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Short Cuts in Computing Ratio Projections of Population

By Helen R. White, Jacob S. Siegel, and Beatrice M. Rosen

The increasing interest of agricultural economists and statisticians in regional population trends and projections was noted in the introduction to an earlier article on projections of the regional distribution of the population. This was published in Agricultural Economics Research in April 1951 (Vol. III, No. 2, p. 41). Many researchers may find, however, that population projections for particular areas in which they are interested are not available, and that all but the roughest methods of projecting population involve somewhat more man-hours of work than can feasibly be expended. The following article presents two short cuts in the projection method described and used in the earlier article – short cuts that simplify the computations and considerably reduce the man-hours required.

THE METHOD generally known as the ratio method is now often used in developing projections of the population of geographic areas within the United States. An article by Hagood and Siegel presented projections of the population of the major geographic divisions to 1975 (4) and a Census Bureau report showed projections for States to 1960 (9). As the method was also recommended or publicized by the International City Managers' Association (5), the Bureau of Foreign and Domestic Commerce (8), and the National Resources Committee (6), it probably is used rather frequently by local estimators. For example, the Schenectady City Planning Commission (1), the California Taxpayers' Association (2), and the Philadelphia City Planning Commission (7) used the method to prepare population projections.

Although the ratio method is relatively simple and requires few data, a good deal of time and work is required to apply it if the procedures described by Hagood and Siegel and the report of the Census Bureau are followed, especially if projections for a number of areas and for several decades into the future are desired.

This paper presents two short cuts in applying the particular variation of the ratio method described by Hagood and Siegel and the Census Bureau report. The procedures suggested reduce considerably the time and work required in computing ratio projections of population; they also have other applications in demographic studies and in other fields.

Essentially, the method involves projecting the ratio of the total population of the area for which a projection is desired to the total popu-

lation of a larger area which contains the first area and for which an acceptable projection of the population is already available. The ratio, or proportion, is projected, in the Census Bureau report and the Hagood-Siegel article, on the basis of two assumptions: (1) That the rate of change in the ratio during the first 12 months of the projection period – that is, the period between the date of the census or of the estimate on which the projections are based, and the date for which a projection is desired – is the same as the average annual rate of change in the ratio for a selected period in the past; and (2) that this rate of change will decrease linearly to zero by some particular future date. If projections are being prepared for all of the subdivisions of the larger area, the projected ratios are adjusted to sum exactly to 1 or 100 percent. The projected ratio or ratios are then applied to the population projection for the larger area to obtain population projections for the smaller areas. For a more detailed discussion of this method, readers are referred to the articles cited.

Two factors keep this procedure from being unqualifiedly simple and brief. First, the derivation of the average annual rate of change in the ratio involves the use of logarithms. Such computations are time-consuming and require technically trained and skilled computers. Second, the procedure requires the computation of both the rate of change in the ratio and the ratio itself for each year of the projection period, even though projections may be desired for only a few particular years. If the populations of 50 areas are to be projected 25 years into the future, 2,500 computations must be made for these

two steps alone. The operations described below eliminate (1) the use of logarithms in computing the average annual rate of change, (2) the computation of the rate of change in the ratio for each year of the projection period, and (3) the computation of the ratio itself for intermediate years.

Approximation of the Average Annual Rate of Change

If we let

r = average annual rate of change in the ratio
 R_a = ratio at the start of the base period
 R_b = ratio at the end of the base period
 t = number of years in the base period,

then the exact value of the average annual rate of change is computed according to the formula:

$$(1 + r)^t = \frac{R_b}{R_a}$$

The use of this formula can be illustrated with figures for the West North Central division, as given by Hagood and Siegel in their article. The base period that they selected for computing the average annual rate of change in the ratio for this area is 1890-1950, and the values of R_a , R_b , and t are

$$\begin{aligned} R_a &= 14.19 \\ R_b &= 9.33 \\ t &= 60 \end{aligned}$$

Hence,

$$\begin{aligned} (1 + r)^{60} &= \frac{9.33}{14.19} = 0.6575 \\ 60 \log (1 + r) &= \log 0.6575 \\ \log (1 + r) &= \frac{9.8178958 - 10}{60} \\ &= \frac{599.8178958 - 600}{60} \\ &= \frac{9.9969649 - 10}{60} \\ r &= -0.00696 \text{ or } -0.696 \text{ percent} \end{aligned}$$

Using the same symbols as those given previously but designating $\frac{R_b}{R_a}$ as y for brevity, we can write the equation for the average annual rate of change

$$(1 + r)^t = \frac{R_b}{R_a}$$

in the form

$$r = \frac{1}{y^t} - 1$$

The right-hand side of this equation can be expanded in an infinite series as follows:

$$\begin{aligned} r &= \frac{1}{t}(y - 1) + \frac{\frac{1}{t}\left(\frac{1}{t} - 1\right)}{2!}(y - 1)^2 \\ &\quad + \frac{\frac{1}{t}\left(\frac{1}{t} - 1\right)\left(\frac{1}{t} - 2\right)}{3!}(y - 1)^3 + \dots \end{aligned}$$

The first term of this series, designated here as r_1 ,

$$r_1 = \frac{1}{t}(y - 1)$$

is a standard approximation for r suggested (with qualifications) in many mathematics texts. This formula is equivalent to the ratio of the annual average amount of change in the proportion to the proportion at the beginning of the base period. It has been used occasionally in demographic analysis as a substitute for the average annual rate of change in population during a period (3). The differences between r_1 and r for selected values of y and t are shown in figure 1. When t or y equals 1, r_1 is equal to r . In general, r_1 is a satisfactory approximation

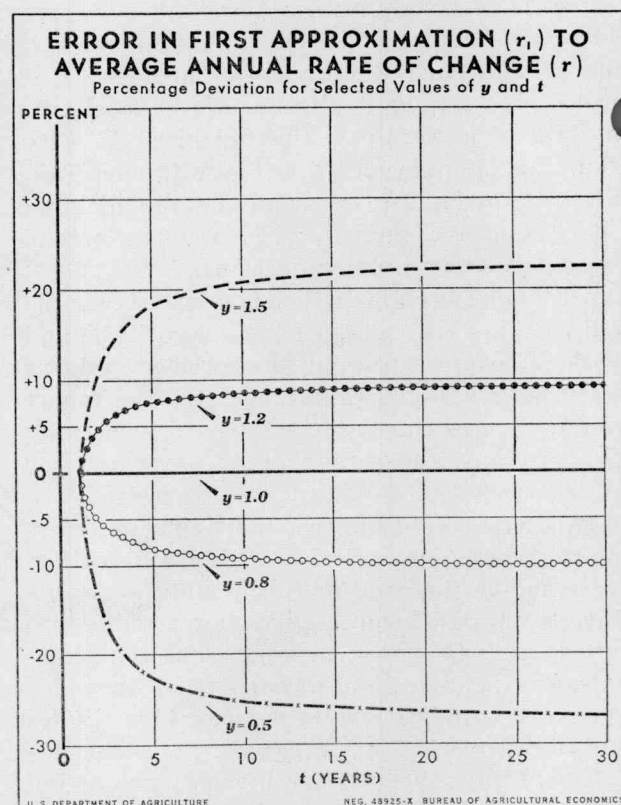


Figure 1

of r (differences of less than 5 percent) only when y falls between 0.9 and 1.1; the difference exceeds 10 percent outside the range $y = 0.8$ and $y = 1.2$ for $t = 5$ years or more. The difference increases with the length of the period and with the deviation of y from 1.0.¹

The use of terms beyond the first, in the series given above, to compute r would reduce the error but it would add considerably to the work, even though logarithms are not immediately or necessarily involved.

Certain explorations suggested that a good approximation of r , designated here as r_2 , could be obtained with comparatively little work from

$$r_2 = \frac{2 (R_b - R_a)}{t (R_b + R_a)}$$

This formula is derived by taking the ratio of the average annual amount of change in the proportion during the base period to the mean of the proportions at the beginning and end of the period:

$$r_2 = \frac{R_b - R_a}{t} \div \frac{R_b + R_a}{2} = \frac{2 (R_b - R_a)}{t (R_b + R_a)}$$

Substituting the figures for the West North Central division, we obtain:

$$r_2 = \frac{2 (9.33 - 14.19)}{60 (9.33 + 14.19)} = \frac{2 (-4.86)}{60 (23.52)} \\ = -0.00689 \text{ or } -0.689 \text{ percent}$$

The resulting value of r_2 differs only slightly from the value for r obtained above. In fact, computations over a selected range indicate that r_2 is generally a rather close approximation to r . The differences between r_2 and r , for selected values of y and t , are shown in figure 2. The differences are less than 5 percent when y falls between 0.5 and 1.4 and t falls between 5 and 30. In general, the relative error without regard to sign decreases as the length of the period increases (at least for the range of t tested) and increases as the deviation of y from 1.0 increases. In general also, r_2 is a much better approximation to r than is r_1 (at least for the range of values tested).² Only in the

¹ Percentage deviations were computed for values of y from 0.5 to 1.5 at intervals of 0.1 and values of t of 1, 2, 5, 10, 20, and 30.

² Test computations indicate that the general pattern of the percentage deviations of r_1 and r_2 from r at $t = 30$, for various values of y from 0.5 to 1.5, as shown in figures 1 and 2, prevails from $t = 30$ to at least $t = 80$.

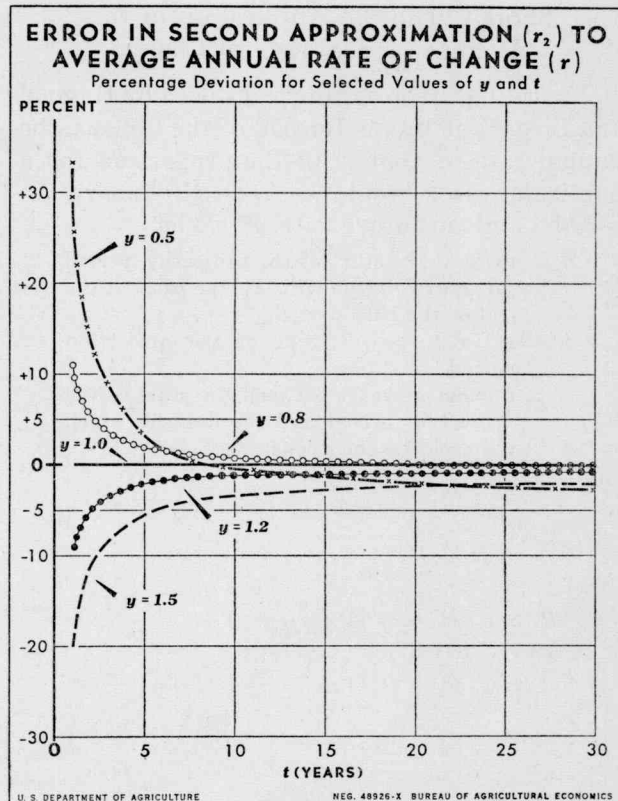


Figure 2

special case of $t = 1$ is r_1 a better approximation — a case in which it is simplest to compute r directly; in the special case of $y = 1$, both r_2 and r_1 equal r .

On the basis of the present analysis, r_2 is recommended as a generally satisfactory approximation for the annual average rate of change. It generally involves only a relatively small error, requires only simple operations, and takes little time to compute.³

³ Herman M. Southworth has suggested an additional formula:

$$r_3 = \frac{R_b - R_a}{t R_c}$$

where R_c is the ratio at the midpoint of the base period. Glenn L. Burrows has shown that, for r positive, r_3 would ordinarily be a better approximation to r than r_2 ; and for r negative, r_2 would be the better approximation to r .

It should be noted that R_c is not always known and that even in the range in which r_3 yields closer approximations than r_2 , r_2 may still be a satisfactory approximation (see curves for $y = 1.5$ and 1.2 in figure 2).

Short Cut in the Application of the Average Annual Rate of Change

According to the assumptions used by Hagood and Siegel and by the Bureau of the Census, the annual rate of change in the proportion for a particular area would be reduced linearly to zero by a given future date. If we let

R_0 = ratio at the start of the projection period
(R_0 may coincide with R_b , the ratio at the end of the base period)

R_i = ratio in the i^{th} year of the projection period

n = number of years between the start of the projection period and the date by which the ratio becomes constant,

then

$$R_1 = R_0 (1 + r)$$

$$R_2 = R_0 (1 + r) \left(1 + \frac{n-1}{n} r\right)$$

$$R_3 = R_0 (1 + r) \left(1 + \frac{n-1}{n} r\right) \left(1 + \frac{n-2}{n} r\right)$$

$$R_i = R_0 (1 + r) \left(1 + \frac{n-1}{n} r\right) \left(1 + \frac{n-2}{n} r\right)$$

$$\dots \left(1 + \frac{n-i+1}{n} r\right)$$

Usually the annual reduction in the rate of change is first computed and then added to, or subtracted from, the initial rate of change serially to get the successive factors in the formula. For the West North Central division, the ratios for 1951, 1952, and 1953, assuming $n = 25$, are obtained as follows:

$$\frac{r}{n} = \frac{-0.00696}{25} = -0.000278$$

$$R_1 = R_0 (1 + r) = 9.33 (1 - 0.00696) = 9.265$$

$$R_2 = R_1 \left(1 + r - \frac{r}{n}\right)$$

$$= 9.265 (1 - 0.00696 + 0.00028) = 9.203$$

$$R_3 = R_2 \left(1 + r - \frac{2r}{n}\right)$$

$$= 9.203 (1 - 0.00696 + 0.00056) = 9.144$$

This chain process is continued until the ratio(s) for the desired year(s) have been computed. The result obtained for 1975 is 8.520. (The final proportion shown in the Hagood-Siegel article - 8.33 - is somewhat different because the projected ratios for all the divisions in the United States were adjusted to sum to 100.00 percent.)

TABLE 1.—Multipliers (c_{ij}) for projecting a population ratio, assuming that the ratio will become constant in 25 years¹

Length of projection period in years (i)	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
1	1.00				
2	1.96	0.9600			
3	2.88	2.7632	0.88		
4	3.76	5.2976	3.31	0.8	
5	4.60	8.4560	7.76	3.6	(2)
6	5.40	12.1360	14.53	9.8	(2)
7	6.16	16.2400	23.75	20.8	10
8	6.88	20.6752	35.45	37.9	30
9	7.56	25.3536	49.50	62.0	50
10	8.20	30.1920	65.73	93.7	90
11	8.80	35.1120	83.85	133.1	150
12	9.36	40.0400	103.51	180.1	220
13	9.88	44.9072	124.33	233.9	320
14	10.36	49.6496	145.89	293.6	430
15	10.80	54.2080	167.73	357.8	560
16	11.20	58.5280	189.41	424.9	700
17	11.56	62.5600	210.48	493.1	850
18	11.88	66.2592	230.50	560.4	1010
19	12.16	69.5856	249.06	625.0	1170
20	12.40	72.5040	265.76	684.7	1320
21	12.60	74.9840	280.26	737.9	1460
22	12.76	77.0000	292.26	782.7	1570
23	12.88	78.5312	301.50	817.8	1670
24	12.96	79.5616	307.78	841.9	1730
25	13.00	80.0800	310.96	854.2	1770

¹ It is assumed that the annual rate of change in the ratio will change linearly. $R_i = R_0 (1 + c_{i1} r + c_{i2} r^2 + c_{i3} r^3 + c_{i4} r^4 + c_{i5} r^5)$.

² Less than 5.

TABLE 2.—Multipliers (c_{ij}) for projecting a population ratio, assuming that the ratio will become constant in 50 years¹

Length of projection period in years (i)	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
1	1.00				
2	1.98	0.980			
3	2.94	2.881	0.9		
4	3.88	5.644	3.6	1	
5	4.80	9.214	8.8	4	(2)
6	5.70	13.534	17.1	12	(2)
7	6.58	18.550	29.0	27	(2)
8	7.44	24.209	45.0	52	(2)
9	8.28	30.458	65.3	90	100
10	9.10	37.248	90.3	144	200
11	9.90	44.528	120.1	216	300
12	10.68	52.250	154.8	310	400
13	11.44	60.367	194.5	427	700
14	12.18	68.832	239.2	571	1000
15	12.90	77.602	288.8	743	1400
16	13.60	86.632	343.1	946	1900
17	14.28	95.880	402.0	1179	2600
18	14.94	105.305	465.3	1444	3300
19	15.58	114.866	532.7	1742	4300
20	16.20	124.526	603.9	2072	5300
21	16.80	134.246	678.6	2435	6600
22	17.38	143.990	756.5	2828	8000
23	17.94	153.723	837.1	3252	9600
24	18.48	163.410	920.1	3704	11300
25	19.00	173.020	1005.1	4182	13300

¹ It is assumed that the annual rate of change in the ratio will change linearly. $R_i = R_0 (1 + c_{i1} r + c_{i2} r^2 + c_{i3} r^3 + c_{i4} r^4 + c_{i5} r^5)$.

² Less than 50.

But such chain computations can be eliminated. It is possible to develop sets of multipliers by means of which the ratios can be computed directly for any desired year—that is, without computing the values of the rates of change or the ratios for the intermediate years. But the step-by-step procedure is still preferable when projections are needed for each year between the base date and some future date.

With the multipliers it is simply necessary (1) to compute the values of the powers of r up to the 4th or 5th power (r, r^2, r^3, r^4, r^5); (2) to take the cumulative product of the powers of r and the appropriate multipliers; and (3) to multiply one plus the result in (2) by the ratio at the beginning of the projection period. Multipliers for the assumption that the annual rate of change will be reduced linearly to zero within 25 years are shown in table 1, and for the assumption that the annual rate of change will be reduced linearly to zero in 50 years are shown in table 2. In these tables, a given row contains the multipliers for a particu-

lar length of projection period; within that row, the first column contains the multiplier for r , the second for r^2 , and so on.

For an illustration of the procedure for deriving the ratio for the West North Central division for 1975, on the assumption that the ratio will cease changing by that year, see table 3. The projected ratio obtained by the use of the multipliers in this example is (except for rounding) the same as that obtained by the exact and longer procedure.

The results obtained by the use of the multipliers shown in tables 1 and 2 are, however, approximations, in that the multipliers have been rounded and multipliers for powers of r higher than r^5 are neglected. On the other hand, the error involved is negligible. In fact, additional digits can be dropped from the multipliers, and the multipliers for r^4 and r^5 can be disregarded in certain cases, depending on the number of significant figures required in the results. (Note that the use of both r^4 and r^5 does not affect the final rounded result in the illustra-

TABLE 3.—Illustration of computation of projected ratio, West North Central division, 1975

Powers of r	Multipliers for 25-year period	Products
$r = -0.00696$ -----	13.00	- 0.090480
$r^2 = +0.000048442$ -----	80.08	+ 0.003879
$r^3 = -0.0000003372$ -----	310.96	- 0.000105
$r^4 = +0.0000000235$ -----	854.20	+ 0.000002
$r^5 = -0.00000000002$ -----	1770.00	- 0.000000
Total-----		- 0.086704

$$R_{25} = (R_0) (1 + \text{sum products})$$

$$R_{25} = (9.33) (1 - 0.086704)$$

$$R_{25} = (9.33) (0.913296) = 8.521$$

tion given above.)

The multipliers are derived as follows:

It may be recalled that

$$R_i = R_0 (1 + r) \left(1 + \frac{n-1}{n} r\right) \left(1 + \frac{n-2}{n} r\right) \dots \left(1 + \frac{n-i+1}{n} r\right)$$

When the indicated multiplications are carried out, all terms involving a specific power of r , represented by r^j (j takes on values from 1 to i), can be collected into a single term, $c_{ij}r^j$; that is, R_i can be represented by the power series

$$R_i = R_0 (1 + c_{i1}r + c_{i2}r^2 + c_{i3}r^3 + \dots + c_{ii}r^i)$$

It is convenient to refer to the coefficients, c_{ij} , for all r^j 's in the power series of all R_i 's, in matrix notation. These coefficients can then be denoted by a matrix, C_{ij} , where, as before, i corresponds to the number of years between the base date and the date for which a projection is desired, and j corresponds to the power to which r is raised. Table 1 is a portion of the matrix for $n = 25$ and table 2 is a portion of the matrix for $n = 50$.

The matrix is formed as follows:

(1) All the elements above the main diagonal are zero.

(2) $c_{11} = 1$

(3) $c_{i1} = \frac{nc_{(i-1)1} + (n-i+1)}{n}$

(4) For all other elements

$$c_{ij} = \frac{n^j c_{(i-1)j} + n^{j-1} (n-i+1) c_{(i-1)(j-1)}}{n^j}$$

The various coefficients may be evaluated suc-

cessively from one another by beginning with c_{11} , c_{21} , and c_{22} . For example, the element c_{82} , for $n = 25$, may be evaluated from equation (4), given above and the data in table 1 as follows:

$$c_{82} = \frac{25^2 c_{72} + 25 (25 - 8 + 1) c_{71}}{25^2}$$

$$c_{82} = \frac{625 (16.24) + 25 (18) (6.16)}{625}$$

$$c_{82} = 20.6752$$

Similar multipliers could be developed on the basis of other assumptions as to the date by which the annual rate of change will be reduced to zero. Multipliers could be developed also on the assumption that the rate of change will equal, at a particular future date, a specified proportion of its size at the base date; for example, that the rate of change will be cut in half by 1975. Similarly, particular values of r other than zero could be assigned for the future date. Multipliers could also be developed for other types of curves (beside a straight line) describing the future trend of the annual rate of change. Once the multipliers are worked out, they may be used again and again to derive the final values of the proportions for the dates desired without computing the intermediate values in chain fashion. Under the particular assumption that the annual rate of change will remain constant, it is possible, of course, to compute the final estimate of the proportion directly, without multipliers, according to the formula

$$R'_i = R_0 \left(t \sqrt{\frac{R_b}{R_a}} \right)^i$$

where

$$\sqrt{\frac{R_b}{R_a}} - 1 = r$$

the exact value of the average annual rate of change.

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Lewis Cecil Gray

Dr. L. C. Gray, aged 70, former assistant chief of the Bureau of Agricultural Economics, land economist, historian, public administrator, and distinguished leader in agricultural economic thought and action, died at his home near Raleigh, N. C., November 18.

Dr. Gray came to the Department of Agriculture in 1919 as the first chief of the Division of Land Economics in the old Office of Farm Management and Farm Economics. When BAE was established in 1922 his division was merged with the new Bureau. In the 1930's, still carrying the responsibilities of his division, he served successively as chief of the Land Policy Section of the Agricultural Adjustment Administration and assistant administrator of the Resettlement Administration in charge of the Land Utilization Division. He was appointed assistant chief of BAE in 1937 and retired for disability in 1941.

Before joining the staff of the Department of Agriculture, Dr. Gray taught agricultural economics at a number of universities, including the University of Wisconsin, where he studied with R. T. Ely, H. C. Taylor, and J. R. Commons, educators whose influence was instrumental in guiding and shaping his career. He was the author of *Introduction to Agricultural Economics*, pioneer text in its field, and *History of Agriculture in Southern United States to 1860*, a 2-volume work of lasting historical significance. He served as president of the American Farm Economic Association in 1928 and as a member of the United States delegation to the International Institute of Agriculture in 1922 and 1928. He was a member of many national and international study groups and committees, including the President's Great Plains Committee and the President's Committee on Farm Tenancy.