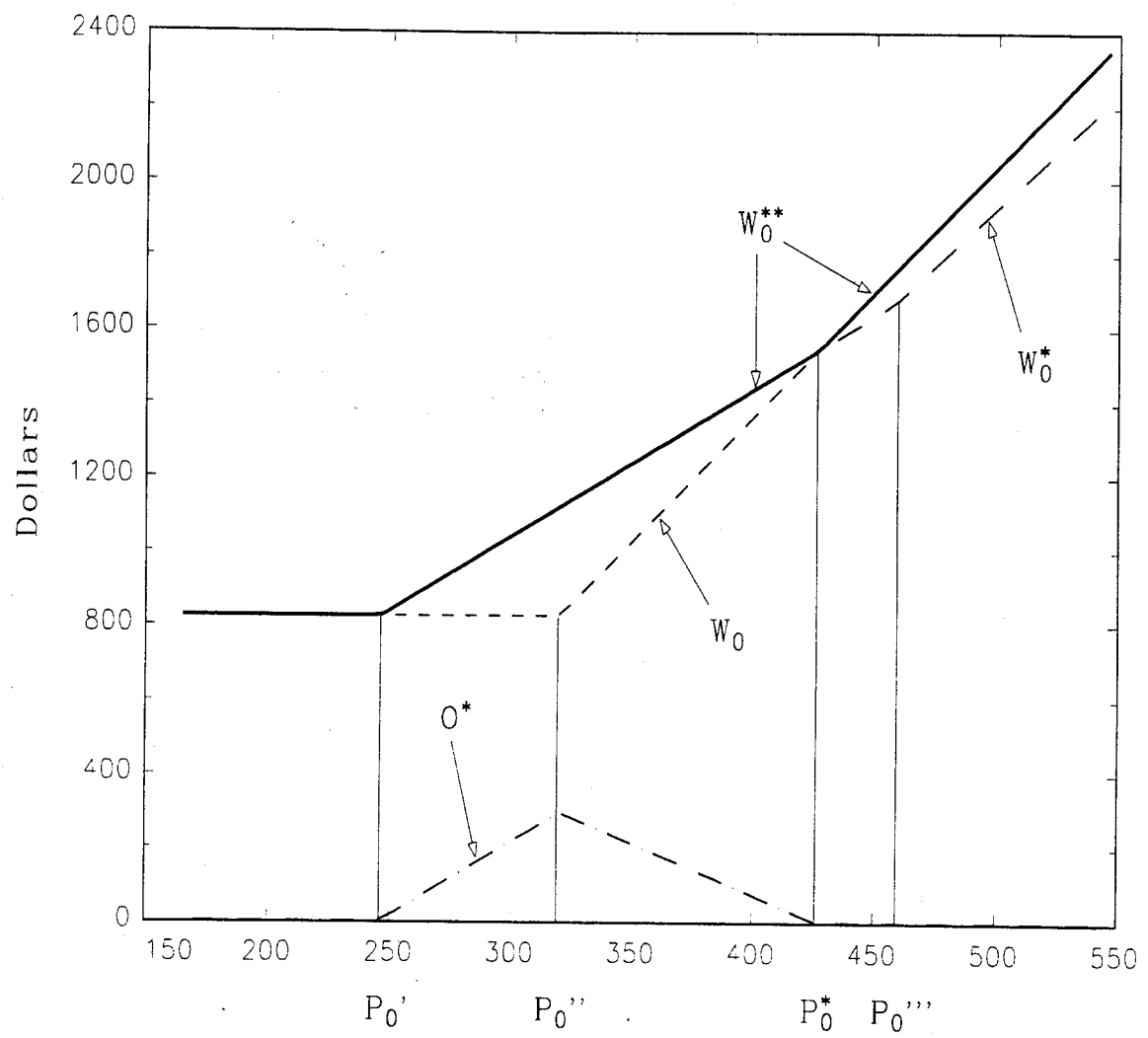


Figure 1. W_0 , W_0^* , W_0^{**} , and O^* as a function of P_0 .

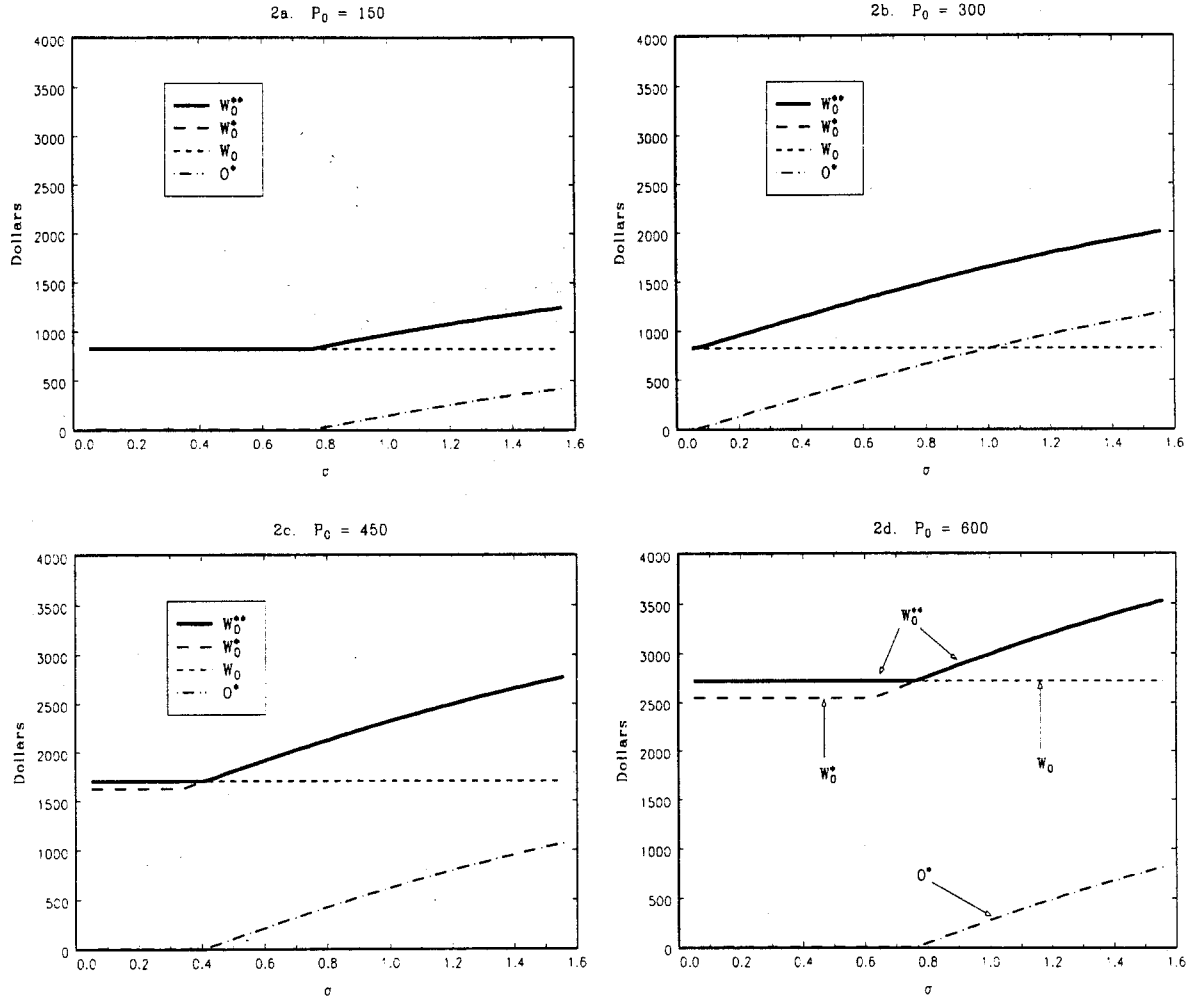


Figure 2. W_0 , W_0^* , W_0^{**} , and O^* as a function of σ .
 $P_0 = 150, 300, 450$, and 600

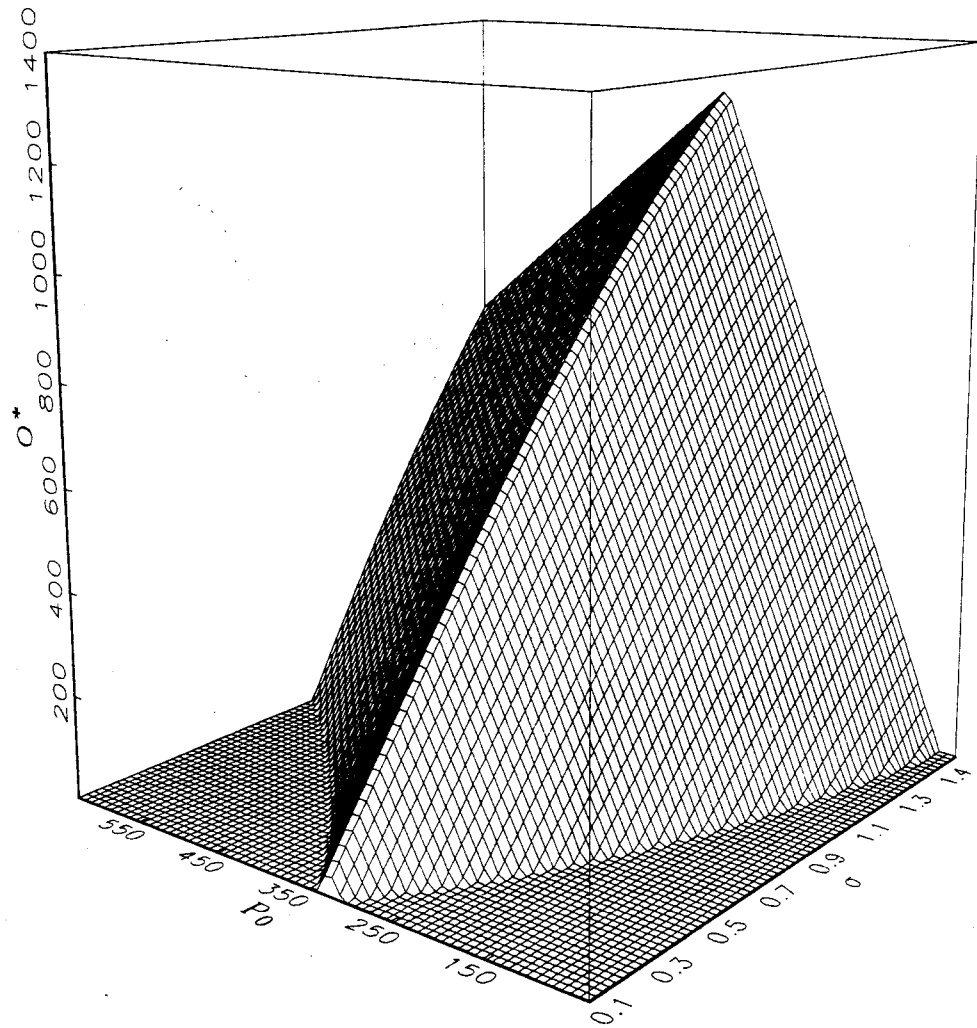


Figure 3. O^* as a function of σ and P_0 .

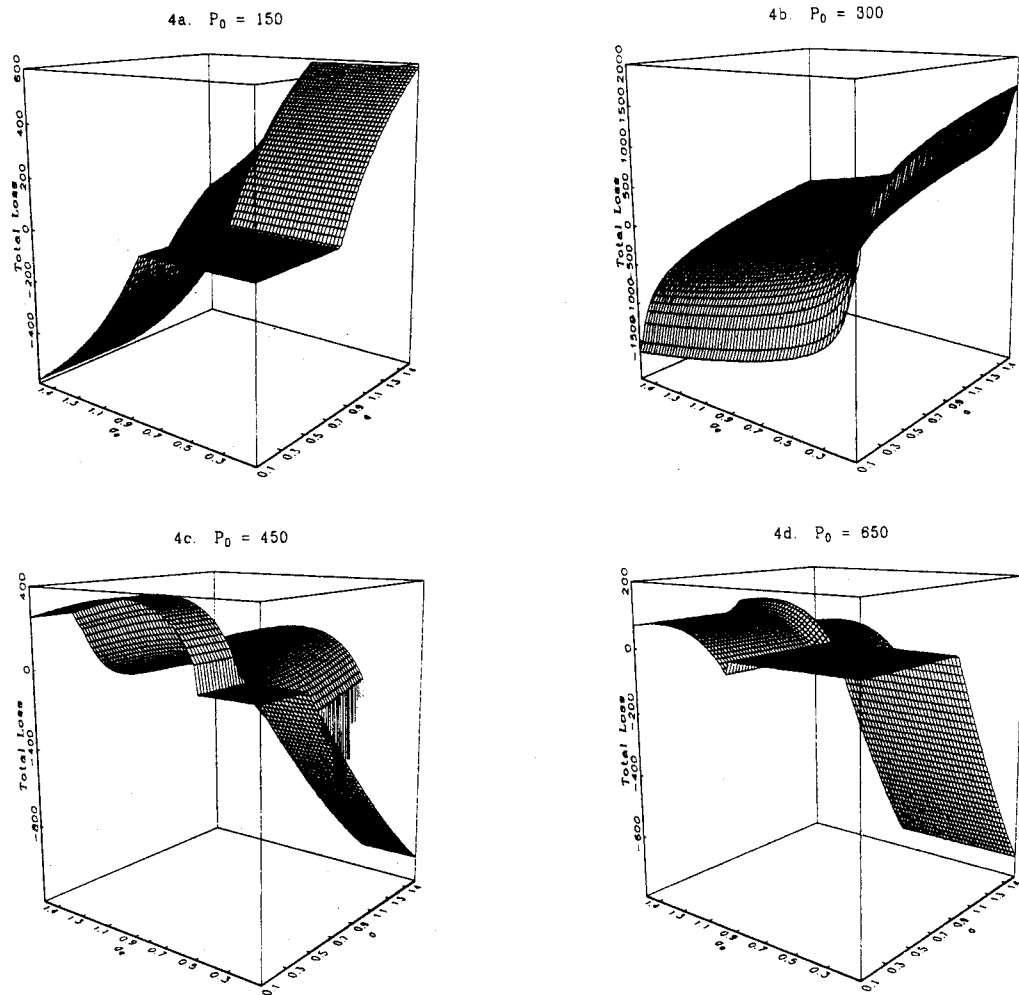


Figure 4. Loss due to mistaken σ_e , as a function of σ and σ_e .
 $P_0 = 150, 300, 450,$ and 650