A Single Matrix Method for Several Problems

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Matrix algebra has become a familiar research tool in recent years, but the teaching and learning problem is still formidable for many individuals. The purpose of this paper is to present a simple general-purpose method of handling matrices for solving simultaneous equations, including those involved in regression and linear programming problems. High-speed computers and different methods are now used in most practical analysis in this field, but teaching must rely on manual approaches to illustrate the mathematical principles. The method presented here is believed to shorten learning time and reduce the memory burden. In short, this is an introduction to matrix algebra in one easy lesson. **The author wishes to thank Rex Daly and Martin Abel for suggestions that have helped to improve this article.**

THIS PAPER SHOWS how a single method of handling matrices can be applied to problems involving simultaneous equations, regression analysis, and linear programming. All of this will be found in standard textbooks, but the conventional solutions for problems in each of these fields have been fragmented along lines that select a most efficient method for each purpose considered independently. The general-purpose approach presented here is a sort of least common denominator which has the pedagogical advantage of bringing out more clearly the interrelationships between the different types of problems.

The method presented employs what might be called a "desired goal approach." No proof of the method is given because this can be found elsewhere (*4*, ch. 1–4).[1] Nor is originality claimed. The procedure uses only a few principles of elementary matrix algebra and anyone who has used signed (positive and negative) numbers will have no difficulty in learning the method. The method is not the fastest one available for every situation. But it does get the job done with a minimum of mental effort.

[1] Italic numbers in parentheses refer to Selected References, p. 100.

The order of presentation is first to outline the procedure step by step using a simple example and then to show the several applications.

The Method

I. Some Definitions

EXHIBIT A

	P_1	P_2	P_3	P_4	P_5	P_6
R_1-----------	6	4	2	1	0	0
R_2-----------	4	9	3	0	1	0
R_3-----------	2	3	5	0	0	1

A. Exhibit A is called a matrix. A matrix is simply a rectangular array of numbers.

B. P_1, P_2, etc., are labels or identifications for the columns.

C. A column, or column vector, is a vertical array of numbers, e.g., column P_1.

D. A row, or row vector, is a horizontal array of numbers, e.g., R_1.

E. A column is sometimes called a column matrix; a row is called a row matrix.

F. An element is any single number in a row, column or matrix. In a matrix, an element in row 2, col. 3 is identified as e_{23}—or 3 in Exhibit A.

G. An identity, or unit column is one containing the numbers, one (1-unity) in one position only, and zeros elsewhere, e.g., columns P_4, P_5, and P_6.

II. Objective

Vectors P_1, P_2, and P_3 are to be transformed into unit vectors like vectors P_4, P_5, and P_6 without disturbing the "relationship" between the rows and vectors (the reason why we want to do this will be clear later).

III. Procedure

A. First we need some information about what we can do without disturbing the relationship between the rows and columns. This information is stated without proof.

1. A row can be multiplied or divided by some number without disturbing the relationship.

2. A row or some multiple of a row (i.e. a row times 2 or row times ½ and so forth) can be added

to another row without disturbing the relationship.[2]

B. Armed with this information we can now proceed toward the objective.

1. We know that we want to get the number 1 or unity where element e_{11} or 6 now stands, hence, we divide row 1 by the number 6, element by element:

	P_1	P_2	P_3	P_4	P_5	P_6
R_1'----	1	.66667	.33333	.16667	0	0

2. We know also that we want to get zeros in the positions of elements e_{21} and e_{31}, i.e., rows 2 and 3 of column P_1. Accordingly, we subtract 4 times row 1′ from row 2, element by element.

	P_1	P_2	P_3	P_4	P_5	P_6
	4	9	3	0	1	0
	4	2.66668	1.33332	.66668	0	0
R_2'------	0	6.33332	1.66668	−.66668	1	0

3. And similarly we subtract 2 times row 1′ from row 3:

	P_1	P_2	P_3	P_4	P_5	P_6
	2	3	5	0	0	1
	2	1.33334	.66667	.33334	0	0
R_3'------	0	1.66666	4.33333	−.33334	0	1

Now we have a completely new matrix in which we have accomplished one-third of our task, i.e., column P_1 is in the desired form and the matrix at this point is as follows:

EXHIBIT B

	P_1	P_2	P_3	P_4	P_5	P_6
R_1'------	1	.66667	.33333	.16667	0	0
R_2'------	0	6.33332	1.66668	−.66668	1	0
R_3'------	0	1.66666	4.33333	−.33334	0	1

Beginning with Exhibit B, let us proceed with the next step of our objective, i.e., to change column P_2 into one like column P_5.

C. Since new column P_2 now has the number 6.33332 in second row, we must divide row 2′ by this number. We also operate on the other two rows in the same way as we did in the first step.

1. Row 2″ below is row 2′ in Exhibit B divided by 6.33332 in order to get 1 in element e_{22}.

	P_1	P_2	P_3	P_4	P_5	P_6
R_2''--------	0	1	.26316	−.10526	.15790	0

2. Row 1″ is obtained by subtracting .66667 times row 2″ from row 1′.

	P_1	P_2	P_3	P_4	P_5	P_6
	1	.66667	.33333	.16666	0	0
	0	.66667	.17544	−.07017	.10527	0
R_1''---	1	0	.15789	.23683	−.10527	0

3. Row 3″ is obtained by subtracting 1.66666 times row 2″ from row 3′.

	P_1	P_2	P_3	P_4	P_5	P_6
	0	1.66666	4.33334	−.33333	0	1
	0	1.66666	.43860	−.17543	.26317	0
R_3''--	0	0	3.89474	−.15790	−.26317	1

Collecting the transformed rows we have a new matrix:

EXHIBIT C

	P_1	P_2	P_3	P_4	P_5	P_6
R_1''-----	1	0	.15789	.23683	−.10527	0
R_2''-----	0	1	.26316	−.10526	.15790	0
R_3''-----	0	0	3.89474	−.15790	−.26317	1

D. With the results in Exhibit C we can proceed to the final step.

1. Divide row 3″ by 3.89474 to get the number 1 in row 3″ of column P_3 and thus obtain a new row 3‴.

	P_1	P_2	P_3	P_4	P_5	P_6
R_3'''----	0	0	1	−.04054	−.06757	.25676

2. The final row, 1‴ is row 1″ minus .15789 times row 3‴:

	P_1	P_2	P_3	P_4	P_5	P_6
	1	0	.15789	.23683	−.10527	0
	0	0	.15789	−.00640	−.01067	.04054
R_1'''--	1	0	0	.24323	−.09460	−.04054

3. The final row, 2‴, is row 2″ minus .26316 times row 3‴.

	P_1	P_2	P_3	P_4	P_5	P_6
	0	1	.26316	−.10526	.15790	0
	0	0	.26316	−.01067	−.01778	.06757
R_2'''-	0	1	0	−.09459	.17658	−.06757

[2] The term column can be substituted for row and these statements are still true. But row and column operations cannot be intermingled. If we start with row operations, we must continue with them to the solution and vice versa.

The following exhibit presents the final rows:

EXHIBIT D—final matrix

	P_1	P_2	P_3	P_4	P_5	P_6
R_1'''-----	1	0	0	.24323	−.09460	−.04054
R_2'''-----	0	1	0	−.09459	.17568	−.06757
R_3'''-----	0	0	1	−.04054	−.06757	.25676

E. The original mission is now completed—vectors P_1, P_2, and P_3 are unit vectors in Exhibit D. In review the steps are:

1. Decide on objective (in the above example, this was the columns to become identity vectors and the elements in these vectors to become the number one or unity).
2. Select pivotal element. This is the element to be transformed to the number one, and is the element that designates the *operating column* and *critical row*. These were selected (so it appears) in arbitrary manner in the preceding example. In particular problems, the pivotal element will be selected by specific criteria.
3. Divide critical row by the pivotal element.
4. Multiply transformed *critical row* by the number located in *operating* column of row 1 and subtract products from row 1. Do this for all rows, except the critical row. This operation transforms all elements in the *operating* column (except the element of the critical row) to zeros.
5. Steps 2 through 4 are repeated for every column or vector that must be transformed into a unit vector.

Once these steps have been learned, it is usually more convenient when using a desk calculator to go directly from one intermediate matrix to another (A to B, B to C and so forth) without writing down the individual row multiplications and subtractions as we have done in this example. It is a good plan when using the direct method to have each successive matrix identified on a long sheet of paper so that the elements can be filled in as computations proceed.

Matrix Inversion Defined

The operations carried out in the preceding section have inverted a matrix. The final vectors P_4, P_5, and P_6 in Exhibit D form an inverse of original vectors P_1, P_2, and P_3 in Exhibit A.

If we let the symbol A stand for vectors P_1, P_2, and P_3 in Exhibit A and let B stand for vectors P_4, P_5, and P_6 in Exhibit D then:

$$AB=I$$

$$\text{where } I=\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}$$

and e_{11} or 1 is obtained by multiplying row 1 in A, by column 1 in B, element by element, and adding. For example, using row 1 of columns P_1, P_2, and P_3 in Exhibit A and column 4 in Exhibit D yields the following:

$$1=[6(0.24323)]+[4(0.09459)]+[2(-0.04054)]$$

In the same way, the zero element, e_{32}, of I is the sum of the inner products of row 3 of A and column 2 of B.

Using the Inverse To Solve Simultaneous Equations

If we let $\bar{y}$ stand for a column vector, $P_o=\begin{matrix}40\\50\\30\end{matrix}$, then we can write:

$$\bar{y}=A\bar{x}$$

where $\bar{x}$ is also a three-element column and A is the 3-column (P_1, P_2, P_3) matrix in Exhibit A. We can also write the above:

$$\underset{y}{\begin{bmatrix}40\\50\\30\end{bmatrix}}=\underset{A}{\begin{bmatrix}6&4&2\\4&9&3\\2&3&5\end{bmatrix}}\underset{\bar{x}}{\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}}$$

Or it can be written as a set of simultaneous equations in conventional algebraic form:

$$40=6X_1+4X_2+2X_3$$

(1) $$50=4X_1+9X_2+3X_3$$

$$30=2X_1+3X_2+5X_3$$

This is an ordinary set of linear simultaneous equations for which unique numbers can be found for X_1, X_2, and X_3 if certain conditions hold.[3]

[3] We usually say that these equations have a unique solution if the matrix is nonsingular, which means that no row or column is some multiple of some other row(s) or column(s). Also, if a matrix has an inverse it is nonsingular. The A matrix has an inverse. Hence, it is nonsingular. But, if a matrix is singular and the method outlined here is used to solve a set of equations or invert a matrix, at some stage in the computations a row of zeros will appear.

The inverse can be used to get these X values once:

$$AB=BA=I$$

Where A is the original matrix and B the inverse (sometimes designated A^{-1}) of that matrix,

$$\text{then } By=BA\bar{x}=\bar{x}$$
$$BA=I \text{ and } I\bar{x}=\bar{x}$$

In terms of our example, the coefficients in the inverted matrix (B) (Exhibit D) times $\bar{y}$ are equal to $\bar{x}$, the values of X as follows:

EXHIBIT E

$$\overset{B}{\begin{bmatrix} .24323 & -.09460 & -.04054 \\ -.09459 & .17568 & -.06757 \\ -.04054 & -.06757 & .25676 \end{bmatrix}} \times \overset{\bar{y}}{\begin{bmatrix} 40 \\ 50 \\ 30 \end{bmatrix}} = \overset{\bar{x}}{\begin{matrix} 3.78300 \\ 2.97330 \\ 2.70270 \end{matrix}}$$

Accordingly, 40 (.24323)+50 (−.09460)+ 30 (−.04054)=3.78300, and so forth.

The Inverse in Input-Output Analysis

This example is also useful to show how the inverse is used in Leontief's input-output analysis. Without going into detail as to how a Leontief matrix is assembled, let us say only that it represents certain relationships within the economy that tie gross output ($\bar{y}$) to net output ($\bar{x}$). Now assume that our B matrix in Exhibit E is such an input-output matrix and the A matrix is its inverse. Accordingly, for any level of net output $\bar{x}$, we can find the corresponding level of gross output needed. If vector $\bar{x}$=3.78300, 2.97330, 2.70270, and represents the level of net output required of goods A, B, and C respectively, then the required gross outputs of A, B, and C would be 40, 50, and 30. For example, 40=6(3.78300)+ 4(2.97330)+2(2.70270) and so forth. In matrix notation, this operation is:

$$\bar{y}=A\bar{x}$$

which looks like the above simultaneous equation problem. But it differs in this respect. For the simultaneous equation problem, the $\bar{y}$ vector and the A matrix are known and we want to find a consistent $\bar{x}$ vector. For the input-output problem, the A matrix is known, the $\bar{x}$ vector is assumed and we want to find a consistent $\bar{y}$ vector.

Other Solutions to Simultaneous Equations

We do not need the inverse in order to solve simultaneous equations, as most readers know. The inverse was used above only to show how it can be used if it is available. Suppose we only want the solution to three equations such as:

EXHIBIT F

$$\begin{matrix} P_0 & & P_1 & & P_2 & & P_3 \\ 40 & = & 6X_1 & + & 4X_2 & + & 2X_3 \\ 50 & = & 4X_1 & + & 9X_2 & + & 3X_3 \\ 30 & = & 2X_1 & + & 3X_2 & + & 5X_3 \end{matrix}$$

We can use the outlined procedure and work with the constants in columns P_0, P_1, P_2, and P_3 only. On carrying the computational procedure to completion, the solution is given by the final P_0 column. After performing the required steps the final matrix is:

EXHIBIT G

P_0	P_1	P_2	P_3
3. 78300	1	0	0
2. 97330	0	1	0
2. 70270	0	0	1

Hence, X_1=3.78300, X_2=2.97330, and X_3= 2.70270; which is the same answer obtained by using the inverse.

Regression Analysis

Let us assume that vectors P_0, P_1, P_2, and P_3 in Exhibit F represent the normal equations in a regression problem with three independent variables, X_1, X_2, and X_3 and the dependent variable Y (Exhibit H).

EXHIBIT H

$$(\Sigma X_1Y=40)=b_1(\Sigma X_1X_1=6)+b_2(\Sigma X_1X_2=4)+b_3(\Sigma X_1X_3=2)$$
$$(\Sigma X_2Y=50)=b_1(\Sigma X_1X_2=4)+b_2(\Sigma X_2X_2=9)+b_3(\Sigma X_2X_3=3)$$
$$(\Sigma X_3Y=30)=b_1(\Sigma X_1X_3=2)+b_2(\Sigma X_2X_3=3)+b_3(\Sigma X_3X_3=5)$$

Then again the values in the P_0 column of Exhibit G constitute the solution to these normal equations, i.e.:

$$3.78300=b_1,\ 2.97330=b_2, \text{ and } 2.70270=b_3$$

Also, the data required to obtain the normal equations or Exhibit H and the data in Exhibit D permit us to derive the standard regression statistics:

$$R^2 1.234=\frac{b_1\Sigma x_1y+b_2\Sigma x_2y+b_3\Sigma x_3y}{y^2}$$

$$S^2 1.234=\Sigma y^2-b_1\Sigma x_1y+b_2\Sigma x_2y+b_3\Sigma x_3y$$

$$S_{b_1}=\sqrt{\frac{S^2 1.234}{c_{11}}}$$

$$S_{b_2}=\sqrt{\frac{S^2 1.234}{c_{22}}}$$

etc.

where $c_{11}=.24323$

and $c_{22}=.17568$

Multiple regression problems can be solved then by these steps:

1. Use the formula

$$\Sigma ix_{1j}x_{1k}=\Sigma iX_{1j}X_{1k}-\frac{\Sigma iX_{1j}\Sigma iX_{1k}}{n}$$

to get the normal equations, where X_{11} stands for Y.

2. Write these down in matrix form with the identity matrix along side, just as was done in Exhibit A.

3. Perform standard steps to reduce X matrix to an identity.

4. Use this final P_0 column to specify regression equations, i.e.:

$$Y=a+3.78300X_1+2.97330X_2+2.70270X_3$$

(Note: We could have assumed some arbitrary numbers for the means and Σy^2 then computed the constant ($a=y-\Sigma ibx_1$) and standard error of the b's in the above equation. However, the purpose of this section is only to relate the computational method to regression analysis, not to give a complete explanation and interpretation.[4])

[4] See for example Anderson and Bancroft (*1*), chapters 13, 14, and 15.

In using this method to solve regression problems, as in other similar methods, it may be prudent to add a row sum or check column to the right of the identity matrix. If the same operations are performed on this column as are done on all the other columns, at any stage in the computations, the sum of all other elements in a row should equal the value of the element in the check column, of the same row. The completed computations can be checked by multiplying the original matrix by the inverse to check that AB=I:

$$a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+\ldots+a_{1n}b_{n1}=1$$

$$a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+\ldots+a_{1n}b_{n2}=0$$

Etc.

In making such checks, it may be found that the sums of the inner products do not equal 0 or 1, but are very near these values. Such discrepancies may be due to the number of decimal places carried in the computations. For most regression problems no more than eight decimal places need be carried. On small problems five or six places may be adequate. However, if there is a high degree of correlation between the independent variables more decimal places may be needed to prevent degeneracy (division by zero).

Linear Programming

With a few additional rules or steps, the procedure outlined in Section III can be used to solve linear programming problems. For programming problems the procedure is usually called the simplex method.

Looking at Exhibit H, let us assume that elements e_{01}, e_{02}, and e_{03} in the P_0 vector represent resources available to a particular firm, for example, 40=hours of labor, 50=hours of machine A time available, and 30=hours of machine B time available. Let the vectors P_1, P_2, and P_3 represent the quantities of each of these resources needed to produce one unit of products X, Y, and Z respectively. In linear programming each resource row must have an identity vector associated with it. Hence we need vectors P_4, P_5, and P_6 of Exhibit H. These vectors can appear in any position in the matrix; first, last or in the middle. And because in linear programming problems the numbers of columns does not need to equal the number of rows, let us add columns P_7, P_8, and P_9. Al-

EXHIBIT I

$C_j \rightarrow$			9	4	7	0	0	0	3	8	5	
$\downarrow$	Solution vector	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	R
0	P_4	40	6	4	2	1	0	0	2	4	2	6.7
0	P_5	50	4	9	3	0	1	0	4	5	3	12.5
0	P_6	30	2	3	5	0	0	1	3	3	2	15.0
	Z_j	0	0	0	0	0	0	0	0	0	0	
	$Z_j - C_j$	0	−9	−4	−7	0	0	0	−3	−8	−5	

though we do not need to, we can assume that these vectors represent alternative ways of producing commodities X, Y, and Z.

Further, in order to have a linear programming problem, we need a profit row. This is usually written above the basic matrix and called the C_j row. We also need other rows, usually called the Z_j and the $Z_j - C_j$. The latter is, of course, the Z_j row minus the C_j row as in Exhibit I.

The Z_j row is computed by multiplying the C_j values of the basis (identity) vectors, i.e., P_4, P_5, and P_6, by each of the vectors P_0 through P_9. For example, Z_j for P_0 is $0(40)+0(50)+0(30)=0$ and Z_j for P_1 is $0(6)+0(4)+0(2)=0$. Because in this case all C_j of the basis vectors are zero, all Z_j values are zero. In numerous programming problems, however, especially minimizing problems, the C_j values of the initial basis are nonzeros. Once the Z_j row and then the $Z_j - C_j$ row have been computed, the Z_j can be omitted from subsequent computations.

Exhibit I is the standard format for linear programming problems. The computational procedure outlined in the first part of this paper can be used to obtain the solution. The objective here is different, however. In words, it is: To find some non-negative levels of the P_1 to P_9 that will maximize net returns, given the resources available, i.e., the P_0 column. Also, the method of selecting rows and columns for sequential operations is different. At the outset we do not know which columns we want to convert to unit vectors and which elements we want to be unity or the number 1. Finally, we need a criterion to tell us when the answer is found. But let's take one thing at a time:

1. The operating column is the column with largest *negative* $Z_j - C_j$ element. In Exhibit I this is P_1, since its $Z_j - C_j$ value is -9.

2. The critical row is the row with smallest positive ratio of P_0 element to operating column element.[5] For example, in Exhibit I, given P_1 as operating column, $40/6=6.7$, $50/4=12.5$, $30/2=15.0$. Hence, row 1 is the critical row.

3. The optimal solution is obtained when all values in $Z_j - C_j$ row are non-negative (i.e., zero or positive). The optimal solution (maximum or minimum) is given by the values in the P_0 column.

The optimal profit solution to Exhibit I is derived by first converting column P_1 to a unit vector, with the number 1 in row one, the critical row. When this step is completed column P_9 or activity P_9 has the largest negative $Z_j - C_j$ value. It, therefore, is the operating column for the next step and row 3 is the critical row. After P_9 has been converted to a unit vector with the number 1 in row 3, all $Z_j - C_j$ values are non-negative, indicating the solution is optimal. These steps are not shown, to save space. The final matrix, after the described steps are completed, is given in Exhibit J.

The solution is as given by the P_0 column: 2.5 units of P_1 (product X), 12.5 units of P_9 (product Z), and 2.5 units of time on machine A unused or left idle. The profit is given in the pivotal element of the P_0 column and the $Z_j - C_j$ row.

Several checks are available to verify the optimal solution. One is the feasibility check which simply checks that resources are available to meet the specified levels of output. This can be checked by matrix multiplication, using the original vectors P_1, P_5, and P_9. We multiply

[5] In some problems this ratio may be zero. Computations can continue even though the ratio is zero. Also, the ratio for two rows may be the same. The selection of either row is permissible in this case.

EXHIBIT J

$C_j \rightarrow$		9	4	7	0	0	0	3	8	5
	Solution vector P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
9 P_1	2.5	1	.25	−.75	.25	0	−.25	−.25	.25	0
0 P_5	2.5	0	4.25	−3.75	−.25	1	−1.25	−.25	.25	0
5 P_9	12.5	0	1.25	3.25	−.25	0	.75	1.75	1.25	1
$Z_j - C_j$	[1] 85.0	0	4.50	2.50	1.00	0	1.50	3.50	.50	0

[1] Net profit.

these by the final P_0 vector and check to see that the product is equal to the original P_0 vector:

$$\begin{bmatrix} 6 & 0 & 2 \\ 4 & 1 & 3 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2.5 \\ 2.5 \\ 12.5 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 40 \\ 50 \\ 30 \end{bmatrix}$$

Multiplying each row of the left-hand matrix sequentially by the right-hand column or solution vector we see that the solution is feasible.

The net profit in the final matrix can be checked simply by multiplying the solution vector by the associated C_j value:

$$9(2.5)+0(2.5)+5(12.5)\overset{?}{=}85$$

Another check is to multiply the original resource levels by the values in the Z_j-C_j row of the final matrix columns P_4, P_5, and P_6, the original identity vectors.

$$1.0(40)+0(50)+1.5(30)\overset{?}{=}85$$

Checking we see that the equality is satisfied.

The final Z_j-C_j values associated with the original identity vectors are the shadow prices of the resources.

Summary

This paper has shown how a relatively simple computational technique can be used to solve several types of problems. Simple matrix algebra principles are stated. Then these principles are used to outline a uniform computational method that is easily memorized. With this method firmly in mind, the student can move easily from one type of problem to another without going to reference books for computational formulas that many times are difficult to follow.

Only hypothetical data are used in the examples presented. The basic theories of input-output analyses, multiple regression, and linear programming were not discussed beyond an attempt to show some of their similarities and dissimilarities. We did not discuss how data are collected and manipulated to build up the several matrices that are needed before computations can begin. Many references are available for those who need information on these subjects. For example, see (*1*), (*3*), (*5*), and (*6*).

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