STOCK UNCERTAINTY IN RENEWABLE RESOURCE THEORY:
THE EXPLOITATION OF AQUIFERS OF UNKNOWN SIZE

Yacov Tsur

DEPARTMENT OF AGRICULTURAL AND APPLIED ECONOMICS
UNIVERSITY OF MINNESOTA
COLLEGE OF AGRICULTURE
ST. PAUL, MINNESOTA 55108
STOCK UNCERTAINTY IN RENEWABLE RESOURCE THEORY: 
THE EXPLOITATION OF AQUIFERS OF UNKNOWN SIZE

Yacov Tsur
Stock Uncertainty in Renewable Resource Theory: The Exploitation of Aquifers of Unknown Size

Yacov Tsur

Department of Agricultural and Applied Economics, University of Minnesota

Abstract:

The theory of stock uncertainty in the utilization of exhaustible resources is extended to renewable groundwater resources. A complete characterization of the exploitation process is presented, paying special attention to the formulation of the transversality conditions. Exploration activities are incorporated and fit smoothly within the framework of analysis. Extensions to other renewable resource situations are outlined.

Corresponding address:

Yacov Tsur, Department of Agricultural and Applied Economics, University of Minnesota, 1994 Buford Avenue, St. Paul, MN 55108.

*The helpful comments of Ted Graham-Tomasi are gratefully acknowledged. The remaining errors are my own.
1 Stock Uncertainty in Renewable Resource Theory: The Exploitation of Aquifers of Unknown Size

1. Introduction

This paper studies the allocation and exploration of groundwater resources under conditions of stock uncertainty. Aquifers, like most exhaustible resource deposits (oil, minerals), lie below the surface, which means that the measurement of their stock is costly. When the aquifer is shallow, such costs are small and the assumption that the aquifer's stock is known, or can easily be calibrated, is reasonable. The situation is different for non-shallow aquifers, where the cost of obtaining stock information can be significant. In such cases, the stock uncertainty, as well as activities aimed at reducing this uncertainty—exploration activities—must be incorporated within the allocation problem.

The problem of extracting (eating) an exhaustible natural resource deposit (cake) of unknown size was studied first by Kemp (1976, 1977) and further investigated by Loury (1978) and Gilbert (1979). Exploration activities were studied by Pindyck (1978), Deshmukh and Pliska (1980, 1985), Arrow and Chang (1982), and more recently by Quyen (1991). This exhaustible resource literature excludes the possibility of recharge, and typically assumes that the unit extraction cost is constant over time. The present work incorporates recharge processes within the theory of extraction and exploration of unknown stock resources. In addition, extraction costs are allowed to change with cumulated net withdrawal—cumulated extractions minus cumulated recharge—a feature common also in exhaustible resource situations.

The general setup is laid out in the next section. The resource stock consists of a known part—the proven reserves—and an uncertain portion characterized by some distribution function. The optimal extraction of the
uncertain portion, assuming that the proven reserves have been depleted, is analyzed first (Section 3). The optimal extraction of the proven stock, assuming that the uncertain stock will be exploited optimally, is then characterized (Section 4). Together, the two sections completely characterize the extraction process without exploration. Special attention is given to formulating transversality conditions and characterizing steady states. Section 5 defines the value of stock information and incorporates exploration activities. The specification of the exploration activities follows that of Quyen (1991); this is a realistic description which fits smoothly within the analysis of Sections 3–4. Section 6 concludes and discusses possible extensions to other renewable resource situations.

2. Problem Formulation

Water is used as an input in a production process represented by the production function \( F(w, x) \), with \( w \) and \( x \) denoting respectively the water input and a vector of other inputs. Given the output price, \( p \), and the price vector of \( x \), \( r \), the function

\[
Y(w) = \max_{x} \left( pF(w, x) - r \cdot x \right)
\]

represents the maximum attainable profit, apart from the cost of water, when the water input is at the level \( w \). We denote \( Y(w) \) the water revenue function and require

**Assumption 1:** \( Y(w) \) is increasing and strictly concave over the relevant range of water input \( w \).

Thus, an additional unit of water input contributes positively to the revenue (increasing) but at a diminishing rate (concave).

Water can be supplied from surface sources (rainfall, reservoirs, stream
flows) and from a ground source—an aquifer. Surface water is assumed to be supplied at no cost and at a fixed rate $S$. The costless assumption can easily be relaxed and involves no loss of generality. The assumption that $S$ is fixed simplifies the analysis and allows us to focus attention on aspects regarding the uncertainty of the aquifer's stock (the case of random surface water supplies with a known aquifer size is treated in Tsur and Graham-Tomasi [1991]).

The aquifer's stock, denoted by $X$, consists of the proven reserves $X_p$ and an uncertain portion $X_u$. Thus, $X = X_p + X_u$, where $X_p \geq 0$ is known with certainty and $X_u$ is a random variable with the distribution function $F_u$ defined over $[0, \bar{X}]$, where $\bar{X} + X_p$ is the (known) maximum conceivable capacity of the aquifer. An aquifer's stock thus is characterized by $(X_p, F_u)$; the notation $(\xi, F)$-aquifer stands for an aquifer with proven reserves $\xi$ and uncertain stock distribution $F$.

Let $G_t$ denote cumulated net water withdrawal as of time $t$, i.e.,

$$G_t = \int_{0}^{t} (g_s - R(G_s)) ds,$$

where $g_t$ is the rate of water extraction and $R(G_t)$ is the rate of recharge at time $t$. Let $h_t = h(G_t)$ measure the distance from the aquifer's table-head to the surface. It is assumed that the function $h(\cdot)$ is known with certainty and observations on $h_t$ are made with no errors. Thus, at each time the state $G_t$ is known with certainty, implying that the recharge $R(G_t)$ is also known. We require

Assumption 2: $R(G)$ is non-decreasing and concave for all $G > 0$.

For non-renewable aquifers, $R(G) = 0$ for all $G \geq 0$; such is the case, for instance, with many of the deep fossil aquifers underlying our globe (see Margat and Saad [1985], Issar [1985] and Tsur et al. [1989]). For renewable aquifers, typically the rate of recharge either remains constant or increases at a diminishing rate as the aquifer's stock shrinks; hence the non-decreasing and
concavity requirements. By letting these properties hold for $G > 0$, rather than $G = 0$, we allow for a discontinuity of $R(G)$ at $G = 0$: the recharge of a full aquifer (when $G = 0$) must be zero, but for $G > 0$, no matter how small $G$ is, the rate of recharge may exceed a certain positive level, denoted $R(0)$.

The cost of extracting groundwater at a rate $g$ is given by $z(G)g$, where $z(G)$ is the unit extraction cost when cumulated net withdrawal equals $G$. It is reasonable to assume the unit extraction cost to be proportional to the distance the pumped water must travel to reach the surface: $z(G_t) = \zeta h(G_t)$, where $\zeta$ is the cost of lifting a water unit (cubic meter, say) one distance unit (meter, say).

We assume $h(\cdot)$ is non-decreasing and convex. The non-decreasing part is obvious. The convexity requirement reflects the common observation that the groundwater table elevation gets more sensitive to changes in stock as the stock diminishes. Thus,

**Assumption 3:** $z(G)$ is non-decreasing and convex for all $G \geq 0$.

An extraction plan consists of the extraction process $g_t$ and the associated withdrawal process $G_t$, $t \geq 0$. A plan is feasible if $g_t \geq 0$, $g_t \leq R(G_t)$ whenever $G_t = X$ (i.e., when the aquifer is empty), and $G_t \leq X$ for all $t \geq 0$. An extraction plan generates a stream of profits $Y(S+g_t) - z(G_t)g_t$, $t \geq 0$, for which the associated benefit is the present value of the profit stream

$$\int_0^{\infty} [Y(S+g_t) - z(G_t)g_t]e^{-\rho t}dt,$$

where $\rho$ is the time rate of discount. We seek the feasible plan that maximizes the expected benefit.

Let $\vartheta(\xi,F)$ be the maximum expected benefit of an $(\xi,F)$-aquifer, and let $\tau$ denote the time at which the proven stock is depleted, i.e., $G_\tau = X_p$. The value of the remaining uncertain stock at time $\tau$ is given by:
\[
\phi(0, F_u) = \max_{g_t \geq 0} \mathbb{E} \left[ \int [Y(S + g_t) - z(X_p + G_t)] g_t e^{-\rho t} dt \mid G_0 = G - X_p = 0 \right] 
\]
subject to \( g_t \leq R(X_p + G_t) \) whenever \( G_t = X_u \) (i.e., when the aquifer is empty), and
\[
G_t = \int [g_s - R(X_p + G_s)] ds \leq X_u \text{ for all } t \geq 0. \quad \text{In (2.1), } g_t^u = g_{T+t} \text{ is the extraction decision at } T+t, G_t^u = G_{T+t} - X_p \text{ is the accumulated net water withdrawal from the uncertain stock } X_u, \text{ and the expectation is taken with respect to the distribution of } G_t^u \text{ induced by the distribution } F_u \text{ of } X_u \text{ via the condition } G_t^u \leq X_u.
\]

The allocation problem over the entire planning horizon can be stated as
\[
\phi(X_p, F_u) = \max_{g_t \geq 0, \tau} \left\{ \int [Y(S + g_t) - z(G_t)] g_t e^{-\rho t} dt + \phi(0, F_u) e^{-\rho t} \mid g_t \geq 0, \tau \right\} 
\]
subject to \( G_t = \int [g_s - R(G_s)] ds \leq X_p, \) \( t \in [0, \tau]. \)

3. Optimal Extraction of the unknown stock

This stage is entered at time \( \tau \), just as the proven stock is depleted, and the management problem concerns the allocation of the remaining uncertain stock \( X_u \). The variable \( t \) represents time distance from \( \tau \). Let \( T \) denote the (random) time of occurrence of the depletion event \( G_t^u = X_u \). For a given extraction plan, the distribution \( F_u \) of the uncertain stock \( X_u \) induces a distribution on the depletion date \( T \) according to \( 1 - F_T(t) = \Pr(Te) = \Pr(X_u \geq G_t^u) = 1 - F_u(G_t^u) \text{ and } f_T(t) = F_T'(t) = F_u(G_t^u) dG_t/dt = f_u(G_t^u) [g_t^u - R(X_p + G_t^u)]. \)

At time \( T \), just as the bottom of the aquifer is reached, the uncertainty regarding the aquifer's stock is resolved: \( X_u = G_T^u \) and \( X = X_p + G_T^u \). The allocation problem over \([T, \infty)\), thus, entails no uncertainty and is relatively simple: since it was optimal to extract the last unit of the stock in addition to
the recharge (for otherwise depletion could not have occurred), it must be
optimal to extract the recharge rate from time \( T \) on. To verify this claim, note
that if it were optimal to extract in excess of recharge just before depletion,
then \( Y'(S+R(X)) \) must exceed \( z(X) \). This is so because \( Y'() \) is strictly
decreasing (Assumption 1), \( z() \) and \( R() \) are nondecreasing (Assumptions 2-3), and
the relation \( Y'(S+g_t) \geq z(g_t) \) must hold along the optimal extraction path, as the
following argument verifies.

Let \( z_t = z(G_t) \) and \( G_t \) be the cost and state processes associated with a
feasible extraction plan \( g_t \) and define \( g_t \) as

\[
\tilde{g}_t = \begin{cases} 
  g_t & \text{if } Y'(S+g_t) \geq z_t \text{ or } t \geq \tau \\
  Y'(S+g_t) - S & \text{if } Y'(S+g_t) < z_t \text{ and } Y'(S) \geq z_t, t \leq \tau \\
  0 & \text{if } Y'(S) < z_t, t \leq \tau 
\end{cases}
\]

That is, whenever \( g_t \) incurs (instantaneous) losses at the margin, it is reduced
to the level that equates marginal revenue to the unit extraction cost. Let \( \tilde{G}_t \)
and \( \tilde{z}_t = z(\tilde{G}_t) \) be the state and cost processes associated with \( \tilde{g}_t \). Clearly, \( \tilde{g}_t \leq g_t \) for all \( t \). Thus, \( \tilde{g}_t \) is feasible, \( \tilde{G}_t \leq G^u_t \) and (by Assumption 3) \( \tilde{z}_t \leq z_t \) for all \( t \). Now, by construction, \( Y(S+g_t) - z_t g_t \leq Y(S+\tilde{g}_t) - z_t \tilde{g}_t \) for all \( t \). Hence

\[
\int_0^\infty (Y(S+g_t) - z_t g_t) e^{-\rho t} dt \leq \int_0^\infty (Y(S+\tilde{g}_t) - z_t \tilde{g}_t) e^{-\rho t} dt, \text{ where equality holds only if the}
\]

two extraction plans coincide. It follows that any extraction plan that permits
\( Y'(S+g_t) < z_t \) can be improved upon and hence cannot be optimal.

Thus, if \( Y'(S+g^u) \geq z(X_p+G^u) \) just before depletion and \( g^u > R(X_p+G^u) = R(X) \),
then \( Y'(S+R(X)) > z(X) \) and it is desirable to extract the recharge \( R(X) \) from time
\( T \) on. Consequently, the value function over \( [T, \omega) \) is specified as
\[ W_T(G^u) = \int_0^\infty \left[ (Y(S+R(X_p+G^u)) - z(X_p+G^u)R(X_p+G^u)) e^{-\rho t} dt \right. \\
\left. = \frac{1}{\rho} \left( Y(S+R(X_p+G^u)) - z(X_p+G^u)R(X_p+G^u) \right) \right] e^{-\rho t} dt 
\]
evaluated at \( G^u = G^u_T = X_u \). The maximand on the right-hand side of (2.1) can thus be written:

\[ T \mathbb{E}\left( \int_0^\infty \left[ (Y(S+g^u_t)-z(X_p+G_t^u)g^u_t) e^{-\rho t} dt + W_T(G^u_T) e^{-\rho T} \right] \mid G^u_0 = G^u < X_u \right), \]

where the expectation is taken with respect to the distribution of \( T \).

We now derive the optimal extraction decisions over \([0,T)\), which in terms of the original time units is the period \([\tau,\tau+T)\), after the depletion of the proven stock and prior to the depletion of the entire stock. With some modifications due to recharge, the derivation follows that of Deshmukh and Pliska (1985, p. 326).

Let \( W(G^u) \) denote the maximum expected benefit starting in state \( G^u \in [0,X_u) \), when accumulated net withdrawal equals \( X_p+G^u \) and withdrawal from the uncertain stock is \( G^u \) (so that \( W(0) = \phi(0,F_u) \));

\[ W(G^u) = \max_{\{g^u_t \geq 0\}} \mathbb{E}\left( \int_0^T \left[ (Y(S+g^u_t)-z(X_p+G^u_t+G^u_t)g^u_t) e^{-\rho t} dt + W_T(G^u_T) e^{-\rho T} \mid G^u_0 = G^u < X_u \right] \right) \]

subject to \( G^u_t = G^u + [g^u_t - R(X_p+G^u+G^u)] ds \leq X_u, \; t \in [0,T) \). Extracting the level \( g^u \) over a short time period \([0,h]\) yields the immediate benefit \( Y(S+g^u_t) - z(X_p+G^u_t)g^u_t)h + o(h) \) and changes the state to \( G^u_h = G^u + [g^u_h - R(X_p+G^u)] h + o(h) \), where \( o(h) \) is such that \( o(h)/h \rightarrow 0 \) as \( h \rightarrow 0 \). The benefit from \( h \) onward depends on whether depletion occurs during \([0,h]\): if depletion does not occur, this benefit, discounted back to the beginning of the period, equals \( W(G^u_h) e^{-\rho h} \); if depletion occurs, the benefit equals \( W_T(G^u_h) e^{-\rho h} \). The expected benefit from \( h \)
onward, thus, equals the sum of these benefits weighted by the probability of their corresponding events.

We calculate these probabilities under the assumption that optimal extraction does not fall short of recharge, so that \( g^u = R(X_p + G^u) \) and \( G_n^u = G^u \).

If this is not so, then it is shown below that the state \( G^u \) should not be reached under the optimal plan, i.e., the system approaches a steady state \( \hat{G}^u \), in which it is optimal to extract exactly the rate of recharge, before it reaches \( G^u \) (i.e., \( \hat{G}^u < G^u \)).

Now, for \( G_n^u = G^u \), the probability that depletion will not occur during \([0,h]\), given that it has not yet occurred, equals \( \Pr(T > h | T > 0) = \)

\[
\Pr(X > G_n^u | X > G^u) = 1 - \frac{f(G_u)}{1 - F(G_u)} dG^u + o(dG^u), \text{ where } dG^u = G^u - G_n^u = [g^u - R(X_p + G^u)]h + o(h).
\]

Letting \( \lambda(G_u) = \frac{f(G_u)}{1 - F(G_u)} \) and noting that \( o(dG^u) = o(h) \), we obtain

\[
\Pr(T > h | T > 0) = 1 - \lambda(G_u)[g^u - R(X_p + G^u)]h + o(h). \text{ The probability of the complement event—that depletion occurs during } (0,h)—is thus } \Pr(T \in (0,h) | T > 0) = 1 - \Pr(T > h | T > 0) = \lambda(G_u)[g^u - R(X_p + G^u)]h + o(h).
\]

The value function \( W(G^u) \), prior to depletion, can now be specified as

\[
W(G^u) = \max \left\{ \left[ Y(S + g^u) - z(X_p + G^u)g^u \right] h + e^{-\rho h} W(T(G_n^u)\lambda(G_u)[g^u - R(X_p + G^u)]h +
\right.
\]

\[
\left. + e^{-\rho h} W(G_n^u)[1 - \lambda(G_u)[g^u - R(X_p + G^u)]h] + o(h) \right\}.
\]

Using the Taylor expansions \( e^{-\rho h} = 1 - \rho h + o(h) \), \( W(G_n^u) = W(G^u) + W'(G^u)[g^u - R(G_n^u)]h + o(h) \), and \( W(T(G_n^u) = W(T(G^u) + W'(G^u)[g^u - R(G^u)]h + o(h) \), the above relation can be written as

\[
0 = \max \left\{ \left[ Y(S + g) - z(X_p + G^u)g^u \right] h + W(T(G^u)\lambda(G_u)[g^u - R(X_p + G^u)]h +
\right.
\]

\[
\left. + W'(G^u)[g^u - R(X_p + G^u)]h - \rho W(G^u)h - W(G)\lambda(G_u)[g^u - R(X_p + G^u)]h + o(h) \right\}.
\]

Upon dividing by \( h \) and letting \( h \) approach zero (from above), we obtain the Bellman equation:
\[ pW(G^u) = \max_{g^u \geq 0} \left\{ Y(S + g^u) - z(X_p + G^u)g^u - \left[ g^u - R(X_p + G^u) \right] \left[ K(G^u) - W'(G^u) \right] \right\}, \quad (3.3) \]

where
\[ K(G^u) = \lambda(G^u) [W(G^u) - W_T(G^u)]. \quad (3.4) \]

In the nonrenewable case, when \( R(G) = 0 \) for all \( G \), \( W_T = 0 \), \( K(G^u) = \lambda(G^u)W(G^u) \), and Equation (3.3) reduces to Equation (16) of Deshmukh and Pliska (1985, p. 334).

Undertaking the maximization on the right-hand side of (3.3) yields the optimal extraction level, given \( G^u \in [0, X_u) \), as the value \( g^u(G^u) \) satisfying:
\[
\begin{cases} 
Y'(S + g^u(G^u)) = z(X_p + G^u) - W'(G^u) + K(G^u) & \text{if } Y'(S) \geq z(X_p + G^u) - W'(G^u) + K(G^u) \\
g^u(G^u) = 0 & \text{otherwise}
\end{cases}
\quad (3.5)
\]

Substituting \( g^u(G^u) \) for \( g^u \) in (3.3) gives the differential equation
\[
pW(G^u) = Y(S + g^u(G^u)) - z(X_p + G^u)g^u(G^u) - \left[ g^u(G^u) - R(X_p + G^u) \right] \left[ K(G^u) - W'(G^u) \right], \quad (3.6)
\]
from which \( W(G^u) \) can be derived (in most cases only numerically) given the boundary (transversality) conditions specified below.\(^3\)

Observing (3.5), one identifies three components that comprise the cost of extracting an additional unit of groundwater. The first is the engineering cost of pumping and distributing water, \( z(X_p + G^u) \). The second is the \textit{in situ} (unextracted) resource price, \(-W'(G^u)\), representing the opportunity cost of current extraction in terms of foregone future benefits. The third part, \( K(G^u) \), is an additional cost due to the fact that extraction changes the probability of the depletion event; it consists of the loss associated with depletion, \( W(G^u) - W_T(G^u) \), times the probability that depletion will occur in the next instant given that it has not yet occurred, \( \lambda(G^u) \). The term \( K(G^u) \) represents an extra cost due to the stock uncertainty. Note that \( K(G^u) \geq 0 \), since \( \lambda(G^u) \geq 0 \) and, observing (3.2), \( W(G^u) \geq W_T(G^u) \). The full cost of a unit of extracted resource is \( (z(G^u)) \).
+ \{-W'(G^u)\} + \{K(G^u)\} = \{\text{unit extraction cost}\} + \{\text{cost of the unextracted resource}\} + \{\text{stock uncertainty cost}\}. \text{ Eq. (3.5) states that groundwater should be extracted to the point where marginal benefits equal full marginal costs. Note that } Y^{-1}(c) \text{ is the derived demand for water available at a price } c. \text{ If the supply of surface water } S \text{ is sufficiently large, the demand price for water falls below the full marginal cost of groundwater and extraction ceases.}

A steady state occurs when the rate of replenishment is equal to the rate of extraction. If it is desirable to exploit the uncertain stock, i.e., if } g^u(0) > R(X_p), \text{ then the system must eventually reach a steady state at some level } G^u \in (0,X_u]. \text{ This is so because both } g^u(G^u_t) \text{ and } R(X_p+G^u_t) \text{ are time-continuous.}^4 \text{ Thus, either (i) } g^u(G^u_t) > R(X_p+G^u_t) \text{ for all } t \in [0,T), \text{ in which case the aquifer is depleted and the steady state is reached at time } T \text{ with an empty aquifer, or (ii) } g^u(G^u_t) = R(X_p+G^u_t) \text{ at some } t < T, \text{ in which case a steady state is reached prior to depletion.}

We turn now to study the conditions under which the uncertain stock admits profitable exploitation, i.e., when it is optimal to set } g^u(0) > R(X_p). \text{ Let } G^u \text{ denote the optimal steady state level under which } g^u(G^u_t) = R(X_p+G^u_t). \text{ If } g^u(0) > R(X_p), \text{ then, as argued above, a steady state must eventually occur after some (or all) of the uncertain stock has been exploited. Conversely, if } G^u > 0 \text{ then } g^u(0) \text{ must exceed } R(X_p), \text{ for otherwise the optimal steady state will never be reached. Thus, } g^u(0) > R(X_p) \text{ if and only if } G^u > 0 \text{ and the conditions under which } g^u(0) > R(X_p) \text{ are the same as those needed for } G^u > 0 \text{ to hold.}

Suppose that } 0 < G^u < X_u, \text{ i.e., the uncertain stock is exploited but not depleted. For the purpose of investigating the conditions under which the uncertain stock admits profitable exploitation, this assumption is harmless (since under depletion it certainly pays to exploit the uncertain stock).}
Observing (3.2), we see that, at the steady state, \( W(G^u) \) is simply the present value of a permanent flow of profits, each of the constant level \( Y(S+R(X_p+G^u)) - z(X_p+G^u)R(X_p+G^u) \). That is,

\[
W(G^u) = \frac{[Y(S+R(X_p+G^u)) - z(X_p+G^u)R(X_p+G^u)]}{p}. \tag{3.7}
\]

Noting (3.1), it is clear that \( W(G^u) = W_T(G^u) \), so that \( K(G^u) = 0 \). Moreover, \( W'(G^u) = W_T'(G^u) \) since \( W(G^u) \) and \( W_T(G^u) \) have the same form. Thus

\[
K(G^u) = K'(G^u) = 0.
\]

Eq. (3.5), therefore, reduces at the steady state to

\[
Y'(S+R(X_p+G^u)) = z(X_p+G^u) - W'(G^u), \tag{3.8}
\]

provided \( 0 < G^u < X_u \). Differentiating (3.7) with respect to \( G^u \), using (3.8), gives

\[
-W'(G^u) = \frac{z'(X_p+G^u)R(X_p+G^u)}{\rho + R'(X_p+G^u)}. \tag{3.9}
\]

Define

\[
J(G) = \frac{z'(G)R(G)}{\rho + R'(G)}, \tag{3.10}
\]

and note that Assumptions 2-3 guarantee that \( J(G) \) is non-decreasing. Eqs. (3.8)-(3.10) imply that at a steady state \( G^u \in (0,X_u) \), the relation

\[
Y'(S+R(X_p+G^u)) = z(X_p+G^u) + J(X_p+G^u) \tag{3.11}
\]

must hold. Thus, for a positive steady state to occur before depletion, Eq. (3.11) must admit a positive solution \( G^u \). As \( Y'(\cdot) \) is decreasing (Assumption 1), and \( z(\cdot), R(\cdot) \) and \( J(\cdot) \) are non-decreasing (Assumptions 2-3), the condition

\[
Y'(S+R(X_p)) > z(X_p) + J(X_p)
\]

is necessary for (3.11) to admit a positive solution. Thus, if \( Y'(S+R(X_p)) \leq z(X_p) + J(X_p) \), there does not exist a positive value \( G^u \) that satisfies (3.11), implying that a positive steady state cannot be optimal.

We can conclude:
Proposition 1: The uncertain stock $X_u$ admits profitable exploitation, i.e.,

$$g^u(0) > R(X_p),$$

only if $Y'(S+R(X_p)) > z(X_p) + J(X_p)$.

Thus, if $Y'(S+R(X_p)) \leq z(X_p) + J(X_p)$, the uncertain stock is never exploited and the problem reduces to that of allocating the proven reserves (see the next section). It is possible that $Y'(S+R(X_p)) > z(X_p) + J(X_p)$ and Eq. (3.11) does not have a finite solution at all. In such cases, it is explained below, depletion is bound to occur, though it may occur only asymptotically at $\tau = \infty$ (this may be the case, for instance, when $S = R = 0$, $Y'(0) = \infty$ and $Pr(X_u=0) > 0$).

We turn now to specify the boundary condition needed to solve for $W(G^u)$ in (3.6) when $Y'(S+R(X_p)) > z(X_p) + J(X_p)$, i.e., when the uncertain stock admits profitable exploitation. To this end, the following extension of Proposition 1 is useful:

**Corollary 1:** If $Y'(S+R(X_p+G^u)) > z(X_p+G^u) + J(X_p+G^u)$ then $g(G_u) > R(X_p+G^u)$ for any $G^u \in [0,X_u)$.

**Proof:** If $g(G_u) \leq R(X_p+G_u)$, then $\hat{G}^u \leq G^u$ and Eq. (3.11) must hold in the steady state (since the steady state must occur before depletion in this case). But under $Y'(S+R(X_p+G^u)) > z(X_p+G^u) + J(X_p+G^u)$ and Assumptions 1-3, there does not exist $\hat{G}^u \leq G^u$ satisfying (3.11). Thus, $g(G_u) \leq R(X_p+G^u)$ cannot be optimal when $Y'(S+R(X_p+G^u)) > z(X_p+G^u) + J(X_p+G^u)$.

Thus, according to Corollary 1, so long as depletion has not yet occurred and $Y'(S+R(X_p+G^u)) > z(X_p+G^u) + J(X_p+G^u)$, the optimal extraction level must exceed the rate of recharge.

Let $G^r$ be the state level satisfying

$$Y'(S+R(X_p+G^r)) = z(X_p+G^r) + J(X_p+G^r), \quad (3.12)$$

if (3.12) admits a finite solution, and $G^r = \infty$ otherwise. When $Y'(S+R(X_p)) >$
z(X_p) + J(X_p), Assumptions 1-3 ensure that G_f is unique. This is clear if either R(·) or z(·) are strictly increasing (since then evaluating (3.12) at any G^u ≠ G_f changes its right-hand side and left-hand side in opposite directions). If both R(·) and z(·) are constants, then J(G) = 0 for all G and the condition Y'(S+R) > 
z + J implies that (3.12) does not admit a finite solution and hence G_f = ∞.

Thus, whenever G_f is finite, it is the unique value satisfying (3.11) and hence must be the desirable steady state. Unfortunately, it may not be feasible. Lacking knowledge of X_u, it is unknown in advance whether G_f is feasible, except when G_f > \bar{X}, in which case it is known with certainty that G_f is infeasible. Recall that \bar{X} is the (known) upper support of X_u, representing the capacity of the uncertain stock, i.e., Pr(X_u ≤ \bar{X}) = 1 and Pr(X_u ≤ a) > 0 for all a < \bar{X}. Since G^u ≤ X_u, G^u ≤ \bar{X} with probability one.

If G_f ≥ \bar{X}, then, as argued below, depletion is optimal and the steady state will occur with an empty aquifer. For a steady state to occur before depletion, it must satisfy Eq. (3.11), but the unique state level that satisfies (3.11) is \hat{G}^u = G_f ≥ \bar{X}; thus, no state level exists that can qualify as a steady state with a non-empty aquifer. The desirable steady state, in this case where G_f ≥ \bar{X}, is set at the level \bar{X}, which yields depletion with probability one.

Another way of reasoning is to note that setting the steady state at a level below \bar{X} may result in the need to update the extraction decisions along the way before the occurrence of depletion. Such updating cannot happen under the optimal plan. The only useful piece of information that can cause updating the optimal extraction plan occurs at the time of depletion. No learning takes place before the occurrence of the depletion event, as there is nothing that could not be anticipated from the outset by assuming that the system attains the current stage. Hence, before depletion, there can be no incentives to update decisions
along the optimal plan. Now, if \( \hat{G}^u \) is set at a level below \( \bar{X} \), it is possible that the state \( \hat{G}^u \) is reached before depletion. If this happens, then, since \( \hat{G}^u < G_f \) (recall that the case \( G_f \geq \bar{X} \) is considered), it must be (Assumptions 1-3) that \( Y'(S+R(X_p+\hat{G}^u)) > z(X_p+\hat{G}^u) + J(X_p+\hat{G}^u) \). Corollary 1, then, implies that it is optimal to extract in excess of the recharge in such circumstances, so that \( \hat{G}^u \) cannot be a steady state. But that would require updating the extraction decisions before depletion, which is not permitted under the optimal plan. Thus, planning a steady state below \( \bar{X} \) cannot be optimal, leaving \( \hat{G}^u = \bar{X} \) as the unique choice.

The value function at the steady state is given by Eq. (3.7). The steady state resource price is thus obtained by differentiating (3.7) with respect to \( \hat{G}^u \), as specified in (3.14) below. Note in this case that depletion occurs with probability one, thus (3.5) may not hold in the steady state, implying that (3.9) may not hold, and hence \( -W'(\hat{G}^u) \) may diverge from \( J(X_p+\hat{G}^u) \).

If, on the other hand, \( G_f < \bar{X} \), then \( \hat{G}^u = G_f \) is the unique level satisfying (3.11) and is therefore the desirable steady state. However, depletion may occur before \( G_f \) is reached (if \( X_u < G_f \)). Nevertheless, as there is no learning involved and no updating of extraction decisions before depletion is allowed under the optimal plan, \( G_f \) must be the planned steady state. If \( \hat{G}^u \) is set at a level below \( G_f \) and depletion has not occurred by the time \( \hat{G}^u \) is reached, then the extraction policy must be updated. This is so because both \( G_f < \bar{X} \) and Assumptions 1-3 require that \( Y'(S+R(X_p+G^u)) > z(X_p+G^u) + J(X_p+G^u) \) for all \( G^u < G_f \). Corollary 1, then, states that starting at a state \( X_p+G^u < X_p+G_f \) before depletion, it must be optimal to extract in excess of the recharge rate. Since the optimal plan cannot be subject to revisions along the way (except at the point of depletion), the possibility of a steady state below \( G_f \) is ruled out.
Selecting a steady state above \( G_f \) is certainly not desirable, since even with unlimited stock it is not optimal to do so. This leaves \( \hat{G}^u = G_f \) as the unique desirable steady state. Noting (3.8)-(3.10), the associated steady state resource price is \(-W'(\hat{G}^u) = J(X_p+\hat{G}^u)\). To summarize this discussion we have:

Proposition 2: Provided \( Y'(S+R(X_p)) > z(X_p) + J(X_p) \), so that \( X_u \) admits profitable exploitation, the desirable steady state is \( \hat{G}^u \) defined by

\[
\hat{G}^u = \min\{G_f, \bar{X}\}, \tag{3.13}
\]

and the steady state resource price is

\[
-W'(\hat{G}^u) = \begin{cases} 
J(X_p+\hat{G}^u) = \frac{z'(X_p+\hat{G}^u)R(X_p+\hat{G}^u)}{\rho + R'(X_p+\hat{G}^u)} & \text{if } G_f < \bar{X} \\
\frac{1}{\rho} \left[ z'(X_p+\hat{G}^u)R(X_p+\hat{G}^u) - (Y'(S+R(X_p+\hat{G}^u))-z(X_p+\hat{G}^u))R'(X_p+\hat{G}^u) \right] & \text{if } G_f = \bar{X}
\end{cases} \tag{3.14}
\]

where the \( G_f \geq \bar{X} \) part of (3.14) is obtained by differentiating (3.7).

Given the boundary conditions (3.13) and (3.14), \( W(\hat{G}^u) \) and \( W'(\hat{G}^u) \), \( \hat{G}^u \in [0, \hat{G}^u] \), are determined by solving Eq. (3.6). In particular, \( W(0) \) and \( W'(0) \) are obtained. These values are then used in (3.5) to determine the optimal extraction \( g^u(0) \), which, in turn, changes the state \( \hat{G}^u \); new values of \( W \) and \( W' \) are evaluated and give rise to a new extraction decision and so on. The process proceeds in this manner until either (i) the bottom of the aquifer is reached, in which case extractions are adjusted to the rate of recharge and the system settles at a steady state with an empty aquifer, or (ii) the state \( \hat{G}^u \), defined in (3.13), is approached prior to depletion, in which case extraction exactly equals the rate of recharge and the system comes to a permanent rest with a non-empty aquifer.
4. Optimal Extraction of the Proven Stock

Equipped with the optimal extraction plan of the uncertain stock and the associated value function \( \theta(O,F_u) = W(0) \), we turn now to characterization of the optimal allocation of the proven stock, as stated in (2.2). For \( t \leq \tau \), let \( V(G_t,t;X_p,\tau) \) be the current value of the aquifer starting at time \( t \) with the state \( G_t \), conditional on the choice of \( \tau \) and on \( G_\tau \leq X_p \):

\[
V(G_t,t;X_p,\tau) = \max_{(g_s \geq 0)} \left\{ \int_0^{\tau-t} \left[ Y(S+g_s)-z(G_t)g_s \right] e^{-\rho s} ds + W(0)e^{-\rho(\tau-t)} \right\}
\]  

subject to \( G_s = G_t + \int_0^s [g_a-R(G_a)] da \leq X_p, \ s \in [0,\tau-t] \). When confusion does not arise, the arguments \( X_p \) and \( \tau \) are suppressed and we write \( V(G_t,t) \).

Extracting the level \( g \) for a short period \([0,h]\) yields the immediate return \([Y(S+g)-z(G_t)g]h + o(h)\) and changes the state to \( G_{t+h} = G_t + [g-R(G_t)]h + o(h) \). The return from \( h \) onward, discounted back to 0, is equal to \( V(G_{t+h},t+h)e^{-\rho h} \). Thus,

\[
V(G_t,t) = \max_{g \geq 0} \left\{ [Y(S+g)-z(G_t)g]h + V(G_{t+h},t+h)e^{-\rho h} + o(h) \right\}.
\]

Expanding \( e^{-\rho h} \) about 1 and \( V(G_{t+h},t+h) \) about \( V(G_t,t) \), one obtains the Bellman equation

\[
\rho V(G_t,t) - V_t(G_t,t) = \max_{g \geq 0} \left\{ Y(S+g)-z(G_t)g + V_0(G_t,t)[g-R(G_t)] \right\},
\]

where \( V_0(G,t) \equiv \partial V(G,t)/\partial G \) and \( V_t(G,t) \equiv \partial V(G,t)/\partial t \).

Undertaking the maximization on the right-hand side of (4.2) gives the optimal extraction for any \( G_t \in [0,X_p] \), \( t \leq \tau \), as the value \( g(G_t) \) satisfying:

\[
\begin{cases}
Y'(S+g(G_t)) = z(G_t)-V_0(G_t,t) & \text{if } Y'(S) \geq z(G_t)-V_0(G_t,t) \\
g(G_t) = 0 & \text{otherwise}
\end{cases}
\]

The quantity \(-V_0(G_t,t)\) is the current-value resource price at time \( t \) when the
remaining proven stock is $X_p - G_t$; this is the opportunity cost of current extraction in terms of foregone future benefits.

Substituting $g(G_t)$ for $g$ in (4.2) gives

$$\rho V(G_t,t) - V_t(G_t,t) = Y(S + g(G_t)) - z(G_t)g(G_t) + V_G(G_t,t) [g(G_t) - R(G_t)].$$ (4.4)

Eq. (4.4) constitutes a differential equation from which $V(G_t,t)$ can be derived (perhaps only numerically), given the boundary condition specified below.

Let $G$ and $t$ denote, respectively, the steady state level and the entrance time to the steady state. If $g(0) > R(0)$, then $G > 0$ and, due to the time-continuity of the extraction and recharge paths, extraction must equal the rate of recharge at the steady state. If $g(0) \leq R(0)$, then $G = 0$; if $R(0) > 0$, it is possible in this case that extraction falls short of recharge at the steady state (of full aquifer). In the former case, when $g(0) > R(0)$, the system must eventually reach a steady state either (i) with $G \in (0,X_p)$, or (ii) with $G > X_p$.

It follows immediately from Proposition 1 that:

**Corollary 2:** (i) If $Y'(S + R(X_p)) \leq z(X_p) + J(X_p)$, then $G \in [0,X_p]$ and $\tau^* = \infty$;

(ii) otherwise, if $Y'(S + R(X_p)) > z(X_p) + J(X_p)$, then $G > X_p$.

We now specify the boundary conditions for Case (i), in which $Y'(S + R(X_p)) \leq z(X_p) + J(X_p)$ and the aquifer is never exploited beyond its proven stock $X_p$. Observe first that if $Y'(S + R(0)) \leq z(0)$ then $g(0) \leq R(0)$ and the aquifer's stock is never diminished. In this case $G = \hat{t} = 0$ and the steady state resource price $-\dot{V}_G \equiv -V_G(\hat{G},\hat{t}) = 0$. To see this, note that $Y'(S + R(0)) \leq z(0)$ and $g(0) > R(0)$ imply (Assumption 1) $Y'(S + g(0)) < z(0)$, which cannot occur under the optimal policy (see the discussion on p. 6 regarding the optimal extraction from time $T$ on). Thus, when $Y'(S + R(0)) \leq z(0)$, extraction never exceeds recharge and a steady state is instantly attained with a full aquifer, i.e., $G = 0$. In such a
case, the demand for groundwater offered at a price $z(0)$ never exceeds the permanent supply (i.e., the recharge rate) and groundwater is not scarce, which means that the in situ resource price must be zero, i.e., $-\dot{V}_G = 0$. The constant extraction level is determined so as to maximize the instantaneous return, as specified in (4.6) below.

If $Y'(S+R(0)) > z(0)$, then $g(0)$ must exceed $R(0)$, for otherwise, if $g(0) \leq R(0)$, then $\dot{G} = 0$ and (as argued above) $-\dot{V}_G = 0$. This, noting that $Y'(S+R(0)) > z(0)$ implies $Y'(S) > z(0)$, violates the necessary condition (4.3). In this case (due to the time-continuity of the extraction and recharge paths), a steady state must occur at some point prior to or at the depletion of the proven stock (recall that Case (i) of Corollary 2 is considered), i.e., $\hat{G} \in (0,X_p]$ and $g(\hat{G}) = R(\hat{G})$.

Eq. (4.4), then, reduces in the steady state to $\rho V(\hat{G},\hat{t}) - V_t(\hat{G},\hat{t}) = Y(S+R(\hat{G})) - z(\hat{G})R(\hat{G})$. Differentiating with respect to $\hat{G}$, while using $Y'(S+g(\hat{G})) - z(\hat{G}) = -\dot{V}_G$ and $\partial V_G/\partial t = 0$, we obtain

$$-V_G(\hat{G},\hat{t}) = J(\hat{G}),$$

where $J(\cdot)$ is defined in (3.10). Condition (4.3), therefore, specializes in a steady state $\hat{G} \in (0,X_p]$ to

$$Y'(S+R(\hat{G})) = z(\hat{G}) + J(\hat{G}). \tag{4.5}$$

Assumptions 1-3 together with $Y'(S+R(0)) > z(0)$ and $Y'(S+R(X_p)) \leq z(X_p) + J(X_p)$ ensure the existence of a solution $\hat{G} \in (0,X_p]$ to (4.5). Since $V(\hat{G},\hat{t}) = \int_0^\infty [Y(S+R(\hat{G}))-z(\hat{G})R(\hat{G})]e^{-\rho t} dt = [Y(S+R(\hat{G}))-z(\hat{G})R(\hat{G})]/\rho$, we use (4.4) and $g(\hat{G}) = R(\hat{G})$ to obtain $V_t(\hat{G},\hat{t}) = \partial V(\hat{G},\hat{t})/\partial t = 0$. We can conclude:
Proposition 3: (i) If \( \gamma'(S+R(0)) \leq z(0) \), then \( \hat{G} = \hat{t} = -V_G(\hat{G},\hat{t}) = V_t(\hat{G},\hat{t}) = 0 \) and extraction remains constant at the level

\[
g(0) = \begin{cases} \gamma'^{-1}(z(0)) - S & \text{if } \gamma'(S) > z(0), \\ 0 & \text{otherwise} \end{cases} \quad (4.6)
\]

(ii) If \( \gamma'(S+R(0)) > z(0) \) and \( \gamma'(S+R(X_p)) \leq z(X_p) + J(X_p) \), then: \( g(0) > R(0) \); a steady state is reached at \( \hat{G} \in (0,X_p] \) defined by (4.5); the (current value) steady state resource price is given by

\[
-V_G(\hat{G},\hat{t}) = J(\hat{G}) = \frac{\gamma'(\hat{G})R(\hat{G})}{\rho + R'(\hat{G})}; \quad (4.7a)
\]

at the entrance time to the steady state

\[
V_t(\hat{G},\hat{t}) = 0; \quad (4.7b)
\]

and

\[
\int_0^\hat{t} [\gamma'^{-1}(z(G_t)+V_G(G_t,t))-S-R(G_t)]dt = \hat{G}. \quad (4.7c)
\]

We turn now to a determination of the boundary conditions when \( \gamma'(S+R(X_p)) > z(X_p) + J(X_p) \) and depletion of the proven stock is desirable. To this end it is expedient to introduce the auxiliary problem

\[
\varphi(G,\tau) = \max_{\{g_t \geq 0\}} \int_0^\tau [\gamma(S+g_t) - z(G_t)g_t]e^{-\rho t}dt \quad (4.8)
\]

subject to \( G_t = \int [g_s - R(G_s)]ds, \ G_0 = 0 \) and \( G_\tau = G \). Observing (4.1), we see that

\[
V(0,0;X_p,\tau) = \varphi(X_p,\tau) + W(0)e^{-\rho \tau}. \quad \text{Thus, noting (2.2) and } W(0) = \vartheta(0,F_u),
\]

\[
\vartheta(X_p,F_u) = \max_{\tau} \{V(0,0;X_p,\tau)\} = \max_{\tau} \{\varphi(X_p,\tau) + W(0)e^{-\rho \tau}\}. \quad \text{The optimal value } \tau^*,
\]

therefore, satisfies the necessary condition

\[
\frac{\partial \varphi(X_p,\tau^*)}{\partial \tau} = \vartheta(X_p,\tau^*) = \rho W(0)e^{-\rho \tau}. \quad (4.9)
\]

Now, splitting (4.8) into the intervals \([0,\tau-h)\) and \([\tau-h,\tau]\), for small \( h \),
and applying the dynamic programming argument yields

$$\varphi_{\tau}(x_p,\tau) = \max_{g \geq 0} \left\{ \left[ Y(S+g) - z(x_p)g \right] e^{-\rho \tau} - \varphi_0(x_p,\tau)[g-R(x_p)] \right\},$$

where $\varphi_0 \equiv \partial \varphi / \partial G$. Undertaking the maximization gives the optimal extraction rule at time $\tau$ as

$$Y'(S+g(x_p)) = z(x_p) + \varphi_0(x_p,\tau)e^{\rho \tau}. \quad (4.10)$$

Substituting back in the equation above (4.10) gives

$$\varphi_{\tau}(x_p,\tau) = [Y(S+g(x_p))-z(x_p)g]e^{-\rho \tau} - \varphi_0(x_p,\tau)[g-R(x_p)]. \quad (4.11)$$

It is clear that, under the optimal plan, the extraction path generated by (4.8) must coincide with the one generated by (4.1) during $t \in [0,\tau]$ (it is verified below that no jumps can occur at $\tau$). Evaluating (4.3) at $t = \tau$ and comparing with (4.10), we see that

$$V^r(x_p,\tau) e^{\rho \tau} = -V_0(x_p,\tau;X_p,\tau), \quad (4.12)$$

so that (4.11) can be written as

$$\varphi_{\tau}(x_p,\tau)e^{\rho \tau} = Y(S+g(x_p)) - z(x_p)g + V_0(x_p,\tau;X_p,\tau)[g-R(x_p)]. \quad (4.13)$$

Condition (4.9), then, gives

$$\rho W(0) = Y(S+g(x_p))-z(x_p)g(x_p) + V_0(x_p,\tau^*;X_p,\tau^*)[g(x_p)-R(x_p)]. \quad (4.14)$$

Now, at $\tau$, $G_\tau = X_p$, $G_0 = 0$ and (4.4) gives

$$\rho V(X_p,\tau;X_p,\tau) - V_t(x_p,\tau;X_p,\tau) = Y(S+g(x_p)) - z(x_p)g(x_p) +$$

$$+ V_0(x_p,\tau;X_p,\tau)[g(x_p)-R(x_p)]. \quad (4.15)$$

Likewise, (3.6) states that

$$\rho W(0) = Y(S+g^u(0)) - z(x_p)g^u(0) + [W'(0)-K(0)](g^u(0)-R(x_p)).$$

It is evident from (4.1) that $V(X_p,\tau;X_p,\tau) = W(0)$ for any $\tau$, particularly for the optimal $\tau^*$. Hence, Eqs. (4.14)-(4.15), evaluated at $\tau^*$, imply that

$$V_t(x_p,\tau^*;X_p,\tau^*) = 0. \quad (4.14)-(4.15), \text{ then, give}$$

$$\rho V(x_p,\tau^*;X_p,\tau^*) = Y(S+g(x_p)) - z(x_p)g(x_p) - [Y'(S+g(x_p))-z(x_p)][g(x_p)-R(x_p)].$$

Similarly, from Eqs. (3.5)-(3.6) we obtain
\[ \rho W(0) = Y(S + g^{u}(0)) - z(X_{p})g^{u}(0) - \left[ Y'(S + g^{u}(0)) - z(X_{p})\right] g^{u}(0) - R(X_{p}) \].

From (4.1), \( V(X_{p}, \tau^{*}; X_{p}, \tau^{*}) = W(0) \), thus the right-hand sides of the above two equations must be the same. Since \( Y(S + g) \) is strictly concave (Assumption 1), the function \( Y(S + g) - zg - \left[ Y'(S + g) - z\right] g - R(X_{p}) \) must be increasing in \( g \) whenever \( g \) exceeds \( R(X_{p}) \), which, according to Corollary 2(ii), is satisfied by both \( g^{u}(0) \) and \( g(X_{p}) \). The equality of the right-hand sides of the above two equations, therefore, requires \( g(X_{p}) = g^{u}(0) \), implying a smooth transition from the certain to the uncertain extraction plans.

Observing (4.3) and (3.5) again, while recalling that \( Y'(\cdot) \) is strictly decreasing, we can conclude that \( -V_{G}(X_{p}, \tau^{*}; X_{p}, \tau^{*}) = K(0) - W'(0) \). Also, at time \( \tau^{*} \) the cumulated net withdrawal must be equal to the proven stock. The boundary conditions we seek can now be stated:

**Proposition 4:** If \( Y'(S + R(X_{p})) > z(X_{p}) + J(X_{p}) \), then

\[
V_{t}(X_{p}, \tau^{*}; X_{p}, \tau^{*}) = 0 \quad \text{and} \quad -V_{G}(X_{p}, \tau^{*}; X_{p}, \tau^{*}) = K(0) - W'(0),
\]

\[
\tau^{*} \int_{0}^{T} \left[ Y^{-1}(z(G_{t}) - V_{G}(G_{t}, t)) - S - R(G_{t}) \right] dt = X_{p}
\]

The optimal extraction policy can be summarized as follows. If \( Y'(S + R(0)) \leq z(0) \), then: extraction remains constant at the level \( g(0) \) specified in (4.6); \( g(0) \leq R(0) \), so that the aquifer's stock is never diminished; as the demand for groundwater never exceeds the recharge rate, the resource is not scarce and its price equals zero.

If \( Y'(S + R(0)) > z(0) \) and \( Y'(S + R(X_{p})) \leq z(X_{p}) + J(X_{p}) \), then: \( g(0) > R(0) \), so that the aquifer's stock admits profitable exploitation; the aquifer, however, is never exploited beyond its proven stock \( X_{p} \); \( V(G_{t}, t) \) is derived by solving (4.4) subject to the boundary conditions (4.7a-c), where \( G \) is defined in (4.5); given
V(G_t,t), the optimal extraction policy is determined by (4.3).

If \( Y'(S+R(X_p)) > z(X_p + J(X_p)) \) then: the proven stock \( X_p \) is eventually depleted; \( V(G_t,t) \) and \( \tau^* \) are determined by solving (4.4) subject to the boundary conditions (4.16)-(4.17) and generate the optimal extraction policy \( g(G_t) \), \( t \in [0,\tau^*] \), according to (4.3); the extraction policy from time \( \tau^* \) on is evaluated using (3.5), where \( W(G^u) \) is derived by solving (3.6) subject to the boundary conditions (3.13)-(3.14).

5. Value of information and exploration

It is often possible to conduct exploratory drilling in order to reduce the uncertainty regarding the aquifer's stock. Such activities entail costs and hence their operation requires careful examination. The effect of discovering additional groundwater stock is twofold: first, it changes the level of the proven reserves; second, it provides information on the distribution of the remaining uncertain stock. The overall benefit associated with a discovery of small (marginal) additional stock is called the value of information. If the value of information is zero, exploration is not desirable. The need to consider exploration activities, then, arises only when the value of information is positive. Accordingly, we first derive the value of information and specify conditions under which it is positive. Then, assuming a positive value of information, we study the exploration problem.

Noting (2.2), the contribution of a marginal increase in the proven stock is given by \( \theta'(X_p,F_u) = \theta(X_p,F_u)/\theta X_p \). Writing \( \theta(X_p,F_u) = \varphi(X_p,\tau^*) + \theta(0,F_u)e^{-\rho\tau^*} \) and differentiating with respect to \( X_p \) gives \( \theta'(X_p,F_u) = \varphi_0(X_p,\tau^*) \), where it is recalled that \( \varphi_0(G,\tau) = \partial \varphi_0/\partial G \). Noting (4.12) and Proposition 4, we obtain

\[
\theta'(X_p,F_u) = -V_0(X_p,\tau^*;X_p,\tau^*)e^{-\rho\tau^*} = (K(0)-W'(0))e^{-\rho\tau^*}. \tag{5.1}
\]
As stated in (5.1), the value of information is the current-value resource price at the time of depletion of the proven stock, discounted back to the beginning of the planning horizon. This value, it should be noted, is in general not the same as \(-V_G(0,0;X_p,\tau^*)\)—the resource price at time zero. The two coincide if the resource price rises exponentially at the rate \(\rho\). It can be shown that this, in general, does not occur, except in the special case where both \(z(G)\) and \(R(G)\) are constants and independent of \(G\). We can see from (5.1) that the value of information is positive only when \(-V_G(X_p,\tau^*;X_p,\tau^*) > 0\) and \(\tau^* < \omega\), for which, noting Corollary 2, \(Y'(S+R(X_p)) > z(X_p) + J(X_p)\) is necessary (this condition is not sufficient because it does not require \(\tau^* < \omega\)). We turn now to exploration activities, assuming a positive value of information; our treatment follows Quyen (1991).

Due to technological and other limitations (financial, hydrological), exploration activities are not completely divisible, which means that their scale has some lower bound, say the scale of a single exploratory drilling. This basic activity is denoted an exploration project. The exploration decisions thus involve (i) the number of exploration projects to operate (i.e., the scale of the operation), and (ii) the timing of each project. We describe the timing decision of a single project; extending the analysis to handle many exploration projects, each corresponding to exploring a different sub-region, is outlined in Quyen (1991, p. 781).

Consider a single exploration project for which a decision must be made on the timing of its operation. Let \(X_e\) represent groundwater discovered by the exploration project. Before the project is carried out, \(X_e\) is a random variable with the distribution function \(F_e\) defined over \([0,\bar{X}_e]\), \(\bar{X}_e \leq \bar{X}\). Let \(F_e(\cdot | X_e)\) denote the conditional distribution of the remaining uncertain stock given \(X_e\) and
let \( t_e \) be the time at which the exploration project is undertaken. We assume that the exploration project is completed instantly and entails a cost of \( C \). At time \( t_e \) with \( G_{t_e} = G_e \), the actual value of \( X_e \) is observed and the value of the aquifer becomes \( \theta(X_p-G_e+X_e,F_u(\cdot|X_e)) - C \). Immediately before \( t_e \), when the realization of \( X_e \) is not yet observed, the value of the aquifer equals

\[
\omega(X_p-G_e,F) = E_e(\theta(X_p-G_e+X_e,F_u(\cdot|X_e))) - C,
\]

where \( E_e \) denotes expectation with respect to \( F_e \), and \( F = F_e,F_u(\cdot|X_e) \). The analyses of Sections 3-4 can be used to evaluate \( \theta(X_p-G_e+X_e,F_u(\cdot|X_e)) \) given \( X_e \), from which \( \omega(X_p-G_e,F) \) is evaluated as specified above.

The decision problem with exploration can be formulated as

\[
\max_{(g_t \geq 0), 0 \leq G_t = X_p, t_e} \left\{ \int_0^{t_e} [Y(S+g_t)-z(G_t)g_t]e^{-\rho t} dt + \omega(X_p-G_e,F)e^{-\rho t_e} \right\} \tag{5.2}
\]

subject to \( G_t = \int_0^t [g_s-R(G_s)]ds, t \in [0,t_e], G_{t_e} = G_e \). The restriction \( G_e \leq X_p \) implies that exploration, if carried out, should not be delayed beyond the time of depletion of the initial proven stock \( X_p \). This restriction simplifies the analysis and is known to be unbinding in some cases (see Theorem 4.1 of Quyen [1990 p. 786]). Lowering \( t_e \) advances the observation of \( X_e \) and permits more informed planning, which acts to increase \( \theta^e \); on the other hand, it advances the payment \( C \), which increases the present value of exploration costs and thereby acts to decrease \( \theta^e \). The choice of \( t_e \), thus, entails a tradeoff between these two conflicting effects and is made so that they cancel each other at the margin.

Letting \( t_e \) play the role of \( \tau \) and \( \omega(X_p-G_e,F) \) of \( \theta(0,F_u) \), it is seen that Problems (5.2) and (2.2) are similar. There is one main difference, however: while in (2.2) the state at time \( \tau \) is given (by the proven stock \( X_p \)), in (5.2) the state at time \( t_e \) is subject to choice. Accordingly, following (4.1), we
define

\[ V^e(G_e, t; G_e, t_e) = \max_{(g_e \geq 0)} \int_0^t [Y(S + g_e) - z(G_e)g_e]e^{-\rho s}ds + \omega(X_p - G_e)e^{-\rho(t_e - t)} \]  \hspace{1cm} (5.3) 

subject to \( G_e = G_e + \int [g_e - R(G_e)]da, \) \( s \in [0, t_e - t], \) and \( G_{te} = G_e. \) (The argument \( F \) is dropped from \( \omega(\cdot) \) for convenience.)

Following the steps that lead to (4.3) and (4.4) we obtain for \( te[0, t_e], \)

\[ Y'(S + g(G_t)) = z(G_t) - V^e(G_t, t) \]  \hspace{1cm} (5.4) 

and

\[ \rho V^e(G_e, t) - V^e(G_e, t) = Y(S + g(G_e)) - z(G_e)g(G_e) + V^e(G_e, t)[g(G_e) - R(G_e)], \]  \hspace{1cm} (5.5) 

where \( V^e_G = \partial V^e / \partial G, \) \( V^e_t = \partial V^e / \partial t, \) and the arguments \( G_e \) and \( t_e \) are suppressed from \( V^e \) whenever confusion does not arise.

The boundary conditions entail the choice of \( t_e \) and \( G_e. \) Observing (4.8), (5.2) and (5.3), we see that \( \Phi^e(X_p, F) = \max_{(G_e, t_e)} \Phi^e(0, 0; G_e, t_e) = \max \{ \Phi^e(G_e, t_e) + \omega(X_p - G_e)e^{-\rho t_e} \}. \) Let \( G_e^* \) and \( t_e^* \) denote the optimal values. If \( G_e^* \) and \( t_e^* \) = 0, then also \( t_e^* = 0 \) and the actual value of \( X_e \) is observed at the beginning of the planning horizon. The decision problem then is identical to that of (2.2) with proven stock \( X_e + X_p \) and uncertain stock distribution \( F_U(\cdot | X_e) \); this case is analyzed in Sections 3 and 4. If \( G_e^* = X_p, \) the problem is identical to (4.1) with \( \omega(X_e, F_U(\cdot | X_e)) \) substituted for \( W(0). \) Again, the analysis of Sections 3-4 is easily extended to cover this case too. The problem is somewhat more involved in the case \( 0 < G_e^* < X_p. \) We outline this case below.

Assuming an interior solution, the optimal values \( G_e^* \) and \( t_e^* \) should satisfy the first order conditions

\[ \frac{\partial \Phi^e(G_e, t_e)}{\partial G_e} = \frac{\partial \Phi^e(G_e, t_e)}{\partial G_e} = \omega'(X_p - G_e)e^{-\rho t_e} \]  \hspace{1cm} (5.6) 

and
Repeating the steps that lead to (4.10)-(4.13), we obtain

\begin{equation}
Y'(S+g(G_e)) = z(G_e) + \varphi_G(G_e, t_e)e^{p t_e},
\end{equation}

\begin{equation}
-V^*_G(G_e, t_e) = \varphi_G(G_e, t_e)e^{p t_e}
\end{equation}

and

\begin{equation}
\varphi_{t_e}(G_e, t_e)e^{p t_e} = Y(S+g(G_e)) - z(G_e)g(G_e) + V^*_G(G_e, t_e)[g(G_e)-R(G_e)].
\end{equation}

Thus, (5.6) gives

\begin{equation}
-V^*_G(G_e, t_e) = \omega'(X_p-G_e^*).
\end{equation}

Observing (5.3), it is clear that \( V^*(G_e, t_e) = \omega(X_p-G_e) \) for any feasible \( G_e \) and \( t_e \), in particular for the optimal \( G_e^* \) and \( t_e^* \). Thus, Eqs. (5.5) and (5.9) evaluated at \( (G_e^*, t_e^*) \) imply \( V^*_G(G_e^*, t_e^*) = 0 \) and give

\begin{equation}
\rho \omega(X_p-G_e) = Y(S+g(G_e^*))-z(G_e^*)g(G_e^*) + V^*_G(G_e^*, t_e^*)[g(G_e^*)-R(G_e^*)],
\end{equation}

where \( g(G_e^*) \) is the value satisfying (5.4) at \( t = t_e^* \), \( G_t = G_e^* \). Also, at time \( t_e^* \) the cumulated net withdrawal must equal \( G_e^* 

\begin{equation}
\int_{t_e^*}^{t_e^*} [Y'^{-1}(z(G_t)+V_G(G_t, t))-S-R(G_t)]dt = G_e^*.
\end{equation}

The value function \( V^*(G_t, t) \) and the boundary values \( G_e^* \) and \( t_e^* \) can now be determined by solving (5.5) subject to the boundary conditions \( V^*_G(G_e^*, t_e^*) = 0 \) and (5.11)-(5.13). The extraction path prior to exploration is generated according to (5.4). At \( t = t_e^* \), the exploration project is undertaken and instantly yields the actual level of \( X_e \). Given \( X_e \), the problem becomes that of managing an aquifer with \( X_p-G_e^*+X_e \) proven reserves and uncertain stock distribution \( F_u(\cdot | X_e) \), which was studied in Sections 3-4.3.

The boundary conditions (5.11)-(5.12) involve \( \omega(\zeta) = E_x(\theta(\zeta+X_e, F_u(\cdot | X_e))) \)

and its derivative \( \omega'(\zeta) = E_x(\theta'(\zeta+X_e, F_u(\cdot | X_e))) \). The function \( \theta(\zeta+X_e, F_u(\cdot | X_e)) \)
is the value of the aquifer at time \( t^* \), immediately after the observation of \( X_e \),
with \( \zeta = X_p - C_e^* \). It can be evaluated using the analysis of Sections 3-4 for any
permissible value \( X_e \). Its derivative, according to (5.1), is

\[
\phi'(\zeta + X_e, F_u(\cdot | X_e)) = -\nu(\zeta + X_e, \tau^*(X_e)) e^{-\rho \tau^*(X_e)} = [K(0; X_e) - W'(0; X_e)] e^{-\rho \tau^*(X_e)}
\]

(5.14)

where \(-\nu(\zeta + X_e, \tau^*(X_e))\) and \( \tau^*(X_e) \) are respectively the current-value resource
price and the time of depletion of the proven stock associated with

\( \phi(\zeta + X_e, F_e(\cdot | X_e)) \), and \( K(0; X_e) \) and \( W'(0; X_e) \) are the \( K(0) \) and \( W'(0) \) corresponding
to \( \phi(0, F_u(\cdot | X_e)) \). The quantity \( \omega'(\xi) = E_e(\phi'(\zeta + X_e, F_u(\cdot | X_e))) \) can now be
evaluated.

The desirability of the exploration project is determined by comparing \( \phi \) and
\( \phi^e \): if \( \phi^e > \phi \), the exploration project generates positive net value and hence
should be undertaken.

6. Concluding Comments

The paper characterizes the exploitation of a renewable groundwater resource
of unknown size. Extraction costs are allowed to vary with net cumulated
extraction. Special attention is given to the specification of the boundary
conditions and to characterization of the steady states. The value of
information regarding the unknown stock is defined; exploration activities are
incorporated and fit smoothly within the framework of analysis.

Two examples immediately come to mind in the context of a renewable resource
with uncertain stock: groundwater and fishery. The present effort deals with
groundwater resources. The analysis here requires that both the recharge rate
and the unit extraction cost are known with certainty at each point of time.

This, in turn, implies that these two processes can depend on the cumulated net
withdrawal but not on the remaining (uncertain) stock. Such a situation is
typical of groundwater resources, where both the recharge rate and the unit extraction cost depend on the elevation of the ground water table, which in turn depends solely on the cumulated net withdrawal. For fishery resources, replenishment is due to the reproduction (growth) of the current (uncertain) stock and the unit extraction (harvest) cost depends on the current stock as well. The fishery case, therefore, requires different analyses (see Clark, 1985, Chapter 6).

The analysis can be modified to study cases where the uncertainty regards an influential event whose probability of occurrence may depend on a (known) resource stock. As an example consider the extinction of a certain species. The extinction event is to a large extent random, depending on factors such as spread of diseases, forest fires or climate conditions. The event probability, in general, depends on the biomass stock—the species is less vulnerable to disastrous events the larger its biomass is. The occurrence of the extinction event is equivalent to the occurrence of the depletion event in the case studied above. With some modifications, the present analysis can be used to study the exploitation of a (known) biomass subject to uncertain extinction conditions.

Another example is the occurrence of breakthrough in a certain backstop technology, e.g., desalination of sea water. Such an event may be independent of the resource stock but its probabilistic presence affects resource allocation decisions in a similar way the uncertain depletion event affects groundwater extraction studied above. Similar examples were analyzed by Deshmukh and Pliska (1985) in an exhaustible resource context. Modifying and extending the present analysis to study these renewable resource examples is left for the future.
References


Margat, J. and K.F. Saad, 1984, Deep-lying aquifers: water mines under the


Footnotes

1 If, as a result of mismanagement, the aquifer was over-exploited in the past and the initial state is such that it is desirable to extract less than the recharge, then the problem is greatly simplified: the uncertainty regarding the aquifer's stock does not play any role, as it is known at the outset that depletion is undesirable. In such a case the problem reduces to that of allocating a known resource stock.

2 Provided $G_h^u = G_u^u$,

$$
Pr(X_u > G_h^u | X_u > G_u^u) = \frac{Pr(X_u > G_h^u \cap X_u > G_u^u)}{Pr(X_u > G_u^u)} = \frac{Pr(X_u > G_h^u)}{Pr(X_u > G_u^u)} = \frac{1-F(G_h^u)}{1-F(G_u^u)} = \frac{1-F(G_u^u)-f(G_u^u)dG^u+o(dG^u)}{1-F(G_u^u)} = 1 - \frac{f(G_u^u)}{1-F(G_u^u)} dG^u + o(dG^u).
$$

3 A precise statement of the dynamic programming equation (3.3) requires showing that there exists a unique solution $W(G_u^u)$ to (3.3), which is differentiable in $G_u^u$ and can be attained by a feasible extraction plan. This would involve technical derivations which are outside the scope of this paper (interested readers can consult Deshmukh and Pliska [1980, 1985] and Benveniste and Scheinkman [1979]). In this work, the value functions $W(\cdot)$ and $V(\cdot)$, the latter is defined in Section 4, are each assumed to be continuously differentiable in their arguments and attainable with a unique feasible extraction plan.
Noting (3.5) and \( \frac{dG_t^u}{dt} = g(G_t^u) - R(X_t^p + G_t) \), the continuity of \( Y'(\cdot), z(\cdot) \)

[Assumptions 1,3] and assuming that \( W'(\cdot) \) and \( K(\cdot) \) are continuous imply that the policy function \( g^u(G_t^u) \) is time-continuous over all time intervals for which \( g^u(G_t^u) > 0 \). The time-continuity of \( R(G_t) \) follows from the continuity of \( R(\cdot) \)

(Assumption 2).

Because (3.5) is derived from (3.3) and the latter permits extraction to exceed recharge, (3.5) may not hold with empty aquifer.

\[ e^{-\rho t} = 1 - \rho h + o(h), \text{ and } V(G_t+h,t+h) = V(G_t,t) + V_0(G_t,t)(G_t+h-G_t) + o(G_t+h-G_t) + V_t(G_t,t)h + o(h). \]

Thus, \( V(G_t+h,t+h)e^{-\rho h} = V(G_t,t) - \rho hV(G_t,t) + V_0(G_t,t)[g-R(G_t)]h + V_t(G_t,t)h + o(h) \).

Collecting terms, dividing by \( h \), and letting \( h \) approach zero, yields (4.2).

This is the familiar transversality condition for the problem

\[ \max_{\tau} \{ V(0,0;X_p,\tau) \}; \text{ see, e.g., Kamien and Schwartz (1980, p. 147)}. \]