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ESTIMATION OF ACTUAL RESPONSE COEFFICIENTS IN THE HILDRETH-HOUCK RANDOM COEFFICIENT MODEL

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1. INTRODUCTION

In a 1968 article Hildreth and Houck [2] investigated the following model

$$\begin{aligned} y_t &= \sum_{k=1}^K z_{tk} b_{tk} \\ &= \sum_{k=1}^K z_{tk} (\beta_k + v_{tk}) \\ &= \sum_{k=1}^K z_{tk} \beta_k + u_t, \end{aligned} \tag{1.1}$$
$$t = 1, 2, \dots, T,$$

where

$$u_t = \sum_{k=1}^K z_{tk} v_{tk}, \quad t = 1, 2, \dots, T, \tag{1.2}$$

and

$$b_{tk} = \beta_k + v_{tk}, \text{ for all } t \text{ and } k. \tag{1.3}$$

The y_t are observed values of a random variable, the z_{tk} are known, non-random values of explanatory variables and v_{tk} are unobserved random variables. The mean response of y to a unit change in the k -th explanatory variable is given by β_k , whilst the actual response for the t -th observation is b_{tk} . It is also assumed that

$$E v_{tk} = 0, \text{ for all } t \text{ and } k, \tag{1.4}$$

and

$$E v_{tk} v_{sj} = \begin{cases} \delta_{kk} & \text{for } t = s \text{ and } k = j, \\ 0 & \text{for } t \neq s \text{ or } k \neq j. \end{cases} \quad (1.5)$$

If the δ_{kk} are known the minimum variance linear unbiased estimator (BLUE) of the β_k is readily attainable. However the "best" estimator (or predictor) of the actual response coefficients (the b_{tk}) is not as obvious.¹ This paper derives the BLUE for the b_{tk} when the δ_{kk} are known.

2. NOTATION

Equation (1.1) can be written as

$$y = Z\beta + u \quad (2.1)$$

where Z is a $T \times K$ matrix of rank K and y, β and u are vectors of orders T, K and T respectively. The vector β contains the mean response coefficients. The actual response coefficients are given by

$$b = L\beta + v \quad (2.2)$$

¹Throughout the paper I will refer to an "estimator" of b_{tk} . It may be more appropriate to think of "predicting" the b_{tk} since, although they have been realized, they are unobservable random variables.

where $L =$
 $TK \times K$

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

(2.3)

$b =$
 $TK \times 1$

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_K \end{bmatrix}$$

, where

$b_k =$
 $T \times 1$

$$\begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{Tk} \end{bmatrix}$$

(2.4)

and

$v =$
 $TK \times 1$

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_K \end{bmatrix}$$

, where $v_k =$
 $T \times 1$

$$\begin{bmatrix} v_{1k} \\ v_{2k} \\ \vdots \\ v_{Tk} \end{bmatrix}$$

(2.5)

Assumptions (1.4) and (1.5) can be written as

$$Ev = 0,$$

(2.6)

and

$$A = E_{TK \times TK} v v' = \begin{bmatrix} \delta_{11} & & & \\ & \delta_{22} & & \\ & & \ddots & \\ & & & \delta_{KK} \end{bmatrix} \quad (2.7)$$

If $Z_k = \text{diagonal } (z_{1k}, z_{2k}, \dots, z_{TK})$ and

$$X = [Z_1, Z_2, \dots, Z_K]_{TK \times TK} \quad (2.8)$$

then $u = Xv.$ (2.9)

Let

$$\textcircled{H} = E u u' = X A X' \quad (2.10)$$

The BLUE for β is

$$\hat{\beta} = (Z' \textcircled{H}^{-1} Z)^{-1} Z' \textcircled{H}^{-1} y. \quad (2.11)$$

3. BEST ESTIMATOR FOR b

The BLUE for b will be unbiased in the sense that the expectation of the estimation (prediction) error is zero, and minimum variance in the sense that the covariance matrix of the estimation error of any other linear unbiased estimator exceeds that of the BLUE by a non-negative definite matrix. That is, if \hat{b} is the BLUE for b , b^* is any other linear unbiased estimator for b and W is a non-negative definite matrix, then

$$E(b^*-b) = 0, \quad (3.1)$$

$$E(\hat{b}-b) = 0, \quad (3.2)$$

and

$$E(b^*-b)(b^*-b)' = E(\hat{b}-b)(\hat{b}-b)' + W. \quad (3.3)$$

In searching for an estimator for $b = L\beta + v$ it seems natural to estimate $L\beta$ with $\hat{L}\hat{\beta}$ given by (2.3) and (2.11) and to look for an estimator for v . Estimates of the elements of u can be found from

$$\hat{u} = y - Z\hat{\beta}, \quad (3.4)$$

and estimates of the elements of v obtained by solving the system of equations,

$$Xv = \hat{u}. \quad (3.5)$$

However no unique solution exists to (3.5) because it involves T equations in TK unknowns. In fact, there are K unknowns in each equation. Equation (3.5) can be written in scalar form as

$$z_{t1} \hat{v}_{t1} + z_{t2} \hat{v}_{t2} + \dots + z_{tK} \hat{v}_{tK} = \hat{u}_t, \text{ for all } t. \quad (3.6)$$

The problem is to find suitable values for $\hat{v}_{t1}, \hat{v}_{t2}, \dots, \hat{v}_{tK}$ from (3.6) or, in other words to allocate some proportion of \hat{u}_t to each of the \hat{v}_{tk} 's. An appealing way to do this is to allocate \hat{u}_t among the $z_{tk} \hat{v}_{tk}$ in the same proportion as the variances of the $z_{tk} v_{tk}$ contribute to the variance of u_t .

The variance of u_t is

$$Eu_t^2 = \sum_{k=1}^K z_{tk}^2 \delta_{kk}, \text{ for all } t, \quad (3.7)$$

while the variance of $z_{tk} v_{tk}$ is

$$E(z_{tk}^2 v_{tk}^2) = z_{tk}^2 \delta_{kk}, \text{ for all } t \text{ and } k. \quad (3.8)$$

Therefore, if \hat{u}_t is allocated between the $z_{tk} \hat{v}_{tk}$ in the same proportion as the variance of u_t is allocated between the variances of the $z_{tk} v_{tk}$, a solution to (3.6) is

$$z_{tk} \hat{v}_{tk} = \frac{z_{tk}^2 \delta_{kk}}{\sum_{k=1}^K z_{tk}^2 \delta_{kk}} \hat{u}_t \quad (3.9)$$

or,

$$\hat{v}_{tk} = \frac{z_{tk} \delta_{kk}}{\sum_{k=1}^K z_{tk}^2 \delta_{kk}} \hat{u}_t, \text{ for all } k \text{ and } t. \quad (3.10)$$

Noting that the denominator in (3.10) is the t -th diagonal element of \textcircled{H} , estimates of the disturbances associated with the k -th coefficient can be written in matrix form as

$$\hat{v}_k = \delta_{kk} z_k \textcircled{H}^{-1} \hat{u}. \quad (3.11)$$

When estimates of all the disturbances are included (3.11) becomes

$$\hat{v} = AX' \textcircled{H}^{-1} \hat{u}. \quad (3.12)$$

This is verified as a solution to (3.5) by substituting (3.12) into (3.5) and recalling that $\textcircled{H} = XAX'$.

We now have as an estimator for b ,

$$\begin{aligned}\hat{b} &= L\hat{\beta} + \hat{v} \\ &= L(Z' \oplus H^{-1} Z)^{-1} Z' \oplus H^{-1} y + AX' \oplus H^{-1} \hat{u}. \quad (3.13)\end{aligned}$$

It will be shown below that this estimator is BLUE.

Using $\hat{u} = [I - Z(Z' \oplus H^{-1} Z)^{-1} Z' \oplus H^{-1}]y$, [1,p.233], \hat{b} can be written as the following linear function of y .

$$\begin{aligned}\hat{b} &= [L(Z' \oplus H^{-1} Z)^{-1} Z' \oplus H^{-1} + AX' \oplus H^{-1} (I - Z(Z' \oplus H^{-1} Z)^{-1} Z' \oplus H^{-1})]y \\ &= Qy. \quad (3.14)\end{aligned}$$

Substituting $Z\beta + u$ for y we have

$$\begin{aligned}\hat{b} - b &= QZ\beta + Qu - L\beta - v \\ &= Qu - v \quad \text{because } QZ = L \\ &= (QX - I)v. \quad (3.15)\end{aligned}$$

Taking expectations,

$$E(\hat{b} - b) = (QX - I) E v = 0 \quad (3.16)$$

and

$$\begin{aligned}E(\hat{b} - b)(\hat{b} - b)' &= (QX - I) E v v' (QX - I)' \\ &= Q \oplus H Q' - QXA - AX'Q' + A. \quad (3.17)\end{aligned}$$

Theorem: The minimum variance linear unbiased estimator of b is given by (3.14) and the variance of the resulting estimation error by (3.17).

Proof: Unbiasedness of \hat{b} is shown in (3.16).

Let $b^* = C^*y$ be any other linear unbiased estimator for b and let $C = C^* - Q$. Then,

$$b^* = (C+Q)y. \quad (3.18)$$

Rewriting (3.18) and subtracting b gives

$$\begin{aligned} b^* - b &= (Q+C)Z\beta + (Q+C)u - L\beta - v \\ &= CZ\beta + Qu + Cu - v. \end{aligned} \quad (3.19)$$

Taking expectations,

$$E(b^*-b) = CZ\beta. \quad (3.20)$$

Since b^* is unbiased (by definition), it follows that $CZ = 0$, and that

$$\begin{aligned} b^*-b &= Qu + Cu - v \\ &= (QX + CX - I)v. \end{aligned} \quad (3.21)$$

The variance of the estimation error of b^* is

$$\begin{aligned} E(b^*-b)(b^*-b)' &= (QX+CX-I)Evv' (QX+CX-I)' \\ &= Q \textcircled{H} Q' + Q \textcircled{H} C' - QXA - AX'Q' \\ &\quad - AX'C' + A + C \textcircled{H} Q' + C \textcircled{H} C' - CXA. \end{aligned} \quad (3.22)$$

Now

$$\begin{aligned}
C(H)Q' &= C(H)(H)^{-1}Z(Z'(H)^{-1}Z)^{-1}L' + C(H)(H)^{-1}XA \\
&\quad - C(H)(H)^{-1}Z(Z'(H)^{-1}Z)^{-1}Z'(H)^{-1}XA \\
&= CXA, \text{ since } CZ = 0.
\end{aligned} \tag{3.23}$$

Therefore, (3.22) will reduce to

$$\begin{aligned}
E(b^*-b)(b^*-b)' &= Q(H)Q' - QXA - AX'Q' + A + C(H)C' \\
&= E(\hat{b}-b)(\hat{b}-b)' + C(H)C'.
\end{aligned} \tag{3.24}$$

Since $C(H)C'$ is non negative definite the theorem is proved.

4. CONCLUSIONS

In the model of (1.1), (1.2) and (1.3) one might suspect that the best estimator of the actual response coefficients is identical to the best estimator of the mean response coefficients. It is shown above that this is not the case.

In practice $\hat{\beta}$ and \hat{b} can seldom be calculated because (H) is unknown. Hildreth and Houck [2] outline several alternative estimators for (H) and note that corresponding to each estimator for (H) is an estimator for β obtained by substituting the estimated (H) for the real (H) in (2.11). In a similar way it is now possible to use any of the estimated (H) 's in (3.14) to estimate b , and we have a number of estimators for b depending on which estimator for (H) is employed. The small sample properties of estimators for β and b obtained in this way are still unknown and need to be investigated to determine whether or not estimation of the variances is worth the extra cost.

REFERENCES

- [1] Goldberger, A.S., Econometric Theory, New York: John Wiley and Sons, Inc., 1964.
- [2] Hildreth, C. and Houck, J.P., "Some Estimators for a Linear Model with Random Coefficients," Journal of the American Statistical Association, 64 (1968), 584-595.