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THE BUFFER VALUE OF GROUNDWATER
WITH STOCHASTIC SURFACE WATER SUPPLIES

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When utilized with a stochastic source of surface water for irrigation, groundwater may serve to mitigate fluctuations in the supply of water. The corresponding benefit is the buffer value of groundwater. We show that the buffer value is positive. Numerical studies reveal that its magnitude can be significant. The paper also offers a characterization of groundwater management in such a setting.

1. Introduction

When groundwater is used for irrigation, it supplements a supply of surface water, such as rainfall. Since the surface water supply typically is stochastic, the stock of groundwater serves two purposes: first, it increases the overall supply of water, and second, its time pattern of use can mitigate undesired fluctuations in the supply of surface water. The total value of a groundwater resource will reflect these two purposes. We call the value of groundwater associated with its second role the buffer value of groundwater. Tsur (1990) analyzed the buffer value in a static context; the purpose of this paper is to investigate the buffer value in a dynamic context.

The buffer value is defined as the difference between the maximal value of a stock of groundwater under uncertainty, and its maximal value under certainty where the supply of surface water is stabilized at its mean. While several authors have studied surface and groundwater systems (Burt (1964a-b), Brown and McGuire (1967), Cummings and Burt (1969), Burt and Cummings (1970), Cummings and Winkelman (1970), Young and Bredehoeft (1972), Bredehoeft and Young [1983], among others), in these analyses the buffer value has remained implicit and not studied as a separate entity.

Why should we be interested in the buffer value as a distinct concept? Suppose that a groundwater development project can be implemented at some cost and that a decision-maker wishes to evaluate such a project using a benefit-cost approach. Clearly, determining the maximum value of the groundwater under certainty is much simpler than when uncertainty is incorporated, since in the latter instance one must solve a complex stochastic-dynamic optimization problem. However, if the buffer value is large relative to the overall value of the groundwater resource, use of the simpler certainty approach provides a poor approximation to true benefits, which could seriously bias assessments of groundwater development projects.

We consider two scenarios regarding the information available in the uncertain situation; in one it is supposed that groundwater extraction in each period takes place before the current realization of surface water is known (the ex-ante case), and in the other it is supposed that extraction decisions are made knowing the current surface water supply (the ex-post case). We determine analytically the sign of the buffer values that arise in these two information scenarios as well as their relative magnitudes. We also provide a partial analytical characterization of optimal groundwater extraction under uncertainty, and compare the size of the shadow prices of groundwater and steady-state groundwater stocks in these three situations (certainty, ex-ante, and ex-post).

We find that the ex-ante buffer value is non-negative, and the shadow price of groundwater in the ex-ante case is at least as great as in the certainty case for all levels of groundwater stock. If marginal extraction costs do not depend on the groundwater stock, then these same two conclusions hold for the ex-post scenario. If the aquifer is confined, so there is no recharge, then the ex-post buffer value is at least as great as the ex-ante buffer value. In general, the steady-state stock levels is smaller and the shadow value of water is larger under either uncertainty regime than under certainty. Finally, we present numerical results for a confined aquifer underlying the Negev region in Israel. These results demonstrate that the buffer value can be large; in one example the buffer value is 84% of the total value of the groundwater stock.

2 Formulation of the Problem

Let F(w,x) be an agricultural production function whose arguments are a water input, w, and a vector of other inputs, x. Given fixed prices of output, p, and of all inputs other than water, r, and given the level of water input, let $x^*(w,p,r)$ represent the x value that maximizes $pF(w,x) - r\cdot x$. The water revenue function is thus given by

$$Y(w,p,r) = pF(w,x^*(w,p,r)) - r\cdot x^*(w,p,r).$$

Henceforth p and r are suppressed in the notation and we write Y(w). We impose

Assumption 1: Y(w) is (i) increasing, (ii) strictly concave and (iii) has a non-negative third derivative over the relevant range of water input w.

Part (i) follows from $\partial F/\partial w > 0$; parts (ii) and (iii) imply respectively that the derived demand for water, $Y'^{-1}(z)$, slopes downward and is convex to the origin.

Water can be supplied from surface (rainfall, stream flows) or ground (aquifer) sources. The stochastic supply of surface water at time t is denoted S_t; the series S_t, $t \geq 0$, is an i.i.d. sequence fluctuating about a mean μ. For simplicity and without loss of generality, the supply of surface water is assumed costless. The (known) stock of groundwater at t is denoted by G_t and its rate of extraction is denoted by g_t. Letting $R(G_t)$ be the (deterministic) rate of water recharge into the aquifer, the groundwater stock evolves according to

$$dG_t/dt = \dot{G}_t = R(G_t) - g_t. \tag{2.1}$$

Letting $\bar{G} < \infty$ represent the capacity of the aquifer, we require

Assumption 2: $R(\bar{G}) = 0$, and R(G) is non-increasing and concave for $0 \leq G \leq \bar{G}$.

The cost of extracting groundwater at a rate g is given by z(G)g, where z(G) is the unit cost of groundwater extraction at a stock level G. We assume

Assumption 3: z(G) is non-increasing and convex.

An extraction plan (or simply a plan) consists of g_t and its associated stock G_t, $t \geq 0$; a plan is feasible if these are non-negative. Associated with any plan is the stochastic process of profits $Y(S_t+g_t)-z(G_t)g_t$, $t \geq 0$. The benefit of a plan is the present value of the profits it generates, given by

$$B(g,S) = \int_0^\infty e^{-\rho t}[Y(S_t+g_t)-z(G_t)g_t]dt \tag{2.2}$$

where ρ is the time rate of discount. We seek the feasible plan that maximizes the expected benefit given the available information.

The information available when extraction decisions are made may or may not include the current surface water realization, S_t. Consequently, we analyze two situations corresponding to whether the extraction decision, g_t, is made before *(ex-ante)* or after *(ex-post)* S_t is observed. The former case is referred to as the a-regime and the latter as the p-regime; the corresponding plans are called the a-plan and the p-plan.

Let E^p and E^a denote expectations conditional on the time path of available information in the *ex-post* and *ex-ante* scenarios, respectively. Define

$$V^p(G) = \text{Max}_g\ E^p\{B(g,S)|\text{stock on hand is } G\}, \tag{2.3p}$$

$$V^a(G) = \text{Max}_g\ E^a\{B(g,S)|\text{stock on hand is } G\}. \tag{2.3a}$$

We define $B(g,\mu)$ as the benefit derived in a certain environment where surface water supplies are stable at μ, with the corresponding maximum payoff

$$V^m(G) = \text{Max}_g\ B(g,\mu) \text{ given stock on hand is } G. \tag{2.3m}$$

Plans associated with this regime are called m-plans.

Under Assumptions 1-3 the value functions $V^j(G)$, $j=m,a,p$, exist and are known to be non-decreasing and differentiable in G (see Benveniste and Scheinkman (1979) for the m-plan and Blume *et al.* (1982) for the a- and p-plans). If, in addition, $z(\cdot)$ is a constant, then the value functions $V^j(G)$, $j=m,a,p$, are also concave.

The total value of a stock of groundwater of size G is given by

$$V_G^j(G) = V^j(G) - V^j(0),\ j=m,a,p. \tag{2.4}$$

This is the maximum value that can be attained starting with a stock of G, less the value starting with an empty aquifer. Note that the latter is not zero due to the existence of surface water supplies and recharge. The buffer value of groundwater in the *ex-post* and *ex-ante* scenarios is defined as

$$BV^j(G) = V_G^j(G) - V_G^m(G), \ j=a,p. \tag{2.5}$$

The buffer value measures the extent to which the value of groundwater in the uncertain environment exceeds the corresponding value obtained in the stable environment. It is a measure of the value associated with the role of groundwater as a buffer that mitigates the uncertain fluctuations in the supply of water. These buffer values are the dynamic analogies of the static concepts introduced by Tsur (1990).

3. Optimal Groundwater Management

In this section we attempt to characterize the optimal plans. Utilizing a dynamic programming approach we obtain, using Eqs. (2.3), the Bellman Equations

$$\rho V^m(G) = \text{Max}_g\Big(Y(g+\mu) - [V^{m\prime}(G)+z(G)](g-R(G))\Big) - z(G)R(G). \tag{3.1m}$$

$$\rho V^a(G) = \text{Max}_g\Big(E\{Y(g+S)\} - [V^{a\prime}(G)+z(G)](g-R(G))\Big) - z(G)R(G) \tag{3.1a}$$

and

$$\rho V^p(G) = E\{\text{Max}_g\Big((Y(g+S) - [V^{p\prime}(G)+z(G)][g-R(G)]\Big)\} - z(G)R(G). \tag{3.1p}$$

The derivation of (3.1m) is straightforward. Regarding (3.1a) and (3.1p), they differ in that in the a-regime the expectation is inside the maximization operator, since extraction must be chosen to maximize expected payoff before the current realization of surface water is known. In the p-regime, on the other hand, the maximization is carried out inside the the expectation, since extraction decisions are made after the realizations of surface water supply are observed. Thus, the difference between the value functions V^p and V^a represents the value of a perpetual flow of perfect information regarding the current supply of surface water.

To verify (3.1p) note that

$$V^p(G) = \mathrm{Max}\ E^p\{\int_0^\infty e^{-\rho t}[Y(S_t+g_t)-z(G_t)g_t]dt|G_0=G\}$$
$$= \mathrm{Max}\ E^p\{\int_0^\tau e^{-\rho t}[Y(S_t+g_t)-z(G_t)g_t]dt + \int_\tau^\infty e^{-\rho t}[Y(S_t+g_t)-z(G_t)g_t]dt|G_0=G\}$$
$$= \mathrm{Max}\ E^p\{\int_0^\tau e^{-\rho t}[Y(S_t+g_t)-z(G_t)g_t]dt + e^{-\rho\tau}V^p(G_\tau)\}.$$

For small τ, $\int_0^\tau e^{-\rho t}[Y(S_t+g_t)-z(G_t)g_t]dt = [Y(S+g)-z(G)g]\tau + o(\tau)$, where g and S stand for g_o and S_o and $o(\tau)$ is such that $o(\tau)/\tau \to 0$ as $\tau \to 0$, $V^p(G_\tau) = V^p(G) + V^{p\prime}(G)\dot{G}\cdot\tau + o(\tau)$ and $e^{-\rho\tau} = 1-\rho\tau+o(\tau)$. Inserting these expressions in the above, using (2.1), collecting terms, dividing by τ and letting τ approach zero (from above), while recalling that under E^p the realization of S is known when g is chosen, yields (3.1p). Equation (3.1a) is derived in a similar manner.

Undertaking the maximization on the right-hand sides of (3.1), the optimal plans $g^m(G)$, $g^a(G)$ and $g^p(G,S)$ satisfy the first order necessary conditions

$$\begin{cases} Y'(g^m(G)+\mu) = V^{m\prime}(G)+z(G) & \text{if } Y'(\mu) \geq V^{m\prime}(G)+z(G) \\ g^m(G) = 0 & \text{otherwise} \end{cases}, \tag{3.2m}$$

$$\begin{cases} E\{Y'(g^a(G)+S)\} = V^{a\prime}(G)+z(G) & \text{if } E\{Y'(S)\} \geq V^{a\prime}(G)+z(G) \\ g^a(G) = 0 & \text{otherwise} \end{cases} \tag{3.2a}$$

and

$$\begin{cases} Y'(g^p(G,S)+S) = V^{p\prime}(G) + z(G) & \text{if } Y'(S) \geq V^{p\prime}(G)+z(G) \\ g^p(G,S) = 0 & \text{otherwise} \end{cases}. \tag{3.2p}$$

Here, $V^{j\prime}(G)$, j=m,a,p, is the shadow price of a unit of groundwater when the stock is G; this is the opportunity cost of current extraction in terms of foregone future benefits. The full marginal cost of extraction is $V^{j\prime}(G)+z(G)$. Hence, (3.2) state that if the marginal net payoffs from water use are positive, then groundwater should be extracted up to the point at which marginal benefits equal full marginal costs. Note that $Y'^{-1}(c)$ is the quantity demanded of irrigation water available at a cost of c. If the current realization of surface water is sufficiently large, the demand price for water falls below the cost of groundwater irrigation, $V^{j\prime}(G)+z(G)$, and extraction ceases.

Extraction is deterministic in the m- and a-regimes. For the p-regime, however, future extractions are stochastic, since they depend on the sequence of realizations of S_t. To determine the expected rate of extraction in the p-regime for positive stock levels (the case of zero stock will be discussed below), note, from (3.2p), that $g^p(G,S) = [K^p(G)-S]I(S \le K^p(G))$, where $K^p(G) = Y'^{-1}(V^{p\prime}(G)+z(G))$ is the total water demanded at a cost $V^{p\prime}(G)+z(G)$, and $I(\cdot)$ is the indicator function that takes the value 1 if its argument is true and zero if it is false. Taking the expectation of $g^p(G,S)$ provides

$$g^p(G) \equiv E\{g^p(G,S)\} = H(K^p(G))K^p(G) - \int_0^{K^p(G)} Sh(S)dS$$
$$= H(K^p(G))\Big(K^p(G) - E\{S|S \le K^p(G)\}\Big),$$

where $H(\cdot)$ and $h(\cdot)$ are the distribution and density functions of S, respectively. Thus, the average extraction in the p-regime at stock level G is given by the demand for groundwater had surface water supply been stable at the level $E\{S|S \le K^p(G)\}$ weighted by the probability that $S \le K^p(G)$.

By defining $Y^m(G) = Y(g^m(G)+\mu)$, $Y^a(G) = E\{Y(g^a(G)+S)\}$ and $Y^p(G) = E\{Y(g^p(G,S)+S)\}$, Eqs. (3.1) can be rendered

$$\rho V^j(G) = Y^j(G) - [z(G)+V^{j\prime}(G)][g^j(G)-R(G)] - z(G)R(G), \; j=m,a,p.$$

Differentiating with respect to G, invoking (3.2) and (2.1) provides for G>0:

$$V^{m\prime}(G) = \Big(d[z(G)+V^{m\prime}(G)]/dt - z'(G)R(G)\Big)/(\rho-R'(G)), \tag{3.3m}$$

$$V^{a\prime}(G) = \Big(d[z(G)+V^{a\prime}(G)]/dt - z'(G)R(G)\Big)/(\rho-R'(G)) \tag{3.3a}$$

and

$$V^{p\prime}(G) = \Big(dE\{z(G)+V^{p\prime}(G)\}/dt - z'(G)R(G)\Big)/(\rho-R'(G)). \tag{3.3p}$$

Eqs. (3.3) can be expressed as

$$\rho V^{j\prime}(G) = \dot{V}^{j\prime}(G) - z'(G)g^j + V^{j\prime}(G)R'(G), \; j=m,a,$$
$$\rho V^{p\prime}(G) = E\{\dot{V}^{p\prime}(G)\} - z'(G)g^p + V^{p\prime}(G)R'(G).$$

The left-hand side is the interest accumulated from selling a unit of stock at a price $V^{j\prime}(G)$ and investing the proceeds. The right-hand side is the gain from delaying extraction of this unit. This consists of the change of the selling price of a unit of stock (the average of this in the p-regime), plus the cost saving from being able to extract from a larger stock, plus the value of the lost recharge due to the larger stock. Thus, the above equation represents an arbitrage condition of asset market equilibrium. Note that if extraction costs are constant and recharge is zero, then this equation reduces to the standard Hotelling rule, which states that the percentage rate of change of the shadow price of the stock (the average of this in the p-regime) equals the interest rate.

We now turn to characterization of the steady-sates of the three regimes. The system is in a steady state if extraction (the average of this in the p-regime) just equals recharge. For each regime there are three situations that might arise: in the first, it is not profitable to exploit the aquifer, and the steady-state has zero extraction and a full aquifer; in the second, the aquifer eventually is depleted and the steady-state stock equals zero; and in the third, a steady-state occurs at some intermediate stock level.

Let $\hat{G}^j$ denote the steady state stock levels satisfying $g^j(\hat{G}^j) = R(\hat{G}^j)$, $j=m,a,p$. In case of multiplicity, $\hat{G}^j$ is the largest stock level under which the recharge and extraction rates are equal. It is straightforward to verify that $\hat{G}^j$, $j=m,a,p$, so defined is locally stable, i.e. a small perturbation causes the stock to move back toward the steady state. Under constant $z(\cdot)$, so that $V^j(G)$ is concave, the steady state is unique.

Our discussion of the steady states is facilitated by defining

$$J(G) = -z'(G)R(G)/[\rho-R'(G)]. \tag{3.4}$$

Evaluating Eqs. (3.3) at a steady state provides

$$V^{j\prime}(\hat{G}^j) = J(\hat{G}^j) \text{ provided } \hat{G}^j > 0,\ j=m,a,p. \tag{3.5}$$

The function $J(G)$ is readily interpreted; the left hand side of (3.5) is the shadow value of the stock, while the right-hand side is the present value of the cost savings of extraction from a larger stock, where the extraction equals the recharge (on average for the p-regime), and the discount rate has been adjusted upward by the amount $R'(G)$ to account for the penalty from lower recharge.

No exploitation

In this first instance we have a boundary steady-state with no extraction (ever) and $\hat{G} = \bar{G}$. Of course, since there is no extraction, $V^j(\bar{G}) = V^j(0)$ and $V^{j\prime}(\bar{G}) = 0$, $j=m,a,p$ (there are no fixed costs in the model). In the m-regime, this occurs when the value of marginal productivity of water (VMP) at zero extraction, $Y'(\mu)$, is no greater than the minimal unit extraction cost $z(\bar{G})$. If the initial stock is less than $\bar{G}$, clearly costs are higher still, and it does not pay to extract.

Similarly for the a-regime, the aquifer will not be exploited if the expected VMP when surface water alone is used does not exceed $z(\bar{G})$. Note that, by the convexity of Y', $E\{Y'(S)\} > Y'(\mu)$, so that it is possible that the aquifer is not exploited under certainty but is exploited under uncertainty.

Finally, regarding the p-regime, it is sure that the aquifer will not be exploited if the largest value that the VMP can take on falls short of the minimal unit extraction cost. This occurs when $Y'(S_L) < z(\bar{G})$, where S_L is the lower support of the distribution of S. Since $Y'(S_L) \geq Y'(S)$ for all feasible values of S, there are situations in which the aquifer is not exploited in the a-regime, but is exploited in the p-regime, due, of course, to the value of information regarding the sequence of surface water supplies. The case of no exploitation is depicted in Figure 1.

Figure 1

Eventual Depletion

Here we have a boundary steady-state at $\hat{G} = 0$. This case will arise when the VMP from surface supplies plus groundwater extractions (which equals the recharge at a zero stock) exceeds the unit extraction cost at zero stock, plus the value of leaving a little more stock in the steady-state, which we have seen to be given by $J(0)$. Since extraction equals the recharge rate, the first-order conditions (3.2) must hold, implying that the steady-state shadow prices in the m-regime and in the a-regime are given by $\hat{V}^{m\prime}(0) = Y'(R(0)+\mu) - z(0)$ and $\hat{V}^{a\prime}(0) = E\{Y'(R(0)+S)\} - z(0)$. These shadow prices must not fall short of $J(0)$, for otherwise a zero stock could not be a steady state. Thus, in the m-regime, we have eventual depletion if $Y'(R(0)+\mu) > z(0)+J(0)$; similarly, depletion occurs in the a-regime if $E\{Y'(R(0)+S)\} > z(0)+J(0)$. Since $Y'(\cdot)$ is convex and $J(\cdot)$ is independent of the regime, eventual depletion is more likely in the a-regime than it is in the m-regime.

Regarding the p-regime, we need to determine $\hat{V}^{p\prime}(0)$ under which the average extraction just equals the recharge rate at a zero stock. With an empty aquifer, the maximization in (3.1p) is carried out subject to the constraint that $g \le R(0)$; it yields the extraction rule:

$$\begin{aligned} g^p(0,S) &= K_0^p - S \quad \text{if } K_0^p - R(0) \le S < K_0^p \\ &= R(0) \quad \text{if } S \le K_0^p - R(0) \\ &= 0 \quad \text{if } S \ge K_0^p, \end{aligned} \tag{3.6}$$

where $K_0^p = Y'^{-1}(z(0)+\hat{V}^{p\prime}(0))$. Thus,

$$\begin{aligned} g^p(0) = E\{g^p(0,S)\} &= \left(K_0^p - E\{S|K_0^p-R(0)\le S<K_0^p\}\right)\Pr\{K_0^p-R(0)\le S<K_0^p\} + \\ &\qquad R(0)\cdot\Pr\{S\le K_0^p-R(0)\} \\ &= K_0^p\left(H(K_0^p) - H(K_0^p-R(0))\right) - \int_{K_0^p-R(0)}^{K_0^p} Sh(S)dS + R(0)\cdot H(K_0^p-R(0)), \end{aligned}$$

where K_0^p is determined so as to equate average extraction with recharge, i.e.,

$$K_0^p\left(H(K_0^p) - H(K_0^p-R(0))\right) - \int_{K_0^p-R(0)}^{K_0^p} Sh(S)dS + R(0)\cdot H(K_0^p-R(0)) = R(0), \qquad (3.7)$$

and $\hat{V}^{p\prime}(0) = Y'(K_0^p)-z(0)$.

As depicted in Figure 2, $\hat{V}^{a\prime}(0) \geq \hat{V}^{m\prime}(0)$ since, due to the convexity of $Y'(\cdot)$, $E\{Y'(R(0)+S)\} \geq Y'(R(0)+\mu)$. Likewise, $\hat{V}^{p\prime}(0) \geq \hat{V}^{m\prime}(0)$. To verify this, note, from (3.6), that

$$\begin{aligned} g^p(0,S) &= (K_0^p-S)I(K_0^p-R(0)\leq S<K_0^p) + R(0)I(S\leq K_0^p-R(0)) \\ &= \left((K_0^p-S)I(K_0^p-R(0)\leq S) + R(0)I(S\leq K_0^p-R(0))\right)I(S<K_0^p) \\ &= \left((K_0^p-S) + [R(0)-(K_0^p-S)]I(S\leq K_0^p-R(0))\right)I(S<K_0^p) \\ &\geq (K_0^p-S)I(S<K_0^p) \\ &\geq K_0^p-S. \end{aligned}$$

Taking expected values on both sides, noting that $E\{g^p(0,S)\} = R(0)$, yields $R(0)\geq K_0^p-\mu$ and establishes the result. The sign of $\hat{V}^{a\prime}(0)-\hat{V}^{p\prime}(0)$ is indeterminate. Note that the conditions under which the aquifer is depleted imply $\hat{V}^{j\prime}(0) \geq J(0)$, $j=m,a,p$.

Figure 2

An interior steady-state

Using (3.5), $\hat{G}^m$ and $\hat{G}^a$ are the stock levels satisfying $Y'(R(\hat{G}^m)+\mu) = z(\hat{G}^m)+J(\hat{G}^m)$ and $E\{Y(R(\hat{G}^a)+S) = z(\hat{G}^a)+J(\hat{G}^a)$. Similarly, $\hat{G}^p$ is the root of

$$K(G)H(K(G)) - \int_0^{K(G)} Sh(S)dS = R(G), \qquad (3.8)$$

where $K(G) = Y'^{-1}(z(G)+J(G))$. Note that the left-hand side of (3.8) is average extractions from a non-empty aquifer in the p-regime.

We summarize the above in:

Proposition 1: Provided Assumptions 1-3 hold:

(m) Either (i) $Y'(R(0)+\mu) \geq z(0)+J(0)$ or (ii) $Y'(\mu) < z(\bar{G})$ or (iii) the equation $Y'(R(G)+\mu) = z(G)+J(G)$ admits a solution on $[0,\bar{G}]$. In Case (i), $\hat{G}^m = 0$ and $V^{m\prime}(\hat{G}^m) = Y'(R(0)+\mu)-z(0)$; in Case (ii), $\hat{G}^m = \bar{G}$ and $V^{m\prime}(\hat{G}^m) = 0$, i.e., the aquifer does not admit profitable extractions; in Case (iii), $\hat{G}^m$ is the root of $Y'(R(G)+\mu) = z(G)+J(G)$ and $V^{m\prime}(\hat{G}^m) = J(\hat{G}^m)$.

(a) Either (i) $E\{Y'(R(0)+S)\} \geq z(0)+J(0)$ or (ii) $E\{Y'(S)\} < z(\bar{G})$ or (iii) the equation $E\{Y'(R(G)+S)\} = z(G)+J(G)$ admits a solution on $[0,\bar{G}]$. In Case (i), $\hat{G}^a = 0$ and $V^{a\prime}(\hat{G}^a) = E\{Y'(R(0)+S)\}-z(0)$; in Case (ii), $\hat{G}^a = \bar{G}$ and $V^{a\prime}(\hat{G}^a) = 0$, i.e., the aquifer does not admit profitable extractions; in Case (iii) $\hat{G}^a$, is the root of $E\{Y'(R(G)+S)\} = z(G)+J(G)$ and $V^{a\prime}(\hat{G}^a) = J(\hat{G}^a)$.

(p) Either (i) Eq. (3.8) does not admit a positive solution or (ii) $Y'(S_L) < z(\bar{G})$, where $S_L \geq 0$ is the lower support of S, or (iii) Eq. (3.8) admits a solution on $[0,\bar{G}]$. In Case (i), $\hat{G}^p = 0$ and $V^{p\prime}(\hat{G}^p) = Y'(K_0^p)-z(0)$, where K_0^p is defined in Eq. (3.7); in Case (ii), $\hat{G}^p = \bar{G}$ and $V^{p\prime}(\hat{G}^p) = 0$, i.e., the aquifer does not admit profitable extractions; in Case (iii), $\hat{G}^p$ is set equal to the solution of (3.8) and $V^{p\prime}(\hat{G}^p) = J(\hat{G}^p)$.

Based on this discussion, the boundary (steady-state) conditions of the three regimes can be summarized as:

$$\hat{G}^j[V^{j\prime}(\hat{G}^j) - J(\hat{G}^j)] = 0,\ j=m,a,p;\quad \hat{V}^{m\prime}(0) = Y'(R(0)+\mu)-z(0);$$
$$\hat{V}^{a\prime}(0) = E\{Y'(R(0)+S)\}-z(0);\ \text{and}\ \hat{V}^{p\prime}(0) = Y'(K_0^p)-z(0), \tag{3.9}$$

where K_0^p is the root of Eq. (3.7). Conditions (3.2) and (3.9) are necessary for the optimal extraction plans.

One notes that Eqs. (3.3) hold along the optimal plans for all t such that $G_t^j>0$, where G_t^j is the optimal stock process, j=m,a,p; as such they determine, using (2.1), differential equations which, together with the boundary conditions (3.9), can be solved (perhaps only numerically) to yield the optimal plans (see the example in Section 5 for analytical solutions).

We conclude our account of the optimal management rules with the following:

Proposition 2: Under Assumptions 1-3,

$$\hat{G}^j \le \hat{G}^m \text{ and } V^{j\prime}(\hat{G}^j) \ge V^{m\prime}(\hat{G}^j),\ j=a,p. \tag{3.10}$$

The proof is given in the appendix. Thus, at the steady state, the stock levels in the uncertain regime fall short of the steady state stock of the certain regime and the unit (marginal) stock values in the uncertain regimes are at least as big as that of the stable regime.

4. The Buffer Value

The buffer value of groundwater (see Eqs. (2.4)-(2.5)) entails comparisons between the uncertain and the certain regimes. Our first result concerns the relationship between the shadow prices of the m- and a-plan.

Proposition 3: Under Assumptions 1-3,

$$V^{a\prime}(G) \ge V^{m\prime}(G) \text{ for all } G \ge 0.$$

The proof utilizes

Lemma 1: Under Assumptions 1-3,

$$V^{a\prime}(G) < V^{m\prime}(G) \text{ implies } V^{a\prime}(\hat{G}^a) < V^{m\prime}(\hat{G}^m).$$

The proof of the Lemma is given in the appendix. Lemma 1 implies that $V^{a\prime}(G) < V^{m\prime}(G)$ contradicts (3.10), and proves Proposition 3. ■

Proposition 3 states that the shadow price of groundwater in the a-regime does not fall short of the shadow price in the m-regime, and that this property holds for all stock levels. We thus expect that the same relation holds for the entire stock of groundwater. This is verified in

Proposition 4: Under Assumptions 1-3,

$$BV^a(G) \ge 0 \text{ for all } G \ge 0;$$

Proof: Proposition 3, recalling (2.4) and (2.5), implies $BV^{a\prime}(G) = V^{a\prime}(G) - V^{m\prime}(G) \ge 0$ for all $G \ge 0$. Noting that $BV^a(0) = 0$, provides $BV^a(G) = \int_0^G BV^{a\prime}(u)du \ge 0$, as asserted. ■

In the special case of a non-replenishable aquifer, i.e., when $R(G) = 0$ for all G, the relation $V^p(0) = V^a(0)$ holds. Thus, $BV^p(G) - BV^a(G) = V^p(G) - V^a(G)$. Because g_t^a is feasible in the p-regime, $V^p(G) \geq V^a(G)$, i.e., the current value of information regarding S_t is non-negative, and we have

Corollary 1: Under assumptions 1 and 3, and if $R(G) = 0$ for all G, then

$$BV^p(G) \geq BV^a(G) \geq 0 \text{ for all } G \geq 0.$$

With a positive recharge, $V^p(0)$ may exceed $V^a(0)$ and not much can be said about the relation between BV^p and BV^a without imposing more structure. A particular case of interest occurs when the unit extraction cost z(G) is constant.

Proposition 5: Provided Assumptions 1-2 hold and if z(G) is constant, then:

(i) $V^{p\prime}(G) \geq V^{m\prime}(G)$ for all $G \geq 0$;

(ii) $BV^p(G) \geq 0$.

Proof: (i) Note first that under constant z:

$$V^p(G) = E\left\{\int_0^\infty e^{-\rho t}[Y(g_t^p+S_t)-zg_t^p]dt\right\}$$

$$< \int_0^\infty e^{-\rho t}[Y(\bar{g}_t^p+\mu)-z\bar{g}_t^p]dt$$

$$\leq \int_0^\infty e^{-\rho t}[Y(g_t^m+\mu)-zg_t^m]dt = V^m(G), \tag{4.1}$$

where $\bar{g}_t^p = E\{g(G_t^p,S_t)\}$ is the average extraction path in the p-regime. The first inequality follows from the strict concavity of $Y(\cdot)$. The second inequality holds since $\bar{g}_t^p$ is feasible in the m-regime.

First note that $V^{m\prime}(G) = 0$ whenever $z > Y'(R(0)+\mu)$. To see this, note that a constant z implies $J(G) = 0$ identically for all G. Now $z > Y'(R(0)+\mu)$ corresponds to Cases (ii) or (iii) of Proposition 1(m), in which a positive steady-state prevails. Proposition 1(m) requires, in such cases, that $V^{m\prime}(\hat{G}^m) = J(\hat{G}^m)$, which we have seen above to vanish. Furthermore, with constant z, $V^m(G)$ is concave, thus $V^{m\prime}(G)$ must vanish for all G. This has an obvious intuition: if the stock does not affect the extraction costs and the recharge rate is always sufficient to satisfy the extraction requirements, then changes in the stock level do not matter.

We therefore need to consider only $z \le Y'(R(0)+\mu)$, in which case $\hat{G}^m = 0$ and $g^m(G_t^m) \ge R(G_t^m)$ for all t. We now show that $V^{m\prime}(G) > V^{p\prime}(G)$ implies $V^p(G) > V^m(G)$ which contradicts (4.1). Suppose $V^{m\prime}(G) > V^{p\prime}(G)$, then the strict concavity of $Y(\cdot)$ implies (see Figure 3) $K^p(G) = Y'^{-1}(V^{p\prime}(G)+z) > Y'^{-1}(V^{m\prime}(G)+z) = K^m(G)$. From Eq. (3.2p), $g^p(G,S) = [K^p(G)-S]I(S \le K^p(G))$ so that $g^p(G,S) \le K^p(G)-S$. Thus, $g^p(G) = E\{g^p(G,S)\} \le K^p(G)-\mu$. From (3.2m), $g^m(G)+\mu = K^p(G)$. We therefore have, using Eqs. (3.1m,p),

$$\rho V^p(G) \ge Y(K^p(G)) - [z+V^{p\prime}(G)](K^p(G)-\mu) + V^{p\prime}(G)R(G)$$

and

$$\rho V^m(G) = Y(K^m(G)) - [z+V^{m\prime}(G)](K^m(G)-\mu) + V^{m\prime}(G)R(G).$$

With the aid of Figure 3, $Y(K^p(G)) - [z+V^{p\prime}(G)](K^p(G)-\mu)$ is given by the area $\{abce\mu 0\}$ and $Y(K^m(G)) - [z+V^{m\prime}(G)](K^m(G)-\mu)$ equals the area $\{abd\mu 0\}$. Likewise, since $g^m(G) \ge R(G)$, $[V^{m\prime}(G)-V^{p\prime}(G)]R(G) \le [V^{m\prime}(G)-V^{p\prime}(G)]g^m(G) = [V^{m\prime}(G)-V^{p\prime}(G)](K^mG)-\mu)$ = area $\{dbfe\}$. Thus

$$\rho[V^p(G) - V^m(G)] \ge \text{area}\{bcf\} > 0$$

which contradicts (4.1) and establishes (i).

Figure 3

(ii) Using $BV^{p\prime}(G) = V^{p\prime}(G)-V^{m\prime}(G) \ge 0$ for all $G \ge 0$ and noting $BV^p(0) = 0$, provides $BV^p(G) = \int_0^G BV^{p\prime}(u)du \ge 0$ and completes the proof of the proposition. ■

The presence of a positive buffer value implies that groundwater is worth more in uncertain environments than in stable ones, the difference being the buffer value. This has immediate implications regarding the development of groundwater resources. The significance of these policy implications depends on the magnitude of the buffer value. In the next section we investigate this magnitude by means of a numerical example and find that it can be substantial.

5. An Illustrative example

We adopt the case studied by Tsur (1990) and evaluate the buffer value to wheat growers of the fossil aquifer underlying the northern Israeli Negev region. The recharge rate of the aquifer is negligible, thus only the *ex-ante* buffer values are derived; by Corollary 1, the *ex-post* buffer value is at least as big as the *ex-ante* one. We assume constant extraction costs, which enables analytical solutions for the forms of the optimal plans.

With constant z and no recharge, (3.3m,a) become

$$\rho V^a(G) = E\{Y(g^a(G)+S)\} - [V^{a\prime}(G)+z]g^a(G)$$

and

$$\rho V^m(G) = Y(g^m(G)+\mu) - [V^{m\prime}(G)+z]g^m(G).$$

Taking derivatives with respect to G, using (3.2m,a), yields $-g^j(G)\cdot V^{j\prime\prime}(G)/V^{j\prime}(G) = \rho$, which, using (2.1), leads to $d(\log V^{j\prime}(G_t^j))/dt = \rho$, $j=a,m$. Thus, the shadow price processes of the a- and m-plans are given by

$$V^{a\prime}(G_t^a) = C^a e^{\rho t} \quad \text{and} \quad V^{m\prime}(G_t^m) = C^m e^{\rho t}, \tag{5.1}$$

where $C^j = V^{j\prime}(G)$, $j=a,m$, and G is the initial stock. A particular value of C^j determines the entire time path of shadow price process, $V^{j\prime}(\cdot)$, and hence the time path of the extraction plan g^j via (3.2j), $j=a,m$. If $z \geq E\{Y'(S)\}$, then no profitable extraction is possible and $C^j = V^{j\prime}(G) = 0$ (note that $E\{Y'(S)\} \geq Y'(\mu)$), i.e., the groundwater stock is of no value in this case (see Proposition 1). We consider the case $z < Y'(\mu)$; thus $C^j > 0$, $j=a,m$, and the shadow price processes increase exponentially in time until the depletion of the aquifer. Conditions (3.2m,a) and the strict concavity of $Y(\cdot)$ require that the optimal extraction plans g_t^a and g_t^m diminish with time and vanish at T^j, $j=a,m$. These end dates satisfy (cf. (3.2a,m))

$$E\{Y'(S)\} = z + C^a e^{\rho T^a} \quad \text{and} \quad Y'(\mu) = z + C^m e^{\rho T^m}. \tag{5.2}$$

With z below $Y'(\mu)$, both plans must exhaust the entire stock. Thus

$$\int_0^{T^j} g_t^j dt = G, \quad j=a,m. \tag{5.3}$$

The parameters $C^j = V^{j\prime}(G)$ and T^j, j=a,m, can now be determined from (5.2) and (5.3). This, in turn, determines the time path of the shadow price processes (cf. (5.1)) and, using (3.2m,a) the extraction plans.

We shall assume that surface water supplies, S, are distributed uniformly over the interval $[\mu-\lambda,\mu+\lambda]$, thus λ represents the variability in the supply of surface water (in particular, $\mathrm{Var}\{S\} = \lambda^2/3$). The water response function takes the form

$$Y(w) = \begin{cases} \alpha - \beta/w & \text{if } w \geq \beta/\alpha \\ 0 & \text{if } w \leq \beta/\alpha \end{cases} . \tag{5.4}$$

The case where water input can fall within the vanishing region of Y, i.e., in the interval $[0,\beta/\alpha]$, requires special treatment (see Tsur [1990]). We shall avoid this difficulty by requiring $\mu-\lambda \geq \beta/\alpha$ so that $\Pr\{S<\beta/\alpha\} = 0$.

The m-plan: Using (3.2m), (5.2) and (5.4) we obtain

$$g_t^m = (\beta/[z+C^m e^{\rho t}])^{1/2} - \mu.$$

From $g_t^m = 0$ at $t = T^m$, it follows that

$$\rho T^m(C^m) = \log\left(\beta/\mu^2 - z\right) - \log(C^m).$$

Defining $u = (z+C^m e^{\rho t})^{1/2}$, g_t^m is integrated (over time) to yield

$$G = \frac{\sqrt{\beta}}{\rho}\,\frac{1}{\sqrt{z}}\left(\log\frac{u-\sqrt{z}}{u+\sqrt{z}}\right)_{u=(z+C^{\mu})^{1/2}}^{u=\sqrt{\beta/\mu}} - \mu T^m(C^m)$$

$$= \frac{\sqrt{\beta}}{\rho}\,\frac{1}{\sqrt{z}}\left(\log\frac{\sqrt{\beta/\mu}-\sqrt{z}}{\sqrt{\beta/\mu}+\sqrt{z}} - \log\frac{(z+C^m)^{1/2}-\sqrt{z}}{(z+C^m)^{1/2}+\sqrt{z}}\right) - \mu T^m(C^m),$$

from which C^m is extracted numerically, given the values of β, μ, ρ and z. Consequently, $g_o^m = (\beta/(z+C^m))^{1/2} - \mu$ is evaluated. Using

$$\rho V^m(G) = \alpha - \beta/(g_o^m+\mu) - g_o^m\beta/(g_o^m+\mu)^2 \text{ and } \rho V^m(0) = \alpha - \beta/\mu$$

(cf. (3.1m) and (3.3m)) the groundwater value function $V_G^m(G) = V^m(G) - V^m(0)$ is calculated.

The a-plan. With S distributed uniformly over the interval $[\mu-\lambda,\mu+\lambda]$,

$$E\{Y'(g_t^a+S)\} = E\{\beta/(g_t^a+S)^2\} = \beta/((g_t^a+\mu)^2-\lambda^2),$$

which, using (3.2a) and (5.1), gives

$$g_t^a = \left(\lambda^2 + \frac{\beta}{z+C^a e^{\rho t}}\right)^{1/2} - \mu.$$

The condition $g_t^a = 0$ at $t=T^a$ induces

$$\rho T^a(C^a) = \log\left(\frac{\beta}{\mu^2-\lambda^2} - z\right) - \log(C^a). \tag{5.5}$$

Note that $T^a=\infty$ only when $\mu=\lambda$, i.e., when the lower support of S is zero. C^a is found as the root of

$$G = \frac{2\beta}{\rho z}[H(C^a) - J(C^a)] - (\mu-\lambda)T^a(C^a), \tag{5.6}$$

where $H(C^a)$ and $J(C^a)$ are defined in Appendix B and $T(C^a)$ in (5.5). Given C^a, we calculate

$g_o^a = (\lambda^2+\beta/(z+C^a))^{1/2}-\mu$ and $\rho V^a(G) = \alpha - \frac{\beta}{2\lambda}\log\frac{g_o^a+\mu+\lambda}{g_o^a+\mu-\lambda} - (z+C^a)g_o^a$. Using $\rho V^a(0) = \alpha - \frac{\beta}{2\lambda}\log\frac{\mu+\lambda}{\mu-\lambda}$, we obtain $\rho V_G^a(G) = \rho[V^a(G)-V^a(0)] = \frac{\beta}{2\lambda}[\log\frac{\mu+\lambda}{\mu-\lambda} - \log\frac{g_o^a+\mu+\lambda}{g_o^a+\mu-\lambda}] - (z+C^a)g_o^a$ from which $BV^a(G) = V_G^a(G)-V_G^m(G)$ is derived.

The following data are employed (see Tsur, 1990): output (wheat) price is \$0.15/kg, the parameters α and β take the values 545.86 and 857484.12, respectively, with $\beta/\alpha = 1570.9$; the time rate of discount is $\rho = 0.1$ and the mean of surface water supplies is $\mu = 3000$ m^3/ha (1 mm rainfall is equivalent to 10 m^3/ha). Table 1 presents values of $V_G^j(G)$, $j=m,a$, $BV^a(G)$ and $BV^a(G)/V_G^a(G)$ for different scenarios characterized by levels of z, G and λ. All units are per hectare. There are about 77551 ha of arable land in the region under consideration; thus total values are determined by multiplying each item of Table 1 by 77551. The extraction paths of the m- and a-plans corresponding to Cases 1 and 4 of Table 1 are depicted in Figures 4 and 5.

With small variability in surface water ($\lambda = 500$) and relatively large aquifer stock (G = 20 billion m^3) the m- and a-plans almost coincide (see Figure 4). The buffer value comprises 5 percent of the value of groundwater in this case. With high variability in the supply of surface water ($\lambda = 1500$) and a smaller aquifer (G = 5 billion m^3) the m- and a-plans depart from each other quite substantially (Figure 5) and the buffer value accounts for 84 percent of the value of groundwater.

Table 1

Figure 4 Figure 5

6. Conclusion

In this paper we have examined the buffer value of a stock of groundwater in a dynamic context. While the details of the model are specific to groundwater with deterministic recharge, and a surface water supply with a stationary probability distribution, the basic idea underlying our investigation has broad applicability.

First, the analysis could be applied with only slight modification to any deterministic stock of water besides one held in an aquifer. Thus, the evaluation of the development of any source of water to supplement irrigation (e.g. desalinization, or an interbasin water transfer) should incorporate the buffer value of the supplementary stock.

Second, it is apparent that the definition of the buffer value, which we base on value functions for an intertemporal optimization problem, could be applied almost independently of the specific form of that problem. Thus, incorporation of stochastic recharge, or a non-stationary surface water distribution requires no modification of the definition of the buffer value, based on value functions for problems with those elements. Similarly, our assumptions of a fixed prices, of constant average costs of groundwater extraction, and of risk neutrality could be dropped. However, obtaining results regarding the sign and relative magnitudes of the buffer values may be more difficult in these more complex settings.

Appendix A: *Proofs*

Notation: The superscript j stands for m, a and p unless otherwise indicated; G_t^j is the stock process associated with $g_t^j = g^j(G_t^j)$, j=m,a, and G_t^p with $g_t^p = g^p(G_t^p,S_t)$; G is the initial stock, i.e., $G_0^j = G$; $V_t^{j\prime} = V^{j\prime}(G_t^j)$; $\dot{V}_t^{j\prime} = dV_t^{j\prime}/dt$; $z_t^j = z(G_t^j)$; $Y^p(G) = E\{Y(g^p(G,S)+S)\}$; $Y^a(G) = E\{Y(g^a(G)+S)\}$; $Y^m(G) = Y(g^m(G)+\mu)\}$; a ^ over a variable indicates its steady state level; T^j is the time at which the steady state is approached; $K(x) = Y'^{-1}(z(x)+J(x))$.

Assumptions 1-3 guarantee that $V_t^j = V^j(G_t^j)$, $g_t^j = g^j(G_t^j)$, j=m,a, and $\bar{g}_t^p = g^p(G_t^p)$ are differentiable in t (see Blume *et al.* (1982), Fleming and Rishel (1975, p. 8)). The proof of Lemma 1 makes use of this property.

Proposition 2: Under Assumptions 1-3,

$$\hat{G}^j \le \hat{G}^m \text{ and } V^{j\prime}(\hat{G}^j) \ge V^{m\prime}(\hat{G}^j),\ j=a,p.$$

Proof: (i) $\hat{G}^a \le \hat{G}^m$: This trivially holds if $\hat{G}^a = \hat{G}^m = 0$. The case $\hat{G}^a > 0$ and $\hat{G}^m = 0$ is impossible since it implies, cf. Proposition 1, $Y'(R(0)+\mu) \ge z(0)+J(0) > E\{Y'(R(0)+S)\}$ and violates the convexity of $Y'(\cdot)$. To verify the claim when both $\hat{G}^a$ and $\hat{G}^m$ are positive, suppose the contrary holds, i.e., $\hat{G}^a > \hat{G}^m > 0$. Then, since $J(\cdot)$ is non-increasing, $z(\hat{G}^m)+V^{m\prime}(\hat{G}^m) = z(\hat{G}^m)+J(\hat{G}^m) \ge z(\hat{G}^a)+J(\hat{G}^a) = z(\hat{G}^a)+V^{a\prime}(\hat{G}^a)$, implying that $Y'(R(\hat{G}^m)+\mu) \ge E\{Y'(R(\hat{G}^a)+S)\}$ which, by virtue of the convexity of $Y'(\cdot)$ and $R'(\cdot) \le 0$, requires $\hat{G}^a \le \hat{G}^m$. Thus the possibility $\hat{G}^a > \hat{G}^m$ is ruled out.

(ii) $\hat{G}^p \le \hat{G}^m$: To rule out the case $\hat{G}^p > 0$ and $\hat{G}^m = 0$ note that it entails $V^{p\prime}(\hat{G}^p) = J(\hat{G}^p)$, $V^{m\prime}(0) \ge J(0)$ and hence (since $J(\cdot)$ is non-increasing) $V^{p\prime}(\hat{G}^p) \le V^{m\prime}(0)$. This, in turn, requires that $z(\hat{G}^p)+V^{p\prime}(\hat{G}^p) \le z(0)+V^{m\prime}(0)$ and, by virtue of the concavity of $Y(\cdot)$, that

$$K(\hat{G}^p) = Y'^{-1}(z(\hat{G}^p)+J(\hat{G}^p)) \ge Y'^{-1}(z(0)+V^{m\prime}(0)) = R(0)+\mu,$$

where equality holds if and only if $z(\cdot)$ is constant over the interval $[0,\hat{G}^p]$. But if $z(\cdot)$ is constant over this interval, then $\hat{G}^p > 0$ cannot be optimal (it is easy to see that it pays to deplete the aquifer under such circumstances). Thus, if $\hat{G}^p>0$ it must be that $z(0) > z(\hat{G}^p)$ and the above relation holds with strict inequality, i.e., $K(\hat{G}^p) > R(0)+\mu$. Now, $K(\hat{G}^p)$ satisfies $E\{[K(\hat{G}^p)-S]I(S\leq K(\hat{G}^p))\} = R(\hat{G}^p)$ and hence, noting that $[K(\hat{G}^p)-S]I(S>K(\hat{G}^p)) \leq 0$,

$$R(\hat{G}^p) = E\{[K(\hat{G}^p)-S]I(S\leq K(\hat{G}^p))\} \geq E\{[K(\hat{G}^p)-S]I(S\leq K(\hat{G}^p)) + [K(\hat{G}^p)-S]I(S>K(\hat{G}^p))\}$$
$$= K(\hat{G}^p) - \mu > R(0),$$

which violates Assumption 2 and rules out the case $\hat{G}^p > 0$ and $\hat{G}^m = 0$. To eliminate the case $\hat{G}^p > \hat{G}^m > 0$, note that this case can prevail only if $z(\hat{G}^p) < z(\hat{G}^m)$ and that $V^{p\prime}(\hat{G}^p) = J(\hat{G}^p) \leq J(\hat{G}^m) = V^{m\prime}(\hat{G}^m)$. Thus, the concavity of $Y(\cdot)$ requires $K(\hat{G}^p) = Y'^{-1}(z(\hat{G}^p)+J(\hat{G}^p)) > Y'^{-1}(z(\hat{G}^m)+J(\hat{G}^m)) = R(\hat{G}^m)+\mu$, implying that $R(\hat{G}^p) \geq K(\hat{G}^p)-\mu > R(\hat{G}^m)$ and violating $R'(\cdot) \leq 0$.

(iii) $V^{j\prime}(\hat{G}^j) \geq V^{m\prime}(\hat{G}^j)$, $j=a,p$: Because $J(\cdot)$ is non-increasing, $V^{a\prime}(\hat{G}^a) \geq V^{m\prime}(\hat{G}^m)$ and $V^{p\prime}(\hat{G}^p) \geq V^{m\prime}(\hat{G}^m)$ whenever $\hat{G}^a > 0$ and $\hat{G}^p > 0$, respectively. Furthermore, if $\hat{G}^a = 0$ and $\hat{G}^m > 0$ then $V^{a\prime}(0) \geq J(0) \geq J(\hat{G}^m) = V^{m\prime}(\hat{G}^m)$. likewise $\hat{G}^p = 0$ and $\hat{G}^m > 0$ entails $V^{p\prime}(0) \geq J(0) \geq J(\hat{G}^m) = V^{m\prime}(\hat{G}^m)$. The case $\hat{G}^a = \hat{G}^m = 0$ entails $V^{a\prime}(0) = E\{Y'(R(0)+S)\}-z(0) \geq Y'(R(0)+\mu)-z(0) = V^{m\prime}(0)$. Finally, $\hat{G}^p = \hat{G}^m = 0$ involves $R(0) \geq K_0^p-\mu$ (cf. the discussion following Eq. (3.7)), implying that $V^{p\prime}(0) = Y'(K_0^p) - z(0) \geq Y'(R(0)+\mu) - z(0) = V^{m\prime}(0)$. ■

Lemma 1: Under Assumptions 1-3,

$$V^{a\prime}(G) < V^{m\prime}(G) \text{ implies } V^{a\prime}(\hat{G}^a) < V^{m\prime}(\hat{G}^m).$$

Proof: Since (3.3m and a) hold along the G_t^m and G_t^a paths, respectively, as long as both stocks are positive, the relation

$$d[V_t^{m\prime}+z_t^m-(V_t^{a\prime}+z_t^a)]/dt = V_t^{m\prime}[\rho-R'(G_t^m)] - V_t^{a\prime}[\rho-R'(G_t^a)] + z'(G_t^m)R(G_t^m) - z'(G_t^a)R(G_t^a) \quad \text{(A1)}$$

holds for all t such that both G_t^m and G_t^p are positive. At t=0, noting that $G_o^m = G_o^a = G$, the right hand side of (A1) equals $[V_o^{m\prime}-V_o^{a\prime}](\rho-R'(G))$. Thus, $V^{m\prime}(G)-V^{a\prime}(G) = \delta > 0$ implies

$$d[V_o^{m\prime}+z_o^m]/dt - d[V_o^{a\prime}+z_o^a]/dt \geq \rho\delta > 0. \tag{A2}$$

Likewise, $V^{a\prime}(G) < V^{m\prime}(G)$ requires (cf. (3.2a and m)) $E\{Y'(g^a(G)+S)\} < Y'(g^m(G)+\mu)$, which together with the convexity of $Y'(\cdot)$ and the concavity of $Y(\cdot)$ imply $g^a(G) > g^m(G)$. Thus, $V_o^{m\prime}+z_o^m - (V_o^{a\prime}+z_o^a) = \delta > 0$, $d[V_o^{m\prime}+z_o^m-(V_o^{a\prime}+z_o^a)]/dt \geq \rho\delta > 0$ and $g_o^a > g_o^m$. By time-continuity these relations hold during some positive time interval $[0,\tau]$. At $t = \tau$, therefore, $V_\tau^{m\prime}+z_\tau^m - (V_\tau^{a\prime}+z_\tau^a) > V_o^{m\prime}+z_o^m - (V_o^{a\prime}+z_o^a) = \delta$ and $G_\tau^a < G_\tau^m$. Since $z(\cdot)$ is non-increasing, so that $z_\tau^a = z(G_\tau^a) > z(G_\tau^m) = z_\tau^m$, one obtains $V_\tau^{m\prime} - V_\tau^{a\prime} > \delta + (z_\tau^a-z_\tau^m) > \delta$. Furthermore, R is non-increasing and concave, $z'R$ is non-decreasing (Assumptions 2-3) and $G_\tau^a < G_\tau^m$. Thus (A2) implies

$$d[V_\tau^{m\prime}+z_\tau^m-(V_\tau^{a\prime}+z_\tau^a)]/dt > \rho\delta. \tag{A3}$$

This process reinforces itself: $V_{to}^{m\prime} - V_{to}^{a\prime} \geq \delta > 0$ and $G_{to}^m \leq G_{to}^a$ imply $V_{to}^{m\prime}+z_{to}^m - (V_{to}^{a\prime}+z_{to}^a) \geq \delta$, $d[V_{to}^{m\prime}+z_{to}^m]/dt - d[V_{to}^{a\prime}+z_{to}^a]/dt \geq \rho\delta$ and $g_{to}^m < g_{to}^m$, which in turn requires $V_t^{m\prime} - V_t^{a\prime} > \delta$ and $G_t^m < G_t^a$ for all $t \in [to, to+\tau]$ for some $\tau > 0$. This process progresses in time as long as both G_t^a and G_t^m are positive. However, this construction does not rule out the possibility that each step τ becomes smaller such that the process progresses only until some time $T < \min(T^a,T^m)$. Suppose this is indeed the case, i.e., $V_t^{m\prime} - V_t^{a\prime} > 0$, $G_t^m > G_t^a$, Eq. (A3) holds for all $0 \leq t < T$ and $V_T^{m\prime} - V_T^{a\prime} = 0$. But if $V_T^{m\prime} - V_T^{a\prime} = 0$, $V_t^{m\prime}-V_t^{a\prime}$ must be decreasing during some time prior to T, which contradicts (A3). Hence this possibility is ruled out and we conclude that $V_o^{m\prime} - V_o^{a\prime} = \delta > 0$ implies for all t such that G_t^m and G_t^a are positive:

$$\text{(i) } d[V_t^{m\prime}+z_t^m-(V_t^{a\prime}+z_t^a)]/dt > \rho\delta, \quad \text{(ii) } V_t^{m\prime} - V_t^{a\prime} \geq \delta,$$
$$\text{(iii) } g_t^m < g_t^a \text{ and (iv) } G_t^m > G_t^a. \tag{A4}$$

The relations in (A4) imply that either $\hat{G}^m$ or $\hat{G}^a$ or both equal zero, for otherwise (A4-i) holds also in the steady states, which is impossible. (A4-iv) then requires that $\hat{G}^a = 0$. Now if $T^m<T^a$, so that $\hat{G}^m>0$, then $\hat{V}^{m\prime} = V^{m\prime}(\hat{G}^m) > V_T^{a}{}_m{}'$ and (A4-i) requires that both $V_t^{a\prime}+z_t^a$ and G_t^a decrease for

$t \in [T^m, T^a]$, which can happen only if $V_t^{a\prime}$ decreases and hence $V^{a\prime}(\hat{G}^a) < V^{m\prime}(\hat{G}^m)$. If, on the other hand, $T^m \geq T^a$ then $V^m_{T^a}{}' + z^m_{T^a} \geq V^{a\prime}(\hat{G}^a) + z(\hat{G}^a) + \delta = V^{a\prime}(0) + z(0) + \delta$ and $V^m_{T^a}{}' \geq V^{a\prime}(0) + \delta$. For $t \in [T^a, T^m]$, Eq. (3.3a) changes to

$$0 = d(V_t^{a\prime} + z_t^a)]/dt = J(0)[\rho - R'(0)] + z'(0)R(0)$$

so that (A1) becomes

$$\begin{aligned} d[V_t^{m\prime} + z_t^m - (V_t^{a\prime} + z_t^a)]/dt &= V_t^{m\prime}[\rho - R'(G_t^m)] - J(0)[\rho - R'(0)] + z'(G_t^m)R(G_t^m) - z'(0)R(0) \\ &\geq \rho[V_t^{m\prime} - J(0)] \\ &\geq \rho\delta, \end{aligned}$$

where the last inequality holds since $V^{a\prime}(0) \geq J(0)$ (cf. Proposition 1). Thus $V_t^{m\prime} + z_t^m$ continues to grow during the time interval $[T^a, T^m]$ and

$$V^m_{T^m}{}' + z^m_{T^m} = V^{m\prime}(\hat{G}^m) + z(\hat{G}^m) \geq V^m_{T^a}{}' + z^m_{T^a} \geq V^{a\prime}(\hat{G}^a) + z(\hat{G}^a) + \delta,$$

implying that $V^{m\prime}(\hat{G}^m) - V^{a\prime}(\hat{G}^a) \geq z(\hat{G}^a) - z(\hat{G}^m) + \delta > 0$. ■

Appendix B. *Evaluating* $\int_0^{T^a} [\left(\lambda^2 + \frac{\beta}{z + C^a e^{\rho t}}\right)^{1/2} - \mu] dt$

We express the integrand as

$$\left(\lambda^2 + \frac{\beta}{z + C^a e^{\rho t}}\right)^{1/2} - \mu = \left(\lambda^2 + \frac{\beta}{z + C^a e^{\rho t}}\right)^{1/2} - \lambda - (\mu - \lambda)$$

and concentrate on

$$\left(\lambda^2 + \frac{\beta}{z + C^a e^{\rho t}}\right)^{1/2} - \lambda = \frac{\beta}{z + C^a e^{\rho t}} \left[\left(\lambda^2 + \frac{\beta}{z + C^a e^{\rho t}}\right)^{1/2} + \lambda\right]^{-1}.$$

Defining $x = \left(\lambda^2 + \frac{\beta}{z + C^a e^{\rho t}}\right)^{1/2}$ and $\gamma = (\lambda^2 + \beta/z)^{1/2}$ yields

$dt = \frac{-2\beta x}{\rho z (x^2 - \lambda^2)(\gamma^2 - x^2)} dx$ and $\frac{\beta}{z + C^a e^{\rho t}} = x^2 - \lambda^2$. Thus

$$\int_0^{T^a} [\left(\lambda^2 + \frac{\beta}{z + C^a e^{\rho t}}\right)^{1/2} - \lambda] dt = \frac{2\beta}{\rho z} \int_\mu^\xi \frac{x}{x + \lambda} \frac{1}{\gamma^2 - x^2} dx,$$

with $\xi = [\lambda^2 + \beta/(z + C^a)]^{1/2}$ (the lower limit is obtained using $\rho T^a = [\log\left(\frac{\beta}{\mu^2 - \lambda^2} - z\right) - \log(C^a)]$). Integrating by parts (with $u = x/(x + \lambda)$, $u' = \lambda/(x + \lambda)^2$, $v' = 1/(\gamma^2 - x^2)$ and $v = \frac{1}{2\gamma} \log \frac{\gamma + x}{\gamma - x}$) yields

$$\int_{\mu}^{\xi} \frac{x}{x+\lambda} \frac{1}{\gamma^2-x^2} dx = \left(\frac{x}{x+\lambda} \frac{1}{2\gamma} \log \frac{\gamma+x}{\gamma-x} \right)_{x=\mu}^{x=\xi} - \int_{\mu}^{\xi} \frac{\lambda}{(x+\lambda)^2} \frac{1}{2\gamma} \log \frac{\gamma+x}{\gamma-x} dx.$$

Denote

$$H(C^a) = \left(\frac{x}{x+\lambda} \frac{1}{2\gamma} \log \frac{\gamma+x}{\gamma-x} \right)_{x=\mu}^{x=\xi} \qquad (B1)$$

and, with $y = x+\lambda$,

$$J(C^a) = \int_{\mu}^{\xi} \frac{\lambda}{(x+\lambda)^2} \frac{1}{2\gamma} \log \frac{\gamma+x}{\gamma-x} dx$$

$$= \frac{\lambda}{2\gamma} \int_{\mu+\lambda}^{\xi+\lambda} \frac{1}{y^2} [\log(\gamma-\lambda+y) - \log(\gamma+\lambda-y)] dy$$

$$= \frac{\lambda}{2\gamma} \left(\frac{2\gamma}{\gamma^2-\lambda^2} \log(y) - \frac{\gamma-\lambda+y}{(\gamma-\lambda)y} \log(\gamma-\lambda+y) + \frac{\gamma+\lambda-y}{(\gamma+\lambda)y} \log(\gamma+\lambda-y) \right)_{y=\mu+\lambda}^{y=\xi+\lambda}. \qquad (B2)$$

Thus $\int_o^{T^a} [\left(\lambda^2 + \frac{\beta}{z+C^a e^{\rho t}}\right)^{1/2} - \lambda] dt = \frac{2\beta}{\rho z} [H(C^a) - J(C^a)]$, yielding Eq. (5.6):

$$\int_o^{T^a} [\left(\lambda^2 + \frac{\beta}{z+C^a e^{\rho t}}\right)^{1/2} - \mu] dt = \frac{2\beta}{\rho z} [H(C^a) - J(C^a)] - (\mu-\lambda) T^a(C^a)$$

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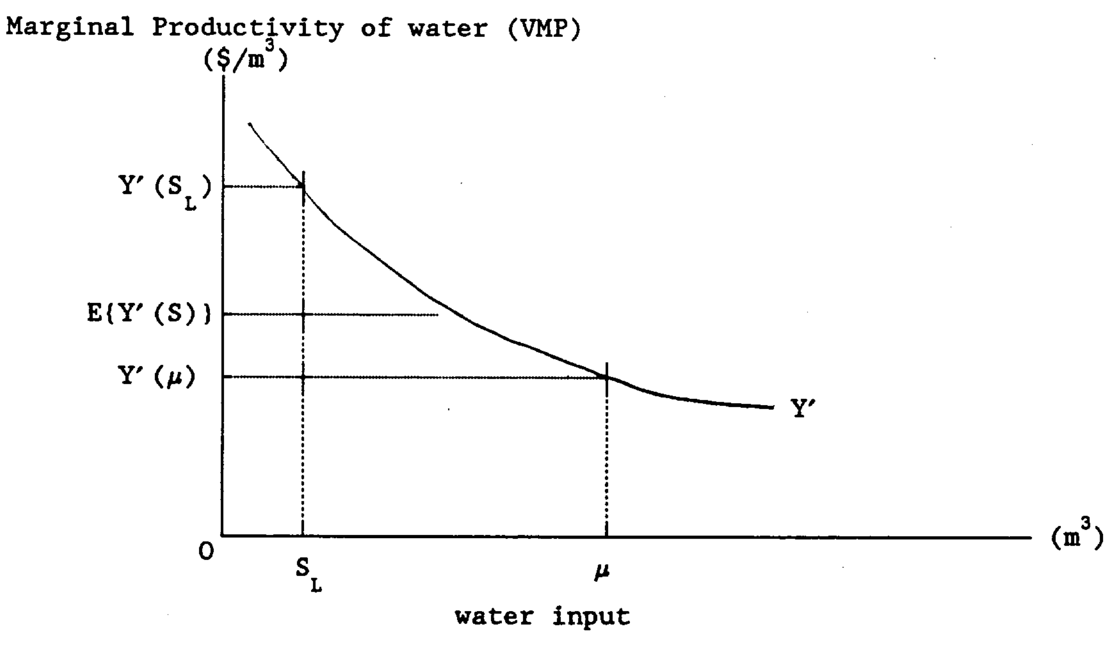
Table 1

Groundwater values and buffer values for several levels of extraction costs (z), aquifer size (G) and variability in surface water supply (λ).

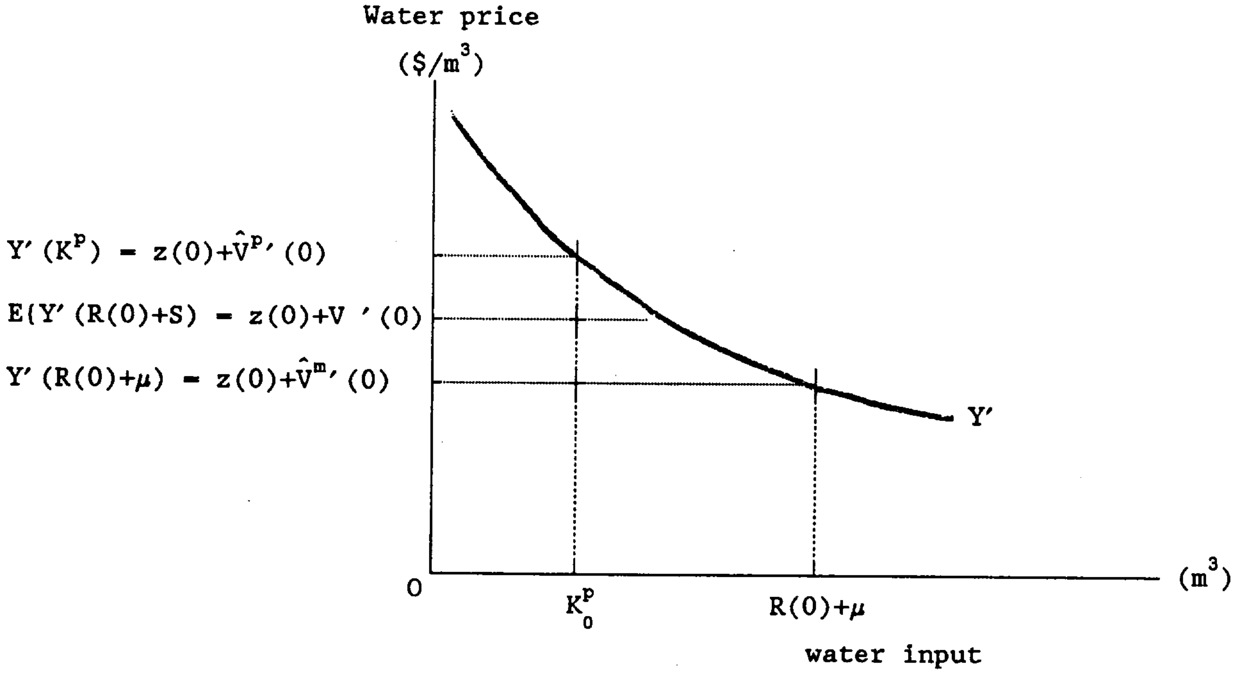
Case	z ($\$/m^3$)	G (m^3)	λ	V_G^m ($/ha)	V_G^a ($/ha)	BV^a ($/ha)	BV^a/V_G^a (%)
1	0.05	20×10^9	500	442.83	467.45	24.66	5
2	0.05	20×10^9	1500	442.83	715.39	272.56	38
3	0.05	5×10^9	1500	437.29	701.37	264.08	38
4	0.1	5×10^9	1500	25.01	154.97	129.96	84

Figure 1



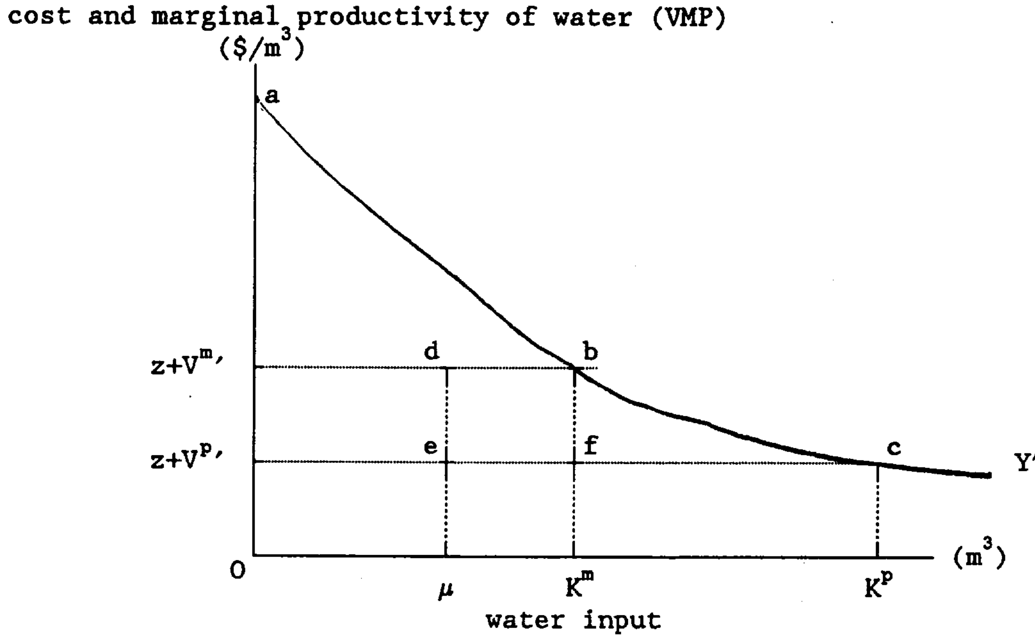
No exploitation: In the m-regime, if $z(\bar{G}) \geq Y'(\mu)$; in the a-regime, if $z(\bar{G}) \geq E\{Y'(S)\}$; in the p-regime, if $z(\bar{G}) \geq Y'(S_L)$

Figure 2



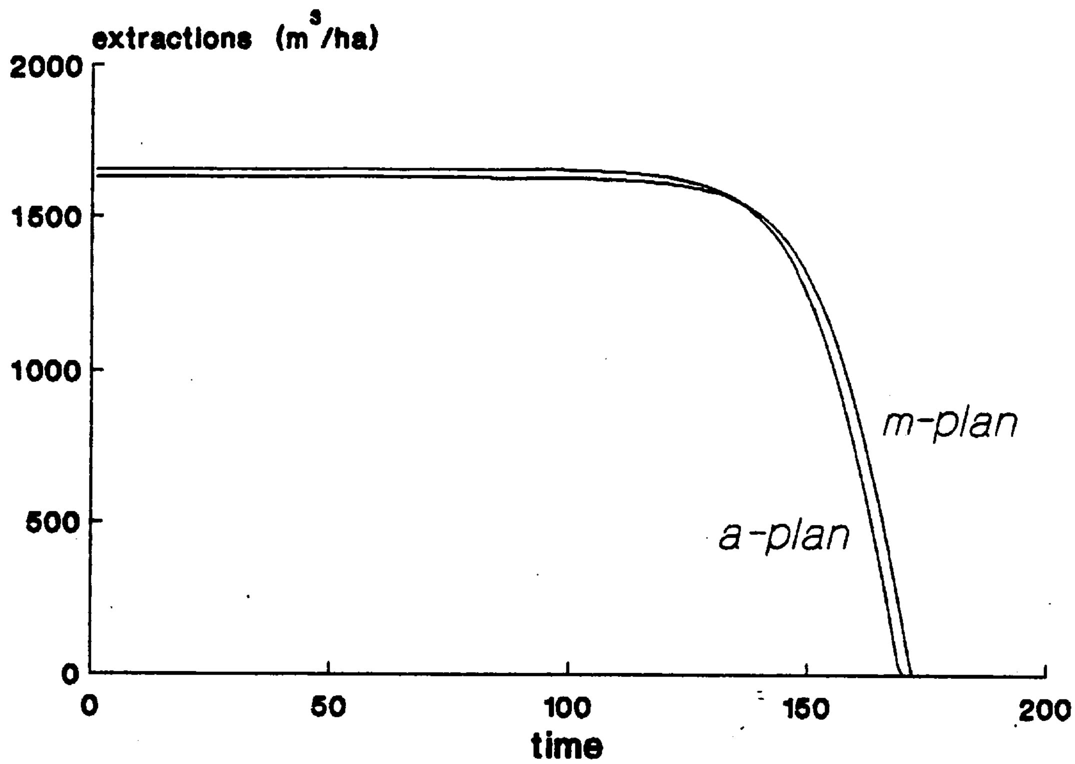
Depletion: In the m-regime, if $z(0)+J(0) \leq Y'(R(0)+\mu)$; in the a-regime, if $z(0)+J(0) \leq E\{Y'(R(0)+S)\}$; in the p-regime, if $z(0)+J(0) \leq Y'(K_0^p)$.

Figure 3



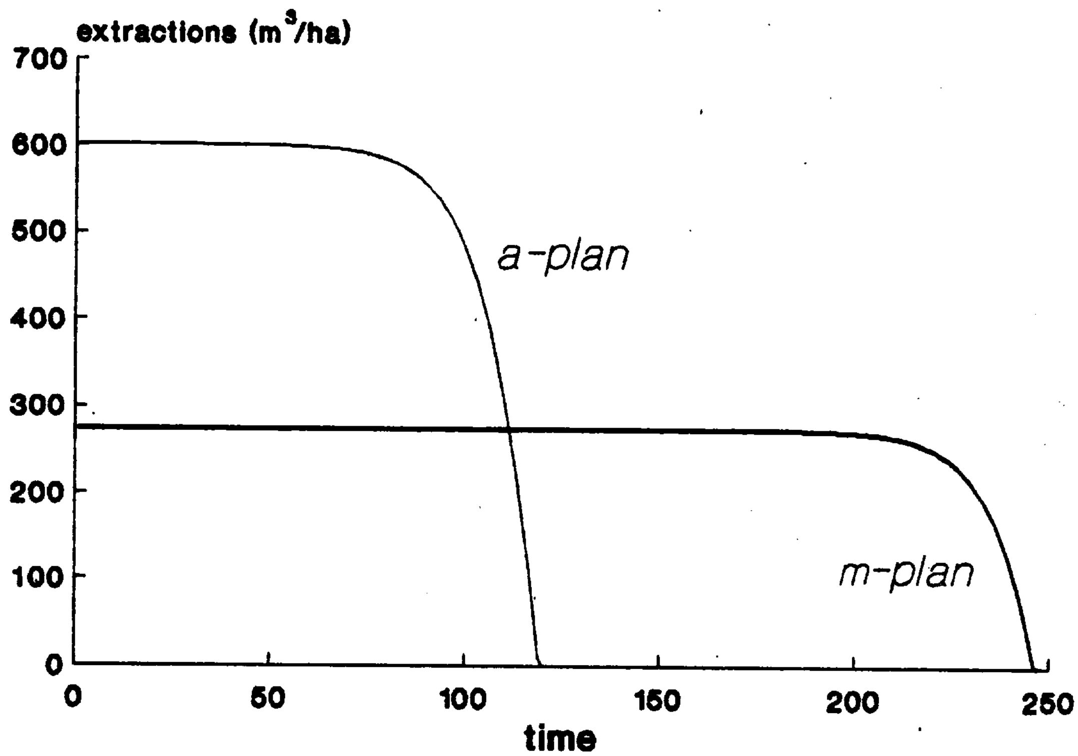
A graphical demonstration that, with constant z, $V^{m\prime} > V^{p\prime}$ implies $V^p > V^m$. $\rho V^p \geq Y(K^p) - (z+V^{p\prime})(K^p-\mu) + V^{p\prime}R$; $\rho V^m = Y(K^m) - (z+V^{m\prime})(K^m-\mu) + V^{m\prime}R$; $Y(K^m) - (z+V^{m\prime})(K^m-\mu) = \text{area}\{abd\mu 0\}$; $Y(K^p) - (z+V^{p\prime})(K^p-\mu) = \text{area}\{ace\mu 0\}$; $(V^{m\prime}-V^{p\prime})(K^m-\mu) = \text{area}\{dbfe\}$. Hence, $\rho(V^p-V^m) \geq \text{area}\{bcf\}$.

Figure 4



Extraction paths under the m- and a-regimes with the data of Case 1.

Figure 5



Extraction paths under the m- and a-regimes with the data of Case 4.