A Polyperiod Model for Estimating the Supply of Milk

By Keith G. Cowling and C. B. Baker

The usual linear programming model for estimating supply functions is a single-period one (*3*).[1] The solution for this period is independent of other periods except as their activities and incomes may be reflected in the resource constraints (*2*). The polyperiod model not only specifies the optimum position toward which the adjustment is presumed to be moving, but also indicates something of the time path and the duration of the adjustment cycle. Methods for developing such a dynamic model are described in this paper. **The writers acknowledge the contributions of C. W. Crickman and others in the Farm Production Economics Division, Economic Research Service, who participated in the Regional Lake States Dairy Adjustment Study which provided much of the basic data. Valuable comments also were supplied by William Gossling, J. C. Headley, and Earl Swanson, all of the University of Illinois.**

THE POLYPERIOD MODEL presented here is sometimes referred to as a dynamic model (*4*). It defines a long-run plan over time with transition plans specified. For plans in a given solution, prices of activity units are assumed constant over the entire planning period. It may be described as a single model "in which the properties of growth have been imbedded (*9*)." It differs from a single period model by specifying activities separately for each production period and by providing for the transfer of resources from one period to the next to maximize discounted net income over the long run. Thus the optimal solution for any one period depends on the optima in other periods in the model and on the needs for consumption within periods. Viewed as a decision model, the polyperiod model yields a series of decisions over time that are jointly optimal. It also can be viewed as a firm growth model in the limited sense of a growth response permitted by capital accumulation at constant prices. Rates of growth can be compared among alternate prices at which solutions are obtained.[2]

In terms of supply analysis, the quantity of milk (m^t) generated by the polyperiod model in time period t is given by

$$m^t = f(p_m{}^t, p_j{}^t, r_i{}^t, a_{ij}{}^t)$$

Where

$p_m{}^t$ is the price of milk in the t^{th} time period

$p_j{}^t$ is the price on each of the alternative products which compete with milk for the use of resources and includes reservation prices on fixed resources to be allocated within the model in the t^{th} time period

$r_i{}^t$ is the quantity of each resource available to the firm in the t^{th} time period

$a_{ij}{}^t$ is the quantity of each resource required per unit of each product produced in the t^{th} time period.

Thus we can derive a three-dimensional supply function with price, quantity and time on the three axes with one plane defining optimal adjustment to the product price at a specified point in time and another plane defining, for a given price, the optimum growth in supply over time.

The model provides for the transfer of three major types of resources from one period to the next: Cash, dairy cows, and building capacity. The amount of cash transferred from one period to the next depends on the income generated in the first period, the level of income constraint for each period (assumed to cover fixed costs and a minimum level of consumption), and the marginal propensity to consume any income above the level of the income constraint.

It is assumed that investment must come from retained earnings and not from borrowing. The number of dairy cows transferred from one period to the next depends on the number of dairy cows entering the solution in the preceding period, the

[1] Italic numbers in parentheses refer to Literature Cited, page 22.

[2] Candler views the polyperiod model as just another example of parametric programming, each period distinguished by different resource constraints, and suggests the use of parametric procedures to lessen the computational burden. (See (*1*).) However, it is difficult to see how such a procedure could be applied to the model used herein. The solutions for the "n" periods are obtained simultaneously so it is impossible to specify the resource constraints for any but the first period, except where these are fixed over the whole planning horizon.

assumption concerning level of replacement, and the relative profitability of using resources in one period for raising heifers to expand the herd in the following period. The assumption is made that in the aggregate it is not realistic to consider that dairy herds can be expanded by purchasing cows. In a single region, this assumption is violated in reality by interregional transfer of dairy cows.

We note that in a polyperiod model the technique of variable price programming cannot be undertaken as we must specify the price of milk in order to be able to define the relevant coefficient in the income equation for each period.

Results reported below relate to specialized dairy farms in the Illinois part of the Chicago milkshed. Most of the data are from a survey of farms found in randomly selected area segments in a 7-county area surrounding Cook County (fig. 1). The 1959 census reported 7,494 farms in the sample counties. We found about 46 percent of the farms in the sample areas to be "dairy farms." Of these, about two-thirds were specialized in the sense that they had no facilities for livestock other than dairy animals. Resource constraints were defined as for a farm typical of those specialized in dairy production. Relevant activities were limited to those specifically related to dairying.[3] Additional input-output data were derived from cost account records for dairy farms in northern Illinois (*4*, *6*) and from experimental data (*8*).

The Model

The model covers 3 periods, each of 3 years. Each period is represented by the middle year. The 3-year periods represent the time required to expand herd size with retained heifers. Each period has 13 equations, together with a variable number of structural activities. In all three periods, the cropping system is taken as given: 188 acres in a rotation of C–C–O–Cl–Cl.[4] Thus 75 acres of corn are available for grain (at 81 bushels per acre) or for silage (at 14.5 tons per acre) and 75 acres of legume are available for hay (at 2.9 tons per acre) or for pasture (at 180 animal-unit days per acre). Grazing is available for 3,479 animal-unit days from permanent pasture in addition to whatever grazing is available by diverting legume meadow from hay harvest. The 38 acres of oats are assumed to yield 1,121 bushels of corn-equivalent feed.

Constraints

In the first period, the harvest of silage is restricted to available silo capacity. Labor available in winter and summer is restricted to that available on the average farm after substracting out the requirements associated with the fixed part of the farm organization—the crop system (except harvesting requirements) and minor amounts for other livestock. The number of milk cows available is 29, the number observed on the average farm. We assume that for the whole of Period 1 there are just sufficient calves, yearling heifers, and springing heifers to maintain herd size. No expansion is possible until Period 2. Cash is restricted to the supply of liquid and semi-liquid assets after account has been taken of the crop acreage requirement. We require that annual income generated in Period 1 be at least $5,000. This is an arbitrary figure; it is intended to represent a minimum consumption level plus an amount that would cover fixed costs. The dairy activities, the corn-selling activity, and a cash-transfer activity contribute to the fulfillment of this constraint. It is also possible, within the model, to sell cows. Any excess income is siphoned off by the savings activity, 45 percent being used for consumption and the remaining 55 percent contribution to cash in Period 2.[5]

Period 2 repeats the same equations used in Period 1, the only differences being in the definition of the milk-cow equation, the cash equation, and the income equation. Milk cows and cash are transferred from Period 1, cows by way of the dairy activities, cash by way of the savings activity. Thus the initial supply of these resources is shown at zero at the beginning of Period 2. The income requirement assumes a higher value than that in Period 1, because we now include the cash requirements of a fixed acreage of crops, whereas in the previous period the cash requirements were subtracted from an initial cash supply.

[3] See forthcoming bulletin of the Illinois Agricultural Experiment Station for a report of supply responses for nonspecialized farms in the areas.

[4] This is the rotation that dominated solutions of a single period model wherein an array of rotations comprised alternatives for the use of cropland.

[5] The implied marginal propensity to consume of 0.45 conforms to an estimate made for farm families in the North Central region in (*5*).

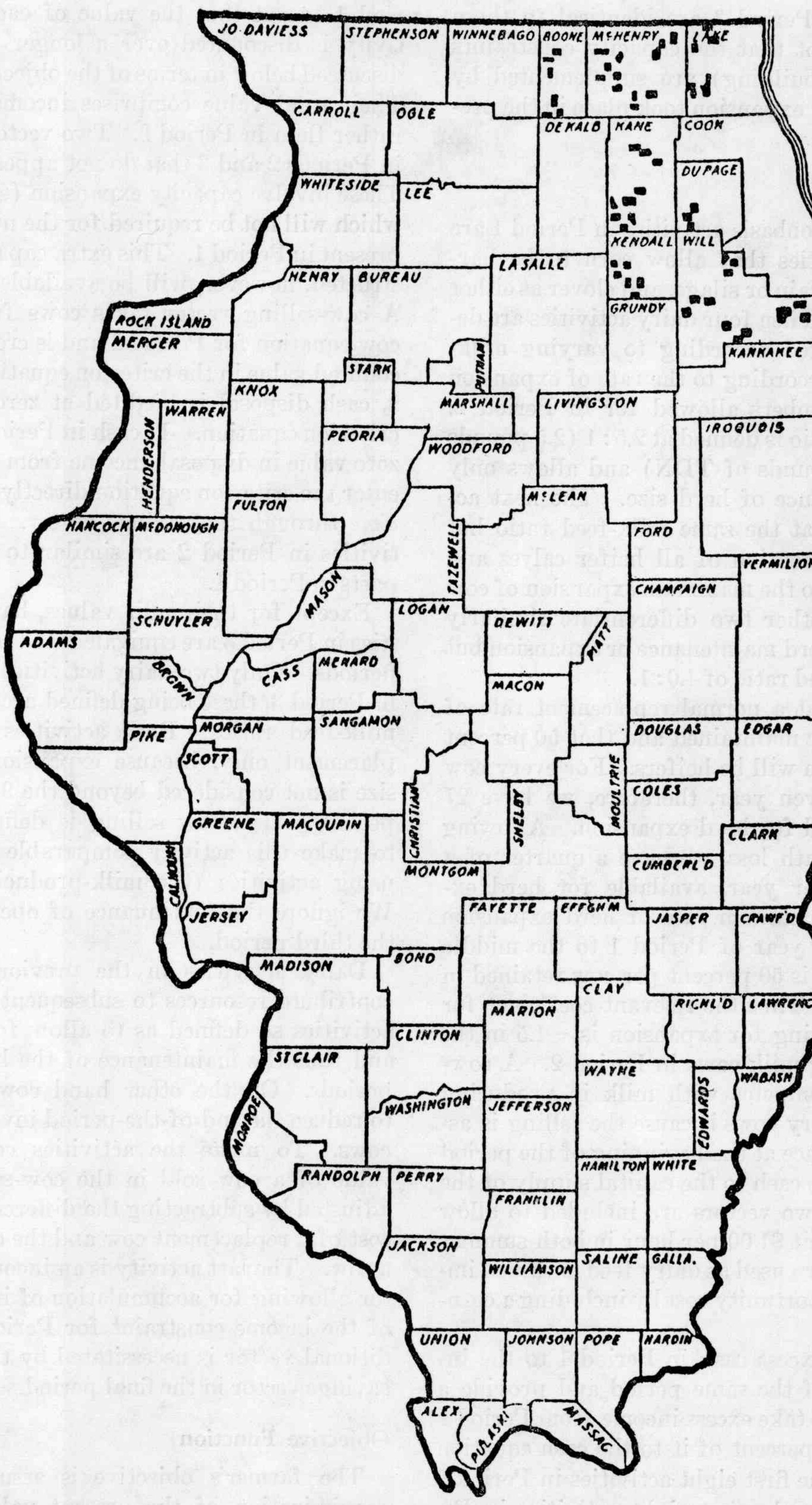

Figure 1. The study area.

Constraints in Period 3 are identical to those in Period 2 except that the capacity constraints (silo and dairy building) are supplemented by whatever capacity expansion took place in the previous period.

Activities

The first four nonbasis activities in Period 1 are harvesting activities that allow corn to be harvested as either grain or silage, and clover as either hay or pasture. Then four dairy activities are defined, differentiated according to varying milk-feed ratios and according to the rate of expansion of dairy cow numbers allowed for in Period 2. One milk-feed ratio is defined at 2.5 : 1 (2.5 pounds of milk to 1.0 pounds of TDN) and allows only for the maintenance of herd size. The next activity is defined at the same milk-feed ratio but allows for the retention of all heifer calves and thus contributes to the maximum expansion of cow numbers. The other two differentiate similarly with respect to herd maintenance or expansion but assume a milk-feed ratio of 4.0 : 1.

We assume that a normal replacement rate of 23 percent will be maintained and that 50 percent of the calves born will be heifers. For every cow retained in a given year, therefore, we have 27 percent of a head for herd expansion. Allowing for 2 percent death loss, we have a quarter of a head per cow per year, available for herd expansion. The maximum rate of herd expansion from the middle year of Period 1 to the middle year of Period 2 is 50 percent per cow retained in the first period. Thus the relevant coefficient for the vectors allowing for expansion is -1.5 in the row representing milk cows in Period 2. A cow-selling vector competes with milk in producing activities for dairy cows because the selling is assumed to take place at the beginning of the period and to contribute cash to the capital supply of the same period. Two vectors are included to allow for labor hiring at $1.00 per hour in both summer and winter. Corn used as dairy feed is valued implicitly at its opportunity cost by including a corn-selling activity.

We transfer excess cash in Period 1 to the income equation of the same period and provide a savings vector to take excess income from Period 1 and transfer 55 percent of it to the cash equation of Period 2. The first eight activities in Period 2 are equivalent to the first eight activities in Period 1 except that the value of each unit of activity is discounted over a longer period (to be discussed below in terms of the objective function). Their total value comprises income in Period 3 rather than in Period 2. Two vectors are present in Periods 2 and 3 that do not appear in Period 1. These involve capacity expansion (silo and dairy) which will not be required for the number of cows present in Period 1. This extra capacity, once constructed, however, will be available for Period 3. A cow-selling vector takes cows from the dairy cow equation for Period 2 and is credited at a discounted value in the criterion equation. In Period 2, cash disposal is credited at zero value in the criterion equation. If cash in Period 2 had a nonzero value in disposal, income from Period 1 could enter the criterion equation directly *for the second time* through the savings vector. The other activities in Period 2 are similar to their counterparts in Period 1.

Except for their unit values, harvesting activities in Period 3 are equivalent to those in the other periods. Only two dairy activities are considered in Period 3, these being defined according to their milk-feed ratios. Both activities allow for replacement only, because expansion of the herd size is not considered beyond the 9-year planning period. Also, cow selling is defined differently to make this activity comparable with the cow-using activities (the milk-producing activities). We ignore the continuance of operations beyond the third period.

Dairy activities in the previous two periods contribute resources to subsequent periods, dairy activities so defined as to allow for replacement, and thus the maintenance of the herd for future periods. On the other hand cow selling serves to reduce the end-of-the-period inventory of dairy cows. To make the activities comparable, the value of a cow sold in the cow-selling vector is adjusted by subtracting the difference between the cost of a replacement cow and the carcass value of a cow. The last activity is an income-dummy vector allowing for accumulation of income in excess of the income constraint for Period 3. This additional vector is necessitated by the absence of a savings vector in the final period.

Objective Function

The farmer's objective is assumed to be the maximization of the present value of a future stream of income. Consistent with this end, we discount (at five percent per year) the value of an activity unit in each period according to the middle year of a particular period. Varying the price of milk entails adjusting not only the C_j values for the dairy activities, but coefficients of the dairy activities in the income equations as well.

Optimal Solutions [6]

The model was solved at selected milk prices over the range from \$2.70 per hundredweight to \$5.10 per hundredweight. At no price within this range was it profitable to contract herd size in the first period. The number of cows entering the dairy activities was constant at 29 irrespective of the price of milk over the range considered. In Period 2 the maximum herd size was 43. For all milk prices except \$2.70, this also was the optimal number. In the third period the optimal number of cows in the dairy activities fell short of the maximum possible, the actual number varying from 39 for the lowest milk price up to 55 for the highest.

The optimal milk-feed ratio proved to be 4.0:1 for milk prices of \$2.70 and \$3.50, with the 2.5:1 ratio optimal for all higher prices. This generalization extends to all periods except that in Period 2, for a milk price of \$3.50, the optimal feed ratio was 2.5:1.

Cows were sold only in Period 2 of the model, with milk priced at \$2.70. The $Z_j—C_j$ for cow selling reflects the value of a cow within the farm organization for a particular period *in addition* to its designated selling price, declining with respect to both milk price and time. The cows are most valuable in Period 1 because the present value of their output exceeds that of subsequent periods, and because they provide the only source of later herd expansion.[7] The magnitude of the $Z_j—C_j$'s in Period 1 is of such proportions that we would expect the solution to remain stable over a relatively wide range of lower prices for milk.

In Period 1 cash did not effectively restrict the program. In Periods 2 and 3, however, it proved to be an effective constraint over an intermediate range of milk prices. At low milk prices, it was not profitable to invest in large-scale expansion, whereas, with high milk prices, the additional income generated was more than ample to meet the additional capital requirements. Within the intermediate range of milk prices, the $Z_j—C_j$'s for cash indicated a return of up to 19 cents for an added dollar.

[6] All income and output figures quoted in this discussion are on an annual basis.

[7] The values for $Z_j—C_j$ in Period 1 would be reduced, of course, had the model included the alternative of buying cows in Periods 2 and 3.

TABLE 1.—*Quantities of milk per year in each of three periods, at selected milk prices: modal farm specialized in milk production, Illinois sector of Chicago milkshed*

Price of milk	Period		
	1	2	3
	Cwt.	*Cwt.*	*Cwt.*
\$2. 70	2, 900	3, 764	3, 854
3. 50	2, 900	4, 569	4, 820
3. 90	3, 164	4, 746	5, 692
4. 30	3, 164	4, 746	5, 692
5. 10	3, 164	4, 746	6, 001

Supply Relationships

The quantities of milk generated by the model at specified milk prices are arranged in table 1. To these discrete points, we fit free hand the curves shown in figures 2 and 3. Figure 2 displays supply curves, abscissas of which are milk quantities expected at each of the price levels in the time period indicated. Figure 3 gives the growth over time of milk supply in response to price levels indicated. To the data we also fitted the following cubic function for each period:

$$Y = a + b_1 p_y + b_2 p^2_y + b_3 p^3_y$$

where Y equal quantity of milk, and p_y is the price of milk.

From this fitted function, estimates of the elasticity of supply with respect to price are given in table 2. Either of the curve-fitting procedures may be criticized from two points of view. First, we are fitting discrete points which are not random variables, and, second, we are fitting a continuous function when programming models postulate that the firm is faced with an array of restricting resources which result in a stepped or discontinuous response function. These arguments may be countered by the fact that we need some uniform method for computing elasticities, and that we are concerned with the aggregative response of a population of firms so that the smoothing operation is perhaps not inappropriate.

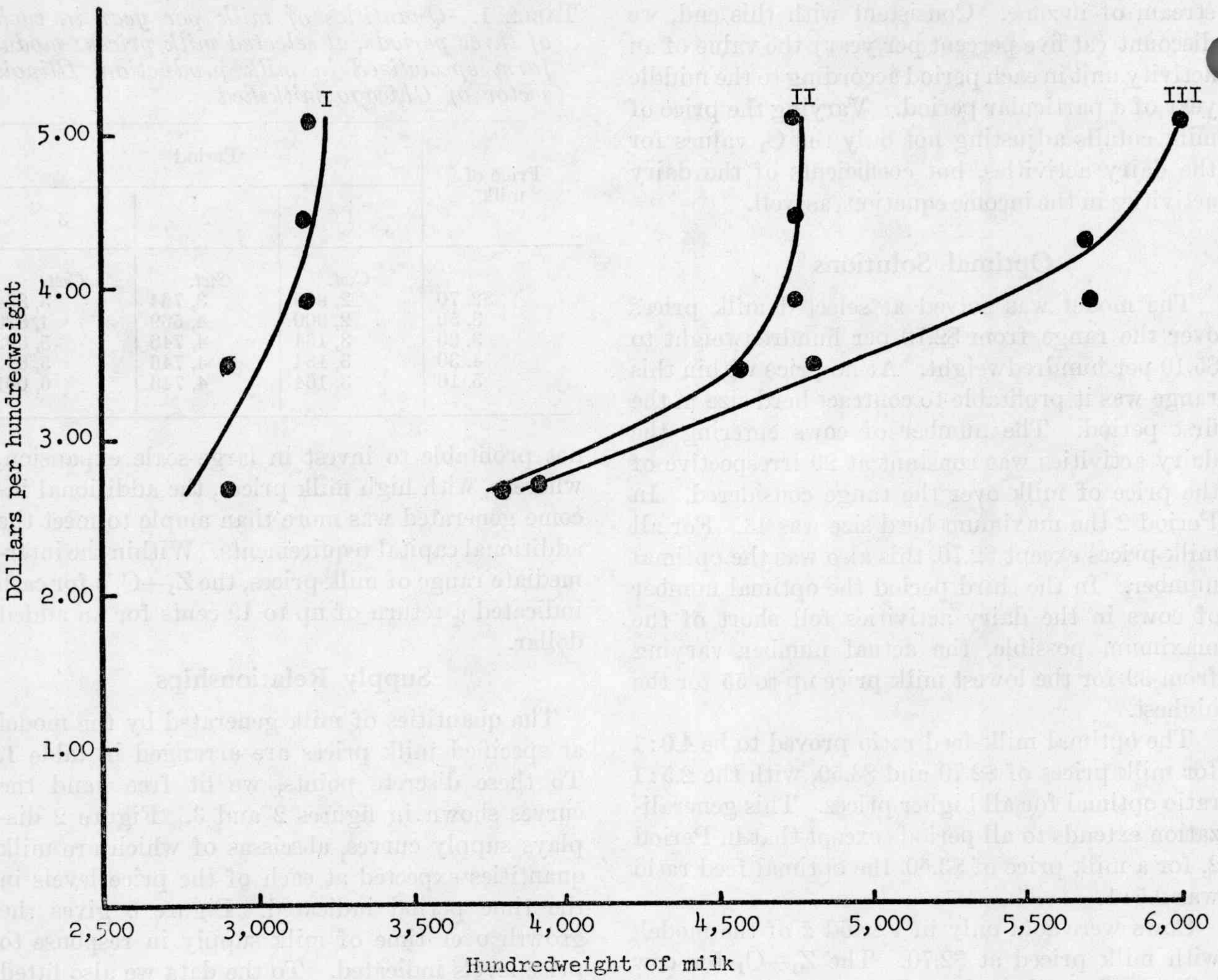

Figure 2. Milk supply response to milk price, by time period for a modal farm.

TABLE 2.—*Short-run elasticities of milk supply with respect to milk price: modal farm specialized in milk production, Illinois sector of Chicago milkshed*

Price of milk	Period 1	Period 2	Period 3
$2.70	0.00	1.15	0.51
3.50	.35	.28	1.22
3.90	.44	.05	.67
4.30	.00	.00	.00
5.10	.00	.00	.00

The elasticities given in table 2 are derived from the cubic functions fitted by least squares procedures and do not necessarily coincide with those implied by the free hand curves, though differences are slight. Given their empiric origin, they are "conventional" in concept. That is, for each year, the elasticity of supply with respect to price is given by

$$\frac{dy}{dp_v}\cdot\frac{p_v}{y}=\frac{1}{y}\left(b_1p_v+2b_2p_v^2+3b_3p_v^3\right).$$

Thus at $2.70, a zero response would be expected within Period 1 to a 1 percent increase in the price of milk (see table 2). Though high enough to prevent sales to reduce cow numbers below the maximum allowed by the facilities, the price is not high enough to induce changes in rations that would increase milk production. However, in Period 2, a substantial response would occur, given an increase in the price of milk. In this period,

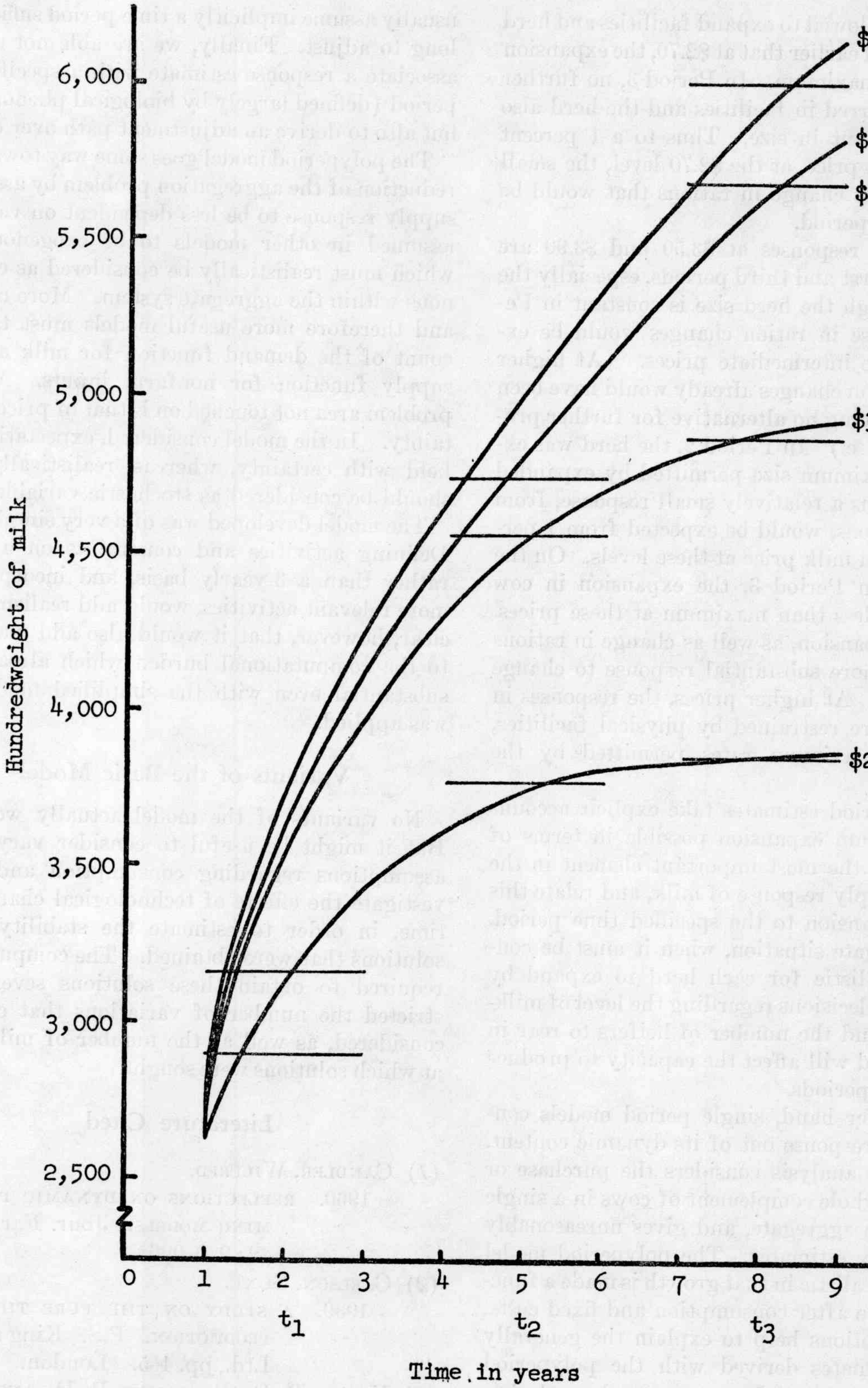

Figure 3. Milk supply response to time, by level of milk price for a modal farm.

investment is allowed to expand facilities and herd size. We noted earlier that at $2.70, the expansion was less than maximum. In Period 3, no further expansion occurred in facilities and the herd also remained constant in size. Thus to a 1 percent change in milk price, at the $2.70 level, the small response reflects change in rations that would be optimal in this period.

In contrast, responses at $3.50 and $3.90 are higher in the first and third periods, especially the third. Although the herd size is constant in Period 1, response in ration changes would be expected at these intermediate prices. (At higher prices, the ration changes already would have been made, thus leaving no alternative for further production response.) In Period 2, the herd was expanded to maximum size permitted by expanded facilities. Thus a relatively small response, from change in rations, would be expected from 1 percent changes in milk price at these levels. On the other hand, in Period 3, the expansion in cow numbers was less than maximum at these prices. Thus herd expansion, as well as change in rations permitted a more substantial response to change in milk price. At higher prices, the responses in each period are restrained by physical facilities, growing at maximum rates permitted by the model.

The polyperiod estimates take explicit account of the maximum expansion possible in terms of cow numbers, the most important element in the long-term supply response of milk, and relate this potential expansion to the specified time period. In the aggregate situation, when it must be considered unrealistic for each herd to expand by buying cows, decisions regarding the level of milking activity and the number of heifers to rear in any one period will affect the capacity to produce in subsequent periods.

On the other hand, single period models consider supply response out of its dynamic content. Such a static analysis considers the purchase or raising of a whole complement of cows in a single period and in aggregate, and gives unreasonably high elasticity estimates. The polyperiod model also is more realistic in that growth is made a function of income after consumption and fixed costs. These assumptions help to explain the generally inelastic estimates derived with the polyperiod model. Also we see that response depends importantly on time, whereas single period models usually assume implicitly a time period sufficiently long to adjust. Finally, we are able not only to associate a response estimate with a specific time period (defined largely by biological phenomena), but also to derive an adjustment path over time.

The polyperiod model goes some way toward the reduction of the aggregation problem by assuming supply response to be less dependent on variables assumed in other models to be exogenous, but which must realistically be considered as endogenous within the aggregate system. More realistic and therefore more useful models must take account of the demand function for milk and the supply function for nonfarm inputs. Another problem area not touched on is that of price uncertainty. In the model considered, expectations are held with certainty, whereas, realistically, they should be considered as stochastic variables.

The model developed was of a very simple form. Defining activities and constraints on a yearly rather than a 3-yearly basis, and incorporating more relevant activities, would add realism. It is clear, however, that it would also add materially to the computational burden which already was substantial even with the simplified model that was applied.

Variants of the Basic Model

No variants of the model actually were run. But it might be useful to consider varying the assumptions regarding consumption and to investigate the effects of technological change over time, in order to estimate the stability of the solutions that were obtained. The computer time required to obtain these solutions severely restricted the number of variations that could be considered, as well as the number of milk prices at which solutions were sought.

Literature Cited

(*1*) Candler, Wilfred.
1960. reflections on dynamic programming models. Jour. Farm Econ. 42: 920–926.

(*2*) Carlson, Sune.
1939. a study on the pure theory of production. P.S. King and Son, Ltd., pp. 4–5. London.

(*3*) Krenz, R. D., Baumann, R. D., and Heady, E. O.

1962. NORMATIVE SUPPLY FUNCTIONS BY LINEAR PROGRAMMING PROCEDURES. Agr. Econ. Res. 14: 13–17. Jan.

(4) LOFTSGARD, L. D., AND HEADY, E. O.
1959. APPLICATION OF DYNAMIC PROGRAMMING MODES FOR OPTIMAL FARM AND HOME PLANS. Jour. Farm Econ. 41: 51–62.

(5) LONGMORE, T. W., AND TAYLOR, C. C.
1951. ELASTICITIES OF EXPENDITURE FOR FARM FAMILY LIVING, FARM PRODUCTION AND SAVINGS, UNITED STATES. Jour. Farm Econ. 33: 1–19.

(6) MUELLER, A. G.
1958. DETAILED COST REPORT FOR NORTHERN ILLINOIS, 1956. Ill. Agr. Econ. AERR–21 (mimeo). Urbana.

(7) MUELLER, A. G., AND VON LANKEN, G. D.
1957. DETAILED COST REPORT OF NORTHERN ILLINOIS. Ill. Agr. Econ. AERR–15 (mimeo). Urbana.

(8) REDMAN, J. C., AND OLSON, R. O.
1956. ECONOMIC PROBLEMS IN FEEDING DAIRY COWS. Ky. Agr. Expt. Sta. Bul. 648.

(9) SWANSON, E. R.
1958. PROGRAMMING OPTIMAL FARM PLANS. In Farm Size and Output Research, Southern Cooperative Series Bul. 56. Okla. Agr. Expt. Sta., pp. 47–58.