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The Approximation of Nonlinear Programming Problems Using Linear Programming

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The Approximation of Nonlinear Programming Problems Using Linear Programming

Jeffrey Apland

Linear programming has been used extensively for analyzing economic problems. Procedurally, linear programming (LP) is an attractive modeling alternative because of the availability of efficient solution algorithms and the accessibility of computer routines which use these algorithms. The applicability of linear programming can be broadened significantly through the use of separable programming -- a technique which allows for the approximation of nonlinear programming problems using LP. Separable programming can be applied to nonlinear programming problems with constraint and objective functions which can be specified as the sums of functions of single variables [Miller]. Separable programming involves the construction of piecewise linear approximations of the nonlinear functions. The result is a LP problem which can be solved using a conventional LP algorithm. In this paper, the technique of separable programming will be discussed. An extension of the approach to nonlinear programs with non-separable functions will be presented also. Finally, examples of separable programming will be provided and discussed.

The General Separable Programming Problem

Consider the following nonlinear programming problem:

$$\max_{X} \; F(X) \qquad (1)$$

$$\text{s.t:}\quad g(X) \le b \tag{2}$$

$$X \ge 0 \tag{3}$$

Where: X is an nx1 vector of instruments, b is an mx1 vector of constraint constants and g(X) is an mx1 vector of constraint functions (with elements $g_1(X)...g_m(X)$). If the objective function F(X) and constraint functions $g_1(X)...g_m(X)$ are separable (i.e. can be expressed as the sum of functions of single variables), the functions can be restated as follows:

$$F(X) = \sum_{j=1}^{n} F_j(X_j) \tag{4}$$

$$g_i(X) = \sum_{j=1}^{n} g_{ij}(X_j);\ i=1...m \tag{5}$$

The nonlinear programming problem ((1), (2) and (3)) then becomes:

$$\text{Max:}\quad \sum_{j=1}^{n} F_j(X_j) \tag{6}$$

$$\text{s.t:}\quad \sum_{j=1}^{n} g_{1j}(X_j) \le b_1 \tag{7.1}$$

$$\sum_{j=1}^{n} g_{2j}(X_j) \le b_2 \tag{7.2}$$

$$\vdots$$

$$\sum_{j=1}^{n} g_{mj}(X_j) \le b_m \tag{7.m}$$

$$X_j \ge 0;\ j=1...n \tag{8}$$

With separable programming, nonlinear functions $F_j(X_j)$ and $g_{ij}(X_j)$ are approximated using linear segments. For the jth single variable function within the objective ($F_j(X_j)$), an approximating function can be derived as...

$$\bar{F}_j(X_j) = F_j(\bar{X}_{jk}) + \frac{F_j(\bar{X}_{jk+1}) - F_j(\bar{X}_{jk})}{\bar{X}_{jk+1} - \bar{X}_{jk}} (X_j - \bar{X}_{jk}) \quad (9)$$

(9) defines a line through coordinates $[\bar{X}_{jk}, F(\bar{X}_{jk})]$ and $[\bar{X}_{jk+1}, F(\bar{X}_{jk+1})]$ which is used to approximate $F_j(X_j)$ for $\bar{X}_{jk} \leq X_j \leq \bar{X}_{jk+1}$. Where $\bar{X}_{jk}$ and $\bar{X}_{jk+1}$ are in the neighborhood of X_j, $\bar{F}_j(X_j) \approx F_j(X_j)$ Any value of X_j such that $\bar{X}_{jk} \leq X_j \leq \bar{X}_{jk+1}$ can be expressed as:

$$X_j = (1-\alpha)\bar{X}_{jk} + \alpha\bar{X}_{jk+1} \quad (10)$$

$$0 \leq \alpha \leq 1 \quad (11)$$

It then follows that...

$$X_j - \bar{X}_{jk} = \alpha(\bar{X}_{jk+1} - \bar{X}_{jk}). \quad (12)$$

The approximated value of $F_j(X_j)$, using (9) and (12) is:

$$f_j(\alpha) = F_j(\bar{X}_{jk}) + \frac{F_j(\bar{X}_{jk+1}) - F_j(\bar{X}_{jk})}{\bar{X}_{jk+1} - \bar{X}_{jk}} (\alpha(\bar{X}_{jk+1} - \bar{X}_{jk})) \quad (13)$$

$$= F_j(\bar{X}_{jk}) + \alpha(F_j(\bar{X}_{jk+1}) - F_j(\bar{X}_{jk})) \quad (14)$$

$$= (1-\alpha)F_j(\bar{X}_{jk}) + \alpha F_j(\bar{X}_{jk+1}) \quad (15)$$

If q values of X_j are used in the approximation of functions $F_j(X_j)$ and $g_{ij}(X_j)$, (call them $\bar{X}_{j1}, \bar{X}_{j2} \ldots \bar{X}_{jq}$), the technique of separable programming calls for the definition of (qxn) special variables, α_{jk}, j=1...n, k=1...q. Each special variable α_{jk} corresponds to the use of the kth value of X_j, $\bar{X}_{jk}$, in the approximation of nonlinear objective and constraint functions. In general, a unique number of special variables and values will be used for functions of each X_j. However for convenience in notation, let $q_1 = q_2 =$

$\ldots q_n = q$. The following is a separable programming formulation of the nonlinear programming problem (6) through (8):

$$\text{Max: } \sum_{j=1}^{n} f_j(\alpha_{j1}\ldots\alpha_{jq}) = \sum_{j=1}^{n}\sum_{k=1}^{q} F_j(\bar{X}_{jk})\alpha_{jk} \tag{16}$$

$$\text{s.t: } \sum_{j=1}^{n}\sum_{k=1}^{q} g_{1j}(\bar{X}_{jk})\alpha_{jk} \le b_1 \tag{17.1}$$

$$\vdots$$

$$\sum_{j=1}^{n}\sum_{k=1}^{q} g_{mj}(\bar{X}_{jk})\alpha_{jk} \le b_m \tag{17.m}$$

$$\sum_{k=1}^{q} \alpha_{1k} = 1 \tag{18.1}$$

$$\vdots$$

$$\sum_{k=1}^{q} \alpha_{nk} = 1 \tag{18.n}$$

$$\alpha_{jk} \ge 0;\ j=1\ldots n;\ k=1\ldots q \tag{19}$$

Note that (16) through (19) is a linear program. With Θ_i the Lagrange multiplier for constraint (17.i) and Φ_j the Lagrange multiplier for constraint (18.j), the Kuhn-Tucker conditions characterizing an optimal solution to (1.16) through (19) are:

$$F_j(X_{jk}) - \sum_{i=1}^{m} \Theta_i g_{ij}(\bar{X}_{jk}) - \Phi_j \le 0 \quad j=1\ldots n;\ k=1\ldots q \tag{20}$$

$$[F_j(x_{jk}) - \sum_{i=1}^{m} \Theta_i g_{ij}(\bar{X}_{jk}) - \Phi_j]\alpha_{jk} = 0 \quad j=1\ldots n;\ k=1\ldots q \tag{21}$$

$$\alpha_{jk} \ge 0 \quad j=1\ldots n;\ k=1\ldots q \tag{22}$$

$$b_i - \sum_{j=1}^{n}\sum_{k=1}^{q} g_{ij}(\bar{X}_{jk})\alpha_{jk} \ge 0 \quad i=1\ldots m \tag{23}$$

$$[b_i - \sum_{j=1}^{n}\sum_{k=1}^{q} g_{ij}(\bar{X}_{jk})\alpha_{jk}]\Theta_i = 0 \quad i=1\ldots m \tag{24}$$

$$\Theta_i \ge 0 \quad i=1\ldots m \tag{25}$$

$$1.0 - \sum_{k=1}^{q} \alpha_{jk} = 0 \qquad j=1...n \tag{26}$$

Using (16) through (19) to approximate (6) through (8), the approximated optimal values of X_j are $\sum_{k=1}^{q} \bar{X}_{jk}\alpha^*_{jk}$, j=1...n (recall that by (18), $\sum_{k=1}^{q} \alpha_{jk} = 1.0$, j=1...n). The approximated optimal value of objective function (1.06) is $\sum_{j=1}^{n} \sum_{k=1}^{q} F_j(\bar{X}_{jk})\alpha^*_{jk}$ and $\sum_{j=1}^{n} g_{ij}(\bar{X}_{jk})\alpha^*_{jk}$ is the approximation of the ith constraint (7.i), i=1...m. θ^*_i is the approximation of the dual of constraint 7.i, i=1...m. The duals of convexity constraints 18.j (Φ^*_j) can be interpreted best in the context of specific classes of economic problems.

Convexity and Separable Programming

If the opportunity set of a mathematical programming problem is compact and nonempty, and the objective function is continuous over the opportunity set, then a global solution to the problem exists [Intrilligator, p.13]. If, further, the opportunity set is convex and the objective function is concave over the opportunity set, a local maximum for the problem is a global maximum [Intrilligator, p.15]. These theorems hold also for the separable programming problem ((6) through (8)) when the subfunctions $F_j(X_j)$ and $g_{ij}(X_j)$ of the original nonlinear program are concave and convex, respectively. The convexity of the constraint functions and the concavity of the objective function have special implications for the separable programming problem.

Recall that the approximating function for $F_j(X_j)$, $f_j(\alpha_j)$, was derived in (9) through (15) for adjacent values of X_j (X_{jk} and X_{jk+1}, where $X_{j1} < X_{j2} < X_{j3} < \ldots X_{jq}$). The convexity constraints of the separable programming problem ((18.1) through (18.n)) do not confine adjacent special activities (α_{jk} and α_{jk+1}, for some k) to sum to 1.0. Solutions involving convex combinations of non-adjacent special variables may be feasible, implying that the convex combination of points used to approximate $F_j(X_j)$ may not represent the intended approximating function. However, if the objective function is concave and the constraint functions are convex, the optimization process will insure that adjacent values of X_{jk} are used in the optimal solution (i.e. $\alpha_{jk} + \alpha_{jk+1} = 1.0$, for every j=1...n and for some k). A proof follows.

Suppose $F_j(X_j)$ is to be approximated using consecutively increasing values X_{jk}, k=1...q. From (16) through (19), we have...

$$f_j(\alpha_j) = \sum_{k=1}^{q} F_j(\bar{X}_{jk})\alpha_{jk} \tag{27}$$

$$\sum_{k=1}^{q} \alpha_{jk} = 1.0 \tag{28}$$

$$\alpha_{jk} \geq 0, \ k=1\ldots q \tag{29}$$

Let α' be a feasible solution to the LP problem, where $\alpha'_{jm-1} + \alpha'_{jm+1} = 1$. So...

$$f_j(\alpha'_j) = F_j(\bar{X}_{jm-1})\alpha'_{jm-1} + F_j(\bar{X}_{jm+1})\alpha'_{jm+1} \tag{30}$$

Assume that $\bar{X}_{jm-1}\alpha'_{jm-1} + \bar{X}_{jm+1}\alpha'_{jm+1} = \bar{X}_{jm}$. If $F_j(X_j)$ is strictly concave, then...

$$F_j(\bar{X}_{jm-1}\alpha'_{jm-1}+\bar{X}_{jm+1}\alpha'_{jm+1}) > F_j(\bar{X}_{jm-1})\alpha'_{jm-1}+F_j(\bar{X}_{jm+1})\alpha'_{jm+1} \tag{31}$$

Where $\alpha''_{jm} = 1$ and thus $f_j(\alpha''_j) = F_j(\bar{X}_{jm})$, and using (30) and (31)...

$$f_j(\alpha''_j) > f_j(\alpha'_j) \tag{32}$$

If α''_j is feasible, α'_j cannot (by (32)) be optimal. When constraint functions $g_{ij}(X_j)$ are convex, then...

$$g_{ij}(X_{jm-1}\alpha'_{jm-1}+X_{jm+1}\alpha'_{jm+1}) < g_{ij}(X_{jm-1})\alpha'_{jm-1}+g_{ij}(X_{jm+1})\alpha'_{jm+1} \tag{33}$$

By (33), constraints for the LP will be no more binding at α''_j than at α'_j. Thus, if α'_j is feasible, α''_j is feasible. If $F_j(X_j)$ is concave but not strictly concave, the more general result is...

$$f_j(\alpha''_j) \geq f_j(\alpha'_j) \tag{34}$$

That is, a solution involving non-adjacent values of X_j will be either nonoptimal or an alternative optimal solution. Figure 1 illustrates the relationship between approximations with non-adjacent points and adjacent points for a strictly concave objective function F(X). Where F(X) is to be maximized subject to the constraint $X \leq 20.0$, an approximation using adjacent points B and C will always give a greater value to the approximated objective function than an approximation using non-adjacent points such as A and C.

The objective function shown with its separable programming approximation in Figure 2 is not concave. If maximized subject to $X \leq 32.66$, the exact solution is at point B. The intended approximation is at point A -- a linear combination of adjacent points. However, a LP solver would select point C by using non-adjacent points, and the intended approximation would be violated.

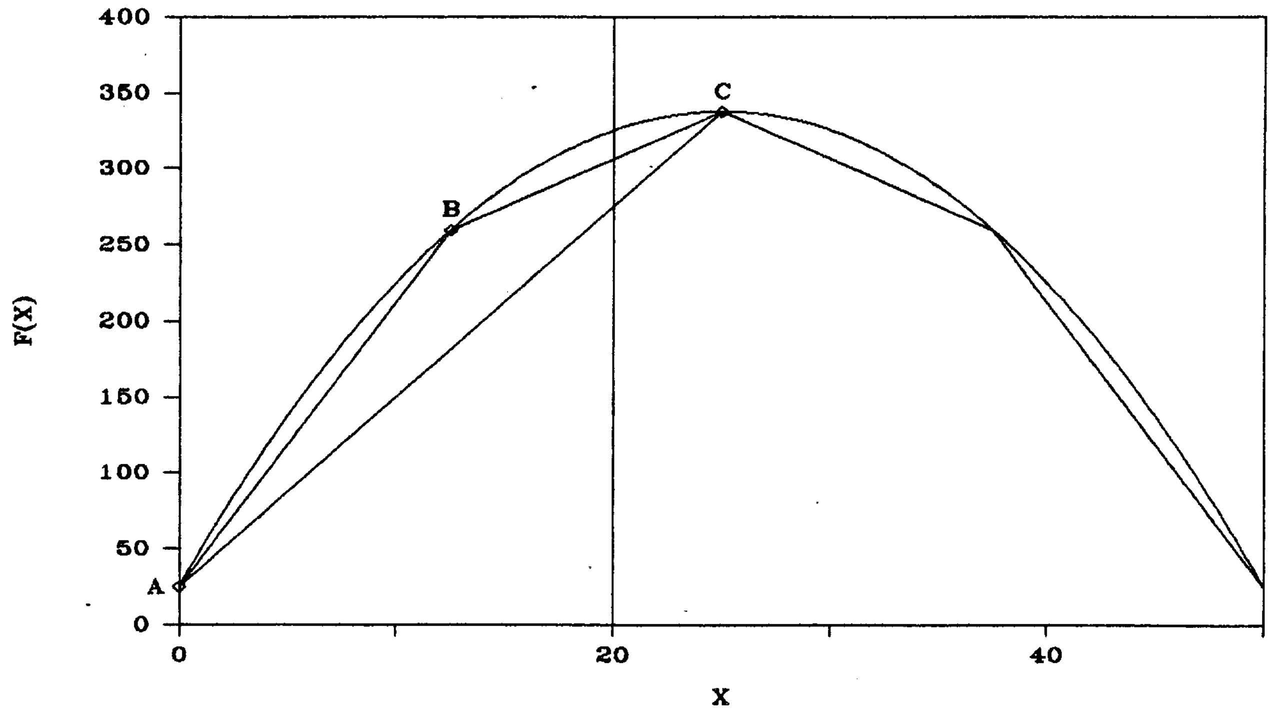


Figure 1: Linear Approximation of a Separable, Strictly Concave Objective Function.

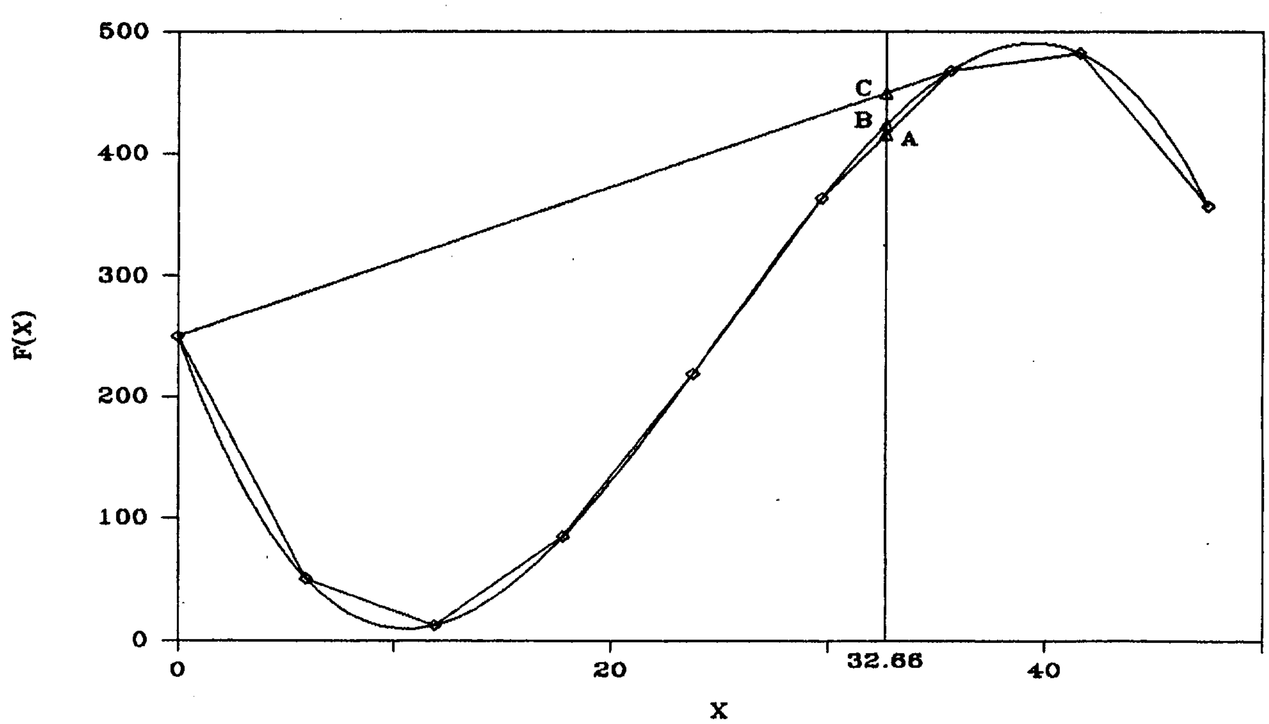


Figure 2: Separable Programming Approximation of an Objective Function Which is Not Concave.

Application of Separable Programming

A few general observations can be made about the implementation of separable programming. The range of values of X_j used in the approximations should, of course, include the optimal value X_j^*, plus and minus an allowance for errors in the approximation. A working knowledge of the problem will usually provide reasonable ranges for the variables. The smaller the ranges over which the functions are approximated, the better the approximation for a given number (q) of special variables.

A better approximation of the functions can be achieved also by using more points in the approximation -- that is, by increasing q, the number of special activities. While this leads to an increase in the number of columns in the LP matrix, no additional rows (constraints) are needed. For many commercial LP solvers, computational costs will not increase significantly as the number of activities increases. The computations associated with matrix preparation, however, may be burdensome. When this is the case, it may be useful to find a first approximation of the solution using a relatively sparce set of values of the variables, then to solve the problem again using the same number of values over a narrower range around the first solution values. Care must be taken so that the range of values is not reduced beyond the range of approximation errors inherent in the first solution.

The solution to a separable programming problem may contain information which is useful in finding formulation, data or solution errors. If an extreme value of a variable X_j ($\bar{X}_{j1}$ or $\bar{X}_{jq}$, where $\bar{X}_{j1} < \bar{X}_{j2} \ldots < \bar{X}_{jq}$) appears in the LP solution such that

$\alpha^*_{j1} = 1.0$ or $\alpha^*_{jq} = 1.0$, the values used may be restricting the optimal solution. If such a restriction is not intended and if there are no alternative optimal solutions, the range should be appropriately altered. If a convex combination of non-adjacent values is used in the solution, errors may exist, also. This may occur when the objective function is not concave, implying the need for adjacency restrictions on the special variables to achieve the intended approximation of the objective function. Similarly, if one or more of the constraint functions are not convex, adjacency restrictions must be imposed to achieve the intended approximation of the constraints. The imposition of adjacency restrictions on a separable programming problem (using additional either-or restrictions with zero-one variables or a specially altered solution algorithm) may be impractical due to the added computational burden. Also, such techniques will typically insure only a local optimum. The convex combinations of non-adjacent values in the optimal solution may also occur if a nonlinear constraint function is non-binding at the optimal solution or if the constraint is not strictly convex. Finally, if the objective function is concave but not strictly concave, an optimal solution may be constructed with a convex combination of non-adjacent points on a flat region of the function.

Linear Approximations of Non-Separable Functions

Nonlinear programming problems with constraint and or objective functions which are not separable may be approximated with piecewise linearization. In some cases, simple algebraic manipulations may be used to transform a nonseparable constraint into a separable form. McCarl and Tice have presented a technique which can be used to transform non-separable quadratic programming problems into separable forms which can be approximated with separable programming. When such transformations are not possible, however, a grid linear approximation of the problem may still be practical.

Consider a nonlinear programming problem with the following constraint:

$$Y - 2.25X_1^{.25}X_2^{.15}X_3^{.05} \leq 0 \tag{35}$$

Discrete values of X_1, X_2, and X_3 may be defined to form a linear approximation of the constraint -- call them X_{ij}, where $X_{i1} < X_{i2} <$ ...X_{iq} and i=1,2,3. A special activity (α_{hkl}) is defined for each combination of values of X_1, X_2 and X_3 to construct the following approximation:

$$Y - \sum_{h=1}^{q} \sum_{k=1}^{q} \sum_{l=1}^{q} (2.25\bar{X}_{1h}^{.25}\bar{X}_{2k}^{.15}\bar{X}_{3l}^{.05})\alpha_{hkl} \leq 0 \tag{36}$$

$$\sum_{h=1}^{q} \sum_{k=1}^{q} \sum_{i=1}^{q} \alpha_{hkl} = 1 \tag{37}$$

Note that in this form, the number of values of the variables used in the approximation has a multiplicative effect on the number of special activities. When the number of special activities must be limited, the technique discussed earlier in the paper of finding a first approximation, then solving again with the approximation constructed over narrower ranges in the variables, may be employed.

A Spatial Equilibrium Model

Quadratic programming (QP) is frequently used to model the equilibrium of spatially separated markets. Such models have been extended to the multiple commodity, multiple time period case [Takayama and Judge], however for purposes of illustration, an n-country, single commodity, static model will be used here. The equilibrium of spatially separated markets can be modeled as the following QP:

$$\text{Max:} \sum_{i=1}^{n} \{[a_{di}X_{di} + .5b_{di}X_{di}^2] - [a_{si}X_{si} + .5b_{si}X_{si}^2]\}$$

$$- \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} c_{ij}T_{ij} \qquad (38)$$

$$\text{s.t:} \quad X_{dj} - \sum_{\substack{i=1 \\ i\neq j}}^{n} T_{ij} \leq 0 \qquad j=1...n \qquad (39)$$

$$- X_{si} - \sum_{\substack{j=1 \\ j\neq i}}^{n} T_{ij} \leq 0 \qquad i=1...n \qquad (40)$$

$$X_{di}, X_{si}, T_{ij} \geq 0 \qquad i=1...n;\ j=1...n\ (j\neq i) \qquad (41)$$

Where: a_{di} is the intercept and b_{di} the slope of the excess demand function for country i, a_{si} is the intercept and b_{si} the slope of the excess supply function for country i, c_{ij} is the unit transportation cost from country i to country j, X_{di} is the excess demand and X_{si} the excess supply in country i, and T_{ij} is total units shipped from country i to country j. The objective function (38) is the producer plus consumer surplus from trade -- explicitly, the sum of the excess demand function integrals minus the sum of the excess supply function integrals and transportation costs. Constraint (39) limits excess demand in each country j to no more than total shipments to that country. Constraint (40) limits shipments from country i to no more than the excess supply in that country. To account for trade distortions, the excess supply and excess demand parameters may be adjusted to reflect tariffs and subsidies and additional constraints may be imposed to reflect quotas and other quantity restrictions. The duals of constraints (39) and (40) are, at the optimal solution, equilibrium import and export prices, respectively.

Note that the contraints of this trade problem are linear and that the objective function is quadratic and separable. Special variables α_{ik} and τ_{ik} can be used to construct the following separable programming approximation of (38) through (41):

$$\text{Max:} \quad \sum_{i=1}^{n} \sum_{k=1}^{q} \{[a_{di}\bar{X}_{dik} + .5b_{di}\bar{X}^{2}_{dik}]\alpha_{ik} - [a_{si}\bar{X}_{sik} + .5b_{si}\bar{X}^{2}_{sik}]\tau_{ik}\}$$

$$- \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} c_{ij}T_{ij} \tag{42}$$

$$\text{s.t:} \quad \sum_{k=1}^{q} \bar{X}_{djk}\alpha_{jk} - \sum_{\substack{i=1 \\ i\neq j}}^{n} T_{ij} \leq 0 \qquad j=1\ldots n \tag{43}$$

$$- \sum_{k=1}^{q} \bar{X}_{sik}\tau_{ik} + \sum_{\substack{j=1 \\ j\neq i}}^{n} T_{ij} \leq 0 \qquad i=1\ldots n \tag{44}$$

$$\sum_{k=1}^{q} \alpha_{ik} = 1 \quad i=1...n \tag{45}$$

$$\sum_{k=1}^{q} \tau_{ik} = 1 \quad i=1...n \tag{46}$$

$$\alpha_{ik},\ \tau_{ik},\ T_{ij} \geq 0 \quad i=1...n;\ j=1...n\ (j \neq i);\ k=1...q \tag{47}$$

The approximation of the demand function integral for a particular country or region in a trade model is illustrated in Figure 3. The graph at the top of Figure 3 shows the quadratic integral and its approximation using five separable programming variables (α). The piecewise linear approximation of the demand function integral in the welfare function implies a step function approximation of the linear demand function as shown in the bottom graph. A more exact approximation may, of course, be achieved by using a larger number of points in the approximation as illustrated in Figure 4.

The Kuhn-Tucker conditions for (42) through (47) are given below.

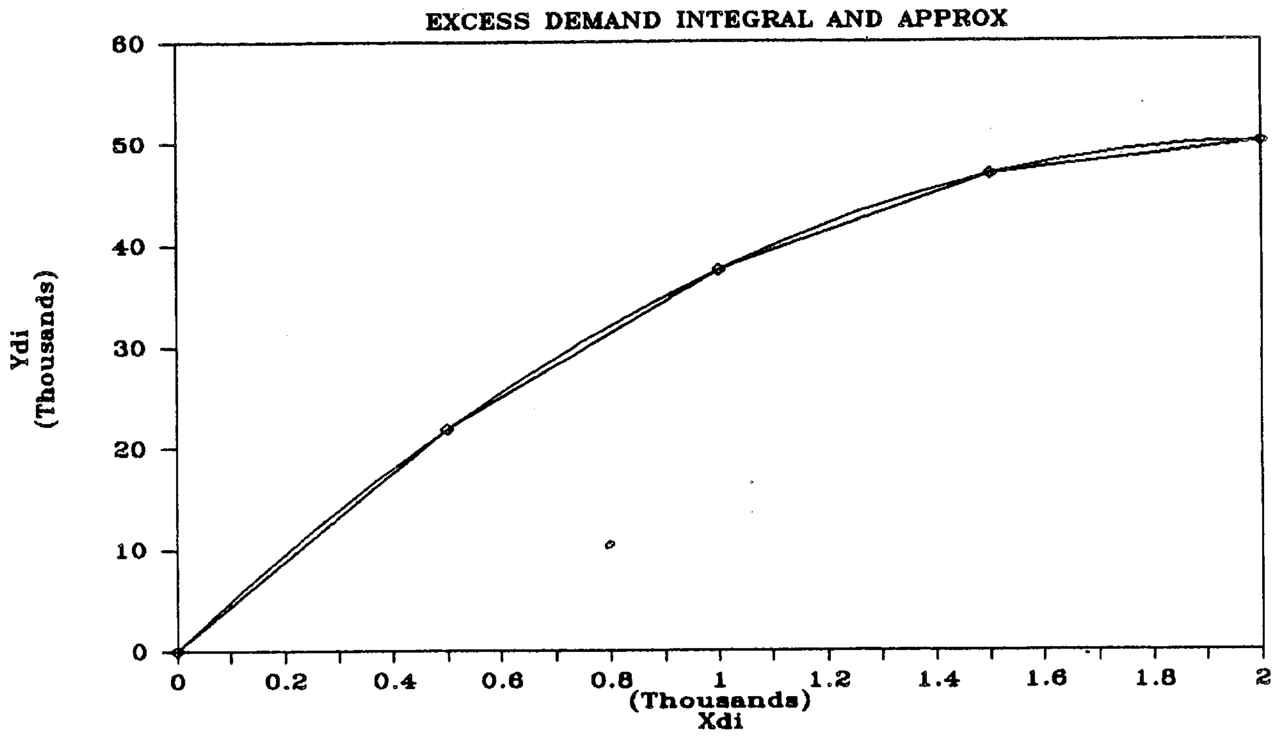
$$[a_{dj}\bar{X}_{djk} + .5b_{dj}\bar{X}^{2}_{djk}] - \theta_{ij}X_{djk} - \Phi_{ij} \leq 0 \quad j=1...n;\ k=1...q \tag{48}$$

$$\Big[[a_{dj}\bar{X}_{djk} + .5b_{dj}\bar{X}^{2}_{djk}] - \theta_{ij}X_{djk} - \Phi_{ij}\Big]\alpha_{jk} = 0 \quad j=1...n;\ k=1...q \tag{49}$$

$$[a_{si}\bar{X}_{sik} + .5b_{si}\bar{X}^{2}_{sik}] - \theta_{zi}X_{sik} - \Phi_{zi} \leq 0 \quad i=1...n;\ k=1...q \tag{50}$$

$$\Big[[a_{si}\bar{X}_{sik} + .5b_{si}\bar{X}^{2}_{sik}] - \theta_{zi}X_{sik} - \Phi_{zi}\Big]\tau_{ik} = 0 \quad i=1...n;\ k=1...q \tag{51}$$

$$-c_{ij} + \theta_{ij} - \theta_{zi} \leq 0 \quad i=1...n;\ j=1...n,\ j \neq i \tag{52}$$



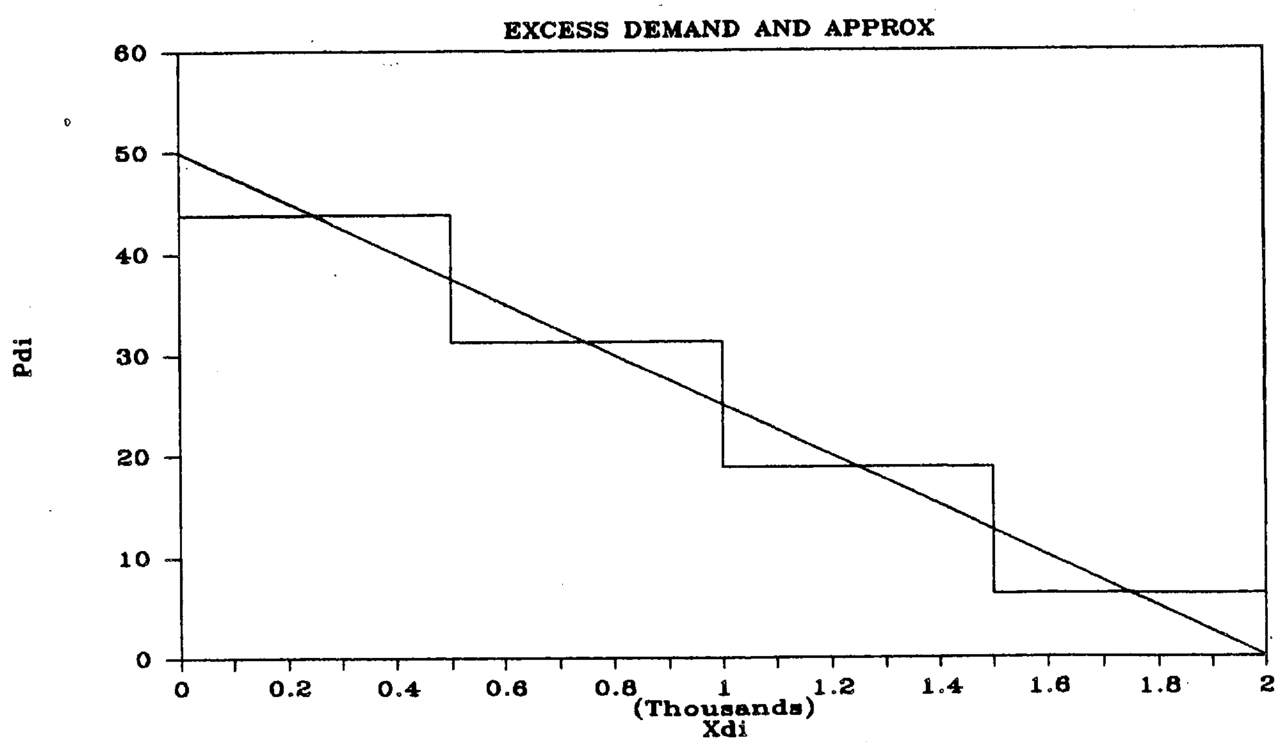
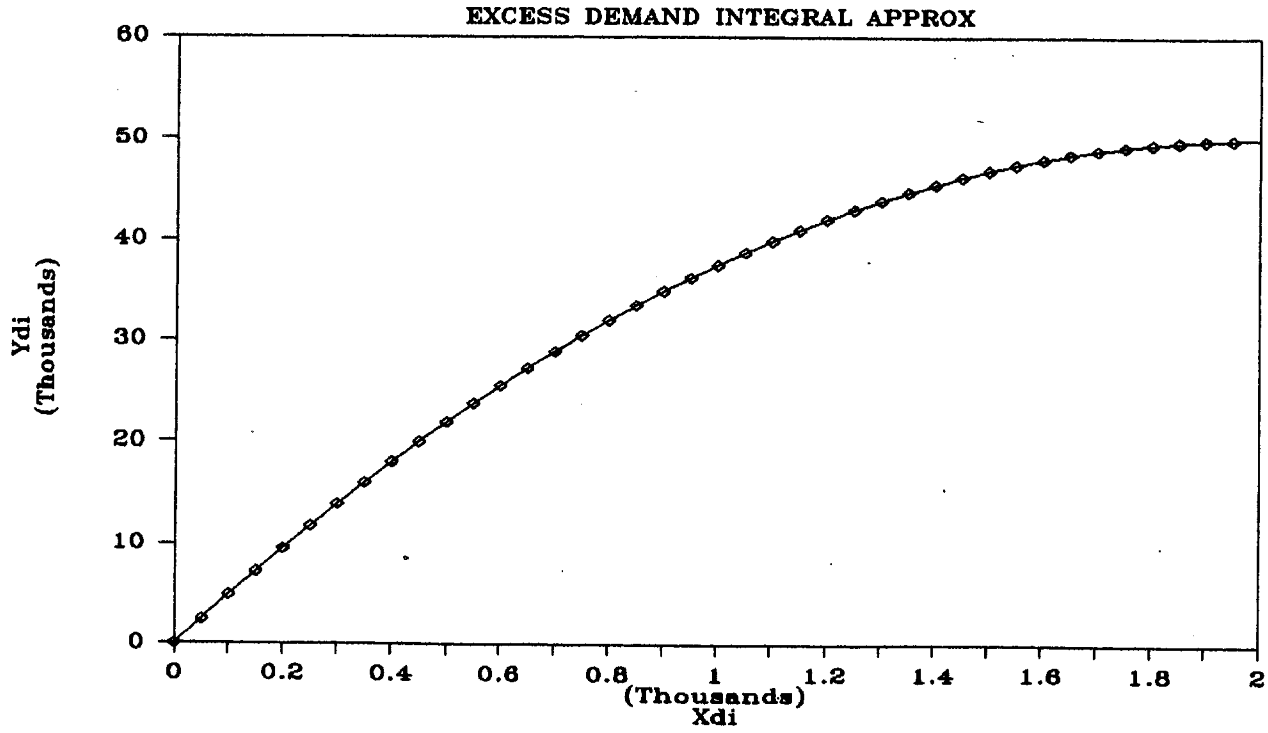


Figure 3: Integral and Demand Function Approximations With Five Special Variables.



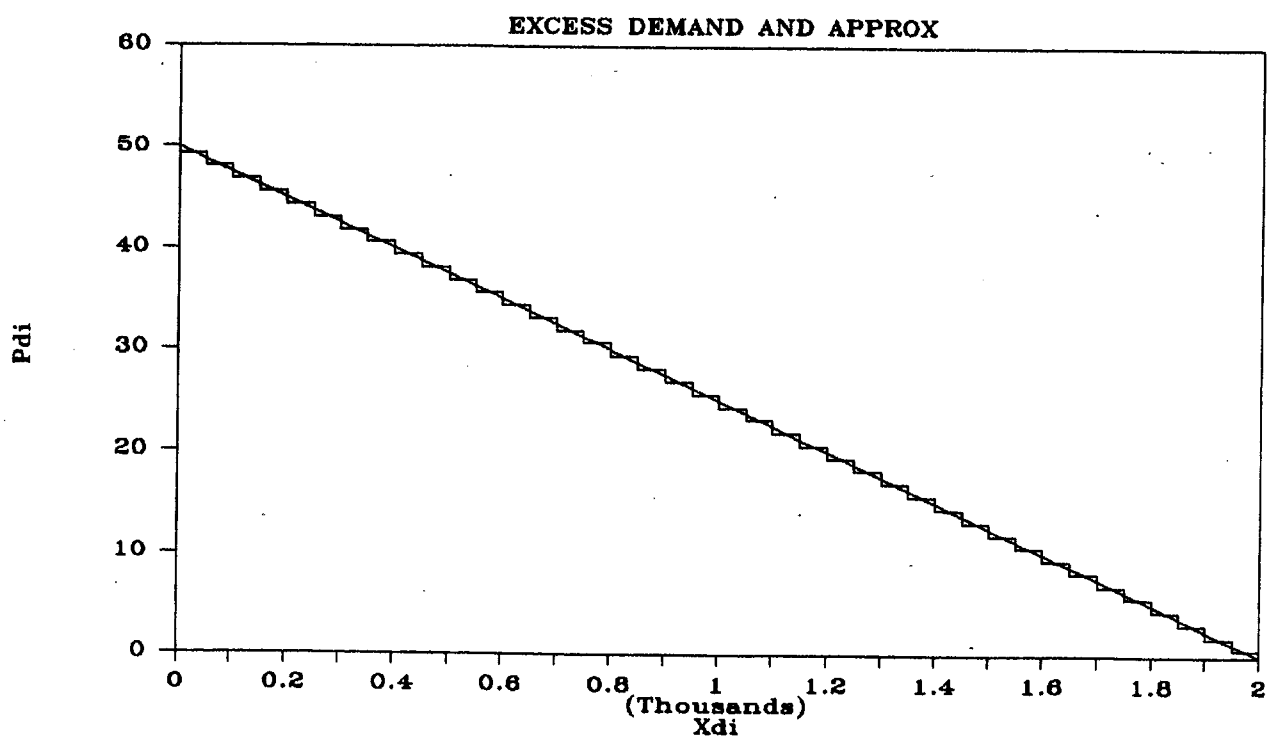


Figure 4: Integral and Demand Function Approximations With Forty Special Variables.

$$[-c_{ij} + \theta_{ij} - \theta_{ei}]T_{ij} = 0 \quad i=1\ldots n;\ j=1\ldots n,\ j\neq i \tag{53}$$

$$\alpha_{ik},\ \tau_{ik},\ T_{ij} \geq 0 \quad i=1\ldots n;\ j=1\ldots n\ (j\neq i);\ k=1\ldots q \tag{54}$$

$$-\sum_{k=1}^{q} \bar{X}_{djk}\alpha_{jk} + \sum_{\substack{i=1 \\ i\neq j}}^{n} T_{ij} \geq 0 \quad j=1\ldots n \tag{55}$$

$$\left[-\sum_{k=1}^{q} \bar{X}_{djk}\alpha_{jk} + \sum_{i=1}^{n} T_{ij}\right]\theta_{ij} \geq 0 \quad j=1\ldots n \tag{56}$$

$$\sum_{k=1}^{q} \bar{X}_{eik}\tau_{ik} - \sum_{\substack{j=1 \\ j\neq i}}^{n} T_{ij} \geq 0 \quad i=1\ldots n \tag{57}$$

$$\left[\sum_{k=1}^{q} \bar{X}_{eik}\tau_{ik} - \sum_{\substack{j=1 \\ j\neq i}}^{n} T_{ij}\right]\theta_{ei} \geq 0 \quad i=1\ldots n \tag{58}$$

$$\sum_{k=1}^{q} \alpha_{ik} = 1 \quad i=1\ldots n \tag{59}$$

$$\sum_{k=1}^{q} \tau_{ik} = 1 \quad i=1\ldots n \tag{60}$$

Suppose that in equilibrium country h is an importer and country l is an exporter. Further, assume that values of X_{di} and X_{ei} used in the approximation are defined such that at the optimal (equilibrium) solution, $\alpha^*_{hm}=1.0$ and $\tau^*_{lp}=1.0$. This assumption is a convenience for the derivations which follow, but does not affect the generality of the results. As with the original quadratic programming formulation, it can be shown that the duals of the commodity balance constraints, θ^*_{dh} and θ^*_{el}, are equilibrium import and export prices, respectively. By (48) and (49), and (50) and (51), the following conditions hold at equilibrium:

$$[a_{dh}\bar{X}_{dhm} + .5b_{dh}\bar{X}^2_{dhm}] - \theta_{1h}X_{dhm} = \Phi_{1h} \qquad (61)$$

$$[a_{s1}\bar{X}_{s1p} + .5b_{s1}\bar{X}^2_{s1p}] - \theta_{21}X_{s1p} = \Phi_{21} \qquad (62)$$

That is, by (61) the dual of the convexity constraint associated with the approximation of the excess demand integral for country j (or in general, any importing country) is the consumer surplus in that country attributable to trade. By (62), the dual of the convexity constraint associated with the approximation of an excess supply function integral is the producer surplus from trade for that country. These results from the optimal solution to the separable programming problem (42) through (47) are not given directly in the solution to the original quadratic programming formulation.

A Nonlinear Nutrient Requirement Constraint

Constraints which embody technological relationships of importance to production problems are often nonlinear. For example the energy requirements of beef cattle are often modeled using the net energy system [National Research Council]. For a given type and size of animal, the ration must have the energy necessary for maintenance of the animal's weight. To achieve a given rate of gain, a given amount of gain energy will be necessary also. Under the net energy system, energy in a portion of the ration is assumed to be used to meet maintenance requirements. Energy in the remainder of the ration is assumed to be used by the animal for gain. For an animal of given type and size and for a targeted daily rate of gain, net energy requirements are specified as follows [Brokken]:

$$\sum_{i=1}^{n} a_{mi} X_i \geq NEM/e \tag{63}$$

$$\sum_{i=1}^{n} a_{gi} X_i \geq NEG/(1-e) \tag{64}$$

$$0 \leq e \leq 1 \tag{65}$$

Where: NEM and NEG are the requirements of maintenance and gain energy, respectively; e is the proportion of the ration which will be used for maintenance requirements (thus (1-e) is the proportion for gain); the amounts of maintenance and gain energy per unit of feed i are a_{mi} and a_{gi}, respectively; and X_i is the quantity of the ith feed in the ration. Since for a ration formulation problem e is endogenous, the net energy constraints are nonlinear. A linear approximation of the constraints may be constructed as follows:

$$\sum_{i=1}^{n} a_{mi} X_i - \sum_{k=1}^{q} (NEM/e_k)\alpha_k \geq 0.0 \tag{66}$$

$$\sum_{i=1}^{n} a_{mi} X_i - \sum_{k=1}^{q} (NEM/(1-e_k))\alpha_k \geq 0.0 \tag{67}$$

$$\sum_{k=1}^{q} \alpha_k = 1 \tag{68}$$

The resulting approximations of the maintenance and gain energy constraints are illustrated in Figure 5.

Computerized Generation of Parameters of a Separable Program

The use of computer programs to generate linear programming matrices can be an efficient approach to the construction of models

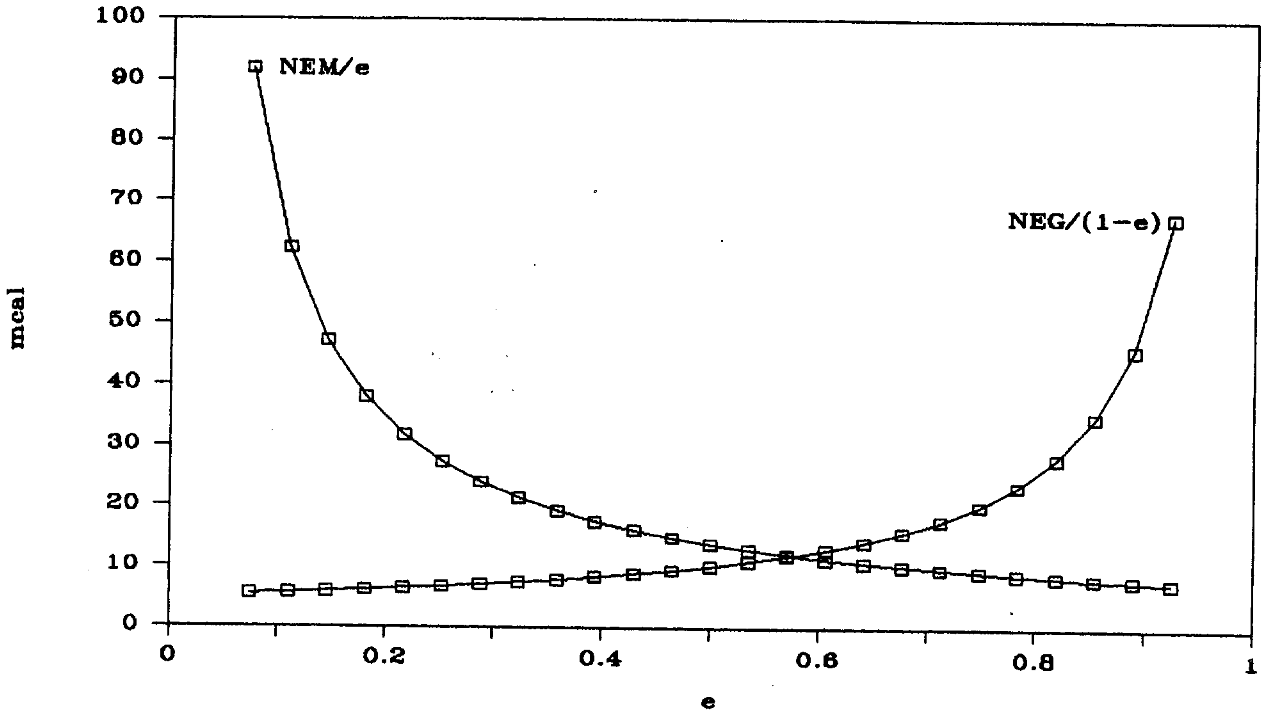


Figure 5: Approximation of Net Energy Requirements.

which will be used repetitively [McCarl and Nuthall]. Computerized matrix generators can be especially effective for calculating the discrete values of variables ($\bar{X}$) and the corresponding function values ($F_j(\bar{X}_{jk})$ and $g_{ij}(\bar{X}_{jk})$) in a separable program. A simple example will serve to illustrate the technique.

For the separable programming approximation of net energy requirements given in (66), (67) and (68), three coeficients (NEM/e, NEG/(1-e), and 1) must be input to the solver for each special variable α_k. If, as illustrated in Figure 5, 25 special variables are used, 75 coefficients must be generated. Within a FORTRAN matrix generating computer program which writes MPS formated matrix data, the lines of code below would create the lines for entering the parameters of (66), (67) and (68). (Note that these parameters are constraint coefficients of the LP which are, in MPS format, entered in the "COLUMNS" data [McRoberts].)

```
      .
      .
      .
      XNEM=6.89
      XNEG=5.06
      EMIN=0.075
      EMAX=0.925
      NQ=25
      ESTEP=(EMAX-EMIN)/(NQ-1)
      DO 10 K=1,NQ
      EK=EMIN+(K-1)*ESTEP
      AIJ1=XNEM/EK
      AIJ2=XNEG/EK
      WRITE(1,20) K,AIJ1
   20 FORMAT(7X,'ALPHA',I2,4X,'NEMMIN',1X,F12.2)
      WRITE(1,21) K,AIJ2
   21 FORMAT(7X,'ALPHA',I2,4X,'NEGMIN',1X,F12.2)
      WRITE(1,22) K
   22 FORMAT(7X,'ALPHA',I2,4X,'CONVEX',10X,'1.0')
   10 CONTINUE
      .
      .
      .
```

The number of points to be used in the approximation (NQ) is specified along with the net energy requirements (NEM and NEG). From this information, the program can calculate the incremental increase in e (ESTEP). And by using a "DO" loop, the parameters of the constraints are calculated and written to the designated file in MPS format. In this example, the LP problem has 75 a_{ij}'s associated with the approximation which are generated with 3 user-provided values -- EMIN, EMAX and NQ. The range of the approximation (EMIN to EMAX) and the number approximating points (NQ) can easily be altered. Given the parameter values in the code above, the following lines of MPS-formated data would be generated:

```
     .          .           .
     .          .           .
     .          .           .
ALPHA 1     NEMMIN       91.83
ALPHA 1     NEGMIN        5.47
ALPHA 1     CONVEX        1.0
ALPHA 2     NEMMIN       62.37
ALPHA 2     NEGMIN        5.69
ALPHA 2     CONVEX        1.0
     .          .           .
     .          .           .
     .          .           .
ALPHA24     NEMMIN        7.74
ALPHA24     NEGMIN       45.82
ALPHA24     CONVEX        1.0
ALPHA25     NEMMIN        7.45
ALPHA25     NEGMIN       67.46
ALPHA25     CONVEX        1.0
     .          .           .
     .          .           .
     .          .           .
```

Summary

Separable prgramming is a technique for approximating the solution of nonlinear programming problems with separable objective and constraint functions using linear programming. When nonlinear programming solvers are not available or impractical to use, linearization and the use of relatively efficient and accessable LP codes is an attractive alternative. In some cases, the grid linear approximation of nonseparable functions may be useful, also.

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