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Towards a Theory of Security Price Adjustment

by

Terry Roe and Mathew Shane

Department of Agricultural and Applied Economics

University of Minnesota
Institute of Agriculture
St. Paul, Minnesota 55101

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I. Introduction

It is generally conceded that economic theory has not been effective in providing an explanation or prediction of security market behavior. Although the mathematical theory of portfolio allocation is well developed [2], the conditions consistent with the theory of individual portfolio allocation have not been utilized to explain observed security price movements.

The analytic foundations of empirical studies of security market behavior are not explicitly stated and tend to relate only to aggregate investment behavior. While some of these studies appear to obtain results consistent with the random walk hypothesis, the lack of an adequate theory to explain security market behavior remains a fundamental problem. This has been pointed out by Granger and Morgenstern [7, p. 1], Moore [10, p. 140], Fama [4, p.36], and others.

This paper will develop a theoretical model of security market behavior. Based on the assumption that security market participants maximize anticipated profits over a finite horizon, seven decision rules are determined. Quantifying the decision rules, individual supply and demand functions are derived. These are aggregated into market supply and demand and excess demand. A nontatonnement adjustment is defined as positively related to excess demand. The characteristics of the market supply and demand and adjustment functions are then utilized to demonstrate the equilibrium and stability properties of the model under two assumptions about expectation changes. In this way an explanation of the security market price behavior is developed.

In the next section, market participant and nonparticipant sets are defined and decision criteria established.

II. Definitions and Notation

In this section the definitions of the participant and nonparticipant sets are introduced along with the concept of security value and quantity. This is followed by the concept of investor costs.

Definition of Investor Sets: Potential investors are divided into two groups denoted as participants and nonparticipants. These are defined as follows:

$\tilde{I} = \hat{I}_t + \bar{I}_t$ where I denotes the total set of potential investors over the entire planning horizon, $\hat{I}_t$ denotes the set of individual participants at time t and $\bar{I}_t$ denotes the set of individuals not participating at time t. Furthermore,

$\hat{I}_t = \hat{I}_{st} + \hat{I}_{dt}$ where $\hat{I}_{st}$ denotes a set of security suppliers and $\hat{I}_{dt}$ denotes a set of security demanders at time t and

$\bar{I}_t = \bar{I}_{kt} + \bar{I}_{\emptyset t}$ where $\bar{I}_{kt}$ denotes a set of individuals holding securities at time t as the consequence of a buy long and/or sell short transaction and $\bar{I}_{\emptyset t}$ denotes a set of individuals who are non-participants and nonholders of a security.

It is assumed that all investors operate over a finite planning horizon where t_o is the initial period, T the horizon and t and $t+\tau$ are assumed to be less than T. The period $t+\tau$ represents an arbitrary point between a given t and the horizon T.

Security Quantity and Value Definitions: The total outstanding quantity of identical issues of the security, Q, is assumed fixed for all t. This total quantity can be divided into a quantity (q_t) traded and (Q_t) not traded at time t, i.e.,

$$Q = Q_t + q_t .$$

Fama [4, p.36] suggests that in addition to the observed price of a security, investors consider another security value, the so called "intrinsic value". We introduce still a third value concept, that of anticipated price. We define these explicitly as follows:

P_t is the price of the security in t;

$E_t^i(P)$ denotes the intrinsic value (expected price) of the security, as evaluated by individual i at time t;

$A_t^i(P_{t+\tau})$ denotes the i^{th} individuals anticipated market price at time t for some future time period $t+\tau \leq T$;

where all of the above are discounted for transaction cost and dividend payments.

The intrinsic value at time t does not involve a specific period within the planning horizon. The derivation of this expected price by the investor can be considered in either of two ways: This value may either be determined by evaluating the factors which effect the earnings of a company directly, or may represent equilibrium prices evolved from some dynamic adjustment process envisioned. It is irrelevant for this paper how these values are determined. The important point is that each individual i of the index set $\tilde{I}$ have such an evaluation.

Since it is reasonable to expect that an individual can conceive of a security price in the future that is different from his current intrinsic value, the concept of an anticipated price is introduced. Implicitly, this assumes that each individual has formulated some concept of the adjustment between current and his intrinsic price. If an investor viewed anticipated price diverging from his intrinsic evaluation, this would suggest a reevaluation of the securities intrinsic value. Thus, it is assumed that the investor's anticipated price is expected to converge to the intrinsic evaluation within his planning horizon. Stated analytically:[1]

$$A^{i}_{t-1}(P_{t+\tau}) \searrow E^{i}_{t}(P), \text{ as } t+\tau \searrow T.^{2/}$$

Investor Costs: It is assumed that each investor has an expected cost function associated with each investment alternative over the planning horizon. The principle components of this cost include a differential risk premium ($r_{t,t+\tau}$) and an opportunity cost ($s_{t,t+\tau}$) each evaluated in time t-1 for the horizon t to $t+\tau \leq T$.

$r_{t,t+\tau}$ is the difference between the risk premium associated with any two investment alternatives. The risk premium for any investment alternative reflects an indifference mapping between the risk associated with a particular security (where risk may be evaluated in terms of past observed variance in the value of a security, or a subjective evaluation of the variance associated with the expected value of a security) and some additional expected return.

The opportunity cost faced by the i^{th} investor is the total expected return from the next best investment alternative from t to $t+\tau$. In terms of an alternative security, this value is equal to the difference between the adjusted sale price of a security less its adjusted purchase price, i.e., ($P_{t+\tau}-P_t$). Thus the i^{th} investor in the t^{th} period is facing the expected cost function

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[1]Fama [4] suggests that prices do adjust to intrinsic values, thus lending additional support to the validity of this assumption.

[2]The arrow ($\searrow$) implies converges to or approaches.

$$\varepsilon^{i}_{t,t+\tau} = s_{t,t+\tau} + r_{t,t+\tau}$$

associated with an investment alternative over the life of this investment, i.e., from t to $t+\tau$.

In the next section, using the concepts of price and cost developed about, seven decision criteria will be presented.

III. The Decision Process

Decision criteria for the following seven investor categories are presented in this section: (1) buy long, (2) sell short, (3) hold long, (4) non-holder, non-participant, (5) sell short, (6) buy short, (7) hold short. These decision criteria are mutually exclusive since for any security, no investor can participate in more than one action at a given time.

Investor decisions are considered to take place during discrete time intervals for purposes of convenience. These time intervals are assumed to occur prior to the transaction process.[3/] The consequence of this decision then defines the set $\hat{I}_{dt}$, $\hat{I}_{st}$, $\bar{I}_{kt}$, $\bar{I}_{\emptyset t}$ to which the investor belongs. Transaction periods (t), $t=1,2,\ldots,T$ are thus defined such that for any given t the "factors" or "conditions" defining membership to the sets $\hat{I}_{dt}$, $\hat{I}_{st}$, $\bar{I}_{kt}$, $\bar{I}_{\emptyset t}$ are determined exogenous to the transaction process. What follows is an elaboration of the particular decision rules.

[3/] Godfrey et. al. [6] provides evidence which supports the assumption that the decision and transaction process are separate.

III.A. Buy Long - Sell Short

The i^{th} investor can be considered a candidate for buying long in time t if over his planning horizon his expected return (his intrinsic value minus his purchase price) is greater than his discounted opportunity cost, i.e., if

$$E^i_{t-1}(P) > P_t + \varepsilon^i_{t,t+\tau} \quad \forall t+\tau \leq T. \tag{III.1.0}$$

Likewise, an investor can be considered a candidate for a sell short position if

$$E^i_{t-1}(P) < P_{t-1} - \varepsilon^i_{t,t+\tau} \quad \forall t+\tau \leq T. \tag{III.2.0}$$

In both (III.1) and (III.2) the investor is considered a candidate for these actions since each represents an expected profit. Although the investor may plan to participate in the market at time t based on previous information in t-1, because of changes which occur in t, he may not actualize his anticipated decision.

The i^{th} investor is indifferent between buying and not buying long (selling short) in time t if the anticipated price change between t and t+τ is equal to the return forgone and the differential risk, i.e., he is indifferent to buying long if

$$A^i_{t-1}(P_{t+\tau}) - P_t = \varepsilon^i_{t,t+\tau}, \forall t+\tau \leq T$$

and indifferent to sell short if

$$A^i_{t-1}(P_{t+\tau}) - P_t = \varepsilon^i_{t,t+\tau}, \forall t+\tau \leq T.$$

Since $\varepsilon^{i}_{t,t+\tau}$, is anticipated and an increasing function of time--at the very least, equal to the interest return from a savings bank--the price must rise by at least $\varepsilon^{i}_{t,t+\tau}$ for the investor to be indifferent between the preferred action A_1 and the next best action A_2. To prefer A_1 to A_2--the buy long (sell short) to the alternative--the anticipated return must be greater than $\varepsilon^{i}_{t,t+\tau}$. This, together with (III.1) and (III.2), provides the decision rule: buying long if

(III.1.1) $$A^{i}_{t-1}(P_{t+\tau}) - P_t \geq \varepsilon^{i}_{t,t+\tau}, \ \forall t+\tau \leq T$$

and selling short if

(III.2.1) $$A^{i}_{t-1}(P_{t+\tau}) - P_t \leq - \varepsilon^{i}_{t,t+\tau}, \ \forall t+\tau \leq T.$$

Remark: From the above relationships, an anticipated profit function for the buy long, $\pi^{BL}_{t,t+\tau}$ and sell short $\pi^{SS}_{t,t+\tau}$ decision is defined by subtracting costs from returns i.e.,

(III.1.2) $$\pi^{BL}_{t,t+\tau} = A^{i}_{t-1}(P_{t+}) - P_t - \varepsilon^{i}_{t,t+\tau}$$

(III.2.2) $$\pi^{SS}_{t,t+\tau} = P_t - A^{i}_{t-1}(P_{t+\tau}) - \varepsilon^{i}_{t,t+\tau}.$$

The optimal profit is achieved for any given t by choosing a τ_* such that (III.1.2) or (III.2.2) is a maximum over $t+\tau \leq T$. Given expectations for each decision, an optimal τ_* is chosen which will define the period of involvement. A similar profit function can be defined for each of the decision rules to be discussed.

III.B. Sell Long and Buy Short

The decisions to sell long (buy short) is the terminating action of the action to buy long (sell short). Based on expectations and anticipations, the objective of the investor is to obtain a preferred position by selling long (buying short) now (action A_1) or by holding for some future point (action A_2).

The i^{th} investor is considered a candidate for selling long in t if

$$(III.3.0) \qquad E^i_{t-1}(P) < P_t + \varepsilon^i_{t,t+\tau}, \quad \forall\ t+\tau \leq T$$

and a candidate for buying short in t if

$$(III.4.0) \qquad E^i_{t-1}(P) > P_t - \varepsilon^i_{t,t+\tau}, \quad \forall\ t+\tau \leq T.$$ [4]

Condition (III.3.0) states that the i^{th} investor's evaluation of the difference between the intrinsic price and the actual price provides a profit (A_2) in t which is less than the anticipated return, adjusted for a risk premium, than can be obtained by selling long in t and investing in the best alternative (A_1).

Condition (III.4.0) can be restated as:

$$P_t - E^i_{t-1}(P) < \varepsilon^i_{t,t+\tau}.$$

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[4] $\varepsilon^i_{t,t+\tau} = s^i_{t,t+\tau} + r^i_{t,t+\tau}$, $t+\tau$ is the anticipated period of time required for the difference $P_t - E^i_{t-1}(P)$ to approach zero, i.e., the anticipated length of time required for the potential profits $P_t - E^i_{t-1}(P)$ to be realized. Therefore, $\varepsilon^i_{t,t+\tau}$ is the discounted return obtainable by participating in an alternative security or action.

This condition states that the difference between the intrinsic value and the actual price, i.e., the potential remaining gross profit (A_2) is less than the adjusted anticipated return that can be obtained by buying short in t and investing in alternative A_1.

We will now state the conditions for an investor's selection of the point in time to actuate his decision. The cost of selling long (buying short) in t is that price increase (decrease) the investor gives up by selling (buying back) now. Investor anticipated returns are maximized by selling long (buying short) when the anticipated returns from holding the issue is less than the return from not holding the issue. Analytically, this amounts to the opposite of the previous buy long-sell short conditions (III.1.1) and (III.2.1). Thus sell long in t if:

(III.3.1) $$A^i_{t-1}(P_{t+\tau}) - P_t \leq \epsilon^i_{t,t+\tau}, \; \forall \; t+\tau \leq T.$$

and buy short in t if

(III.4.1) $$A^i_{t-1}(P_{t+\tau}) - P_t \leq -\epsilon^i_{t,t+\tau}, \; \forall \; t+\tau \leq T.$$

Remark: The anticipated profit function over the remaining period $t+\tau$ for the sell long position π_{SL_t} and buy short position π_{BS_t} is derived by comparing the purchase price with the current and future expected prices:

(III.3.2) $$\pi_{SL_t} = A^i_{t-1}(P_{t+\tau}) - P_t - \epsilon^i_{t,t+\tau}$$

(III.4.2) $$\pi_{BS_t} = P_t - A^i_{t-1}(P_{t+\tau}) - \epsilon^i_{t,t+\tau}.$$

The difference $A^i_{t-1}(P_{t+\tau}) - P_t$ and $P_t - A^i_{t-1}(P_{t+\tau})$ are potential gross profits remaining in the sell long and buy short positions anticipated over τ periods. These gross profits must be equal to or less than

$\varepsilon^{i}_{t,t+\tau}$ to effectuate a transaction activity consistent with (III.3.1) and (III.4.1).

III.C. Hold Long - Hold Short

Anticipated profits are maximized from a hold long (short) position in time period t if the investor anticipates that price will increase (decrease) within a period τ by an amount greater than the adjusted returns $\varepsilon^{i}_{t,t+\tau}$ from his next best alternative. In other words, hold long if:

$$A^{i}_{t-1}(P_{t+\tau}) \geq P_t + \varepsilon^{i}_{t,t+\tau} \text{ for some } t+\tau \tag{III.5.0}$$

and hold short if:

$$A^{i}_{t}(P_{t+\tau}) \leq P_t - \varepsilon^{i}_{t,t+\tau} \text{ for some } t+\tau \tag{III.6.0}$$

By requiring that this condition holds <u>for some</u> and <u>not all</u> $t+\tau$, we allow for anticipated prices within the period t to $t+\tau - 1$ to be less than P_t, in the case of the hold long position and greater than P_t in the hold short position. This, of course, precludes either a buy long or sell long action from the hold long position and a sell short or buy short activity from the hold short position.

The necessary condition for holding long is:

$$E^{i}_{t-1}(P) > P_t + \varepsilon^{i}_{t+\tau}, \tag{III.5.1}$$

while that for holding short is:

$$E^{i}_{t-1}(P) < P_t - \varepsilon^{i}_{t,t+\tau}. \tag{III.6.1}$$

III.D. Non-Holder, Non-Participant

For an investor to be a non-holder, non-participant he must believe that he will make a higher return by neither buying long or selling short. This could occur for several reasons: (1) an investor does not expect a sufficient change in price to warrant an action, (2) returns from current investments are sufficiently high that even with a "substantial" expected change in price, he is better off continuing his current activity, (3) "market conditions" are such that future price changes are extremely difficult to predict and therefore risky, (4) an investor may face a restrictive investment budget restraint.

The conditions for the non-holding--non-participating investor can be obtained by reversing the inequalities in the buy long relationships (III.1.0) and (III.1.1) and the sell short relationships (III.2.0) and (III.2.1). For an investor to fall into this category, these relationships must hold simultaneously. Thus,

$$P_t - \varepsilon^i_{t,t+\tau} \leq E^i_{t-1}(P) \leq P_t + \varepsilon^i_{t,t+\tau} \qquad \text{(III.7.0)}$$

$$P_t - \varepsilon^i_{t,t+\tau} \leq A^i_{t-1}(P_{t+\tau}) \leq P_t + \varepsilon^i_{t,t+\tau}. \qquad \text{(III.7.1)}$$

This concludes the set of decision rules. This set is exhaustive of all possible decision positions facing the investor and are based on the assumption of profit maximization. Using these qualitative rules, the behavioral index sets $\hat{I}_{st}$, $\hat{I}_{dt}$, $\bar{I}_{kt}$ and $\bar{I}_{\emptyset t}$ defined loosely in section II can now be rigorously stated.

IV. Specification of Index Sets

In this section, the conditions for membership to the set of security market suppliers, $\hat{I}_{st}$, demanders, $\hat{I}_{dt}$, holders, $\bar{I}_{kt}$, and non-holder-nonparticipants, $\bar{I}_{\emptyset t}$, at time t will be specified.

IV.A. The Set of Demand Individuals

The set of security market demanders in time period t is the union of the class of individuals who want to buy long (I_{BL_t}) and buy short (I_{BS_t}). Using assumptions (III.1.0) and (III.1.1) gives:

(IV.1.0) $$I_{BL_t} = \{i\varepsilon\tilde{I}/A^i_{t-1}(P_{t+\tau}) - P_t \geq \varepsilon^i_{t,t+\tau} \;\; \forall\, t+\tau \leq T,\; E^i_{t-1}(P) > P_t + \varepsilon^i_{t,t+\tau}\}.$$

From (III.4.0) and (III.4.1), we get:

(IV.1.1) $$I_{BS_t} = \{i\varepsilon\tilde{I}/A^i_{t-1}(P_{t+\tau}) - P_t \leq -\varepsilon^i_{t,t+\tau},\; \forall\, t+\tau,\; E^i_{t-1}(P) > P_t - \varepsilon^i_{t,t+\tau}\}.$$

The union of (IV.1.0) and (IV.1.1) is the index of the demand set ($\hat{I}_{dt}$):

(IV.1.2) $$\hat{I}_{dt} = I_{BL_t} \cup I_{BS_t}.$$

The supplier set can be derived by using the conditions of selling long and selling short. From (III.3.0) and (III.3.1) we get the selling long (I_{SL_t}) index set:

IV.2.0) $$I_{SL_t} = \{i\varepsilon\tilde{I}/A^i_{t-1}(P_{t+\tau}) - P_t \leq \varepsilon^i_{t,t+\tau} \;\; \forall\, t+\tau \leq T,\; E^i_{t-1}(P) < P_t + \varepsilon^i_{t,t+\tau}\}.$$

From (III.2.0) and (III.2.1) we get the index of selling short set (I_{SS_t}):

$$\text{(IV.2.1)} \qquad I_{SS_t} = \{i\varepsilon\tilde{I}/A^i_{t-1}(P_{t+\tau}) - P_t \le -\varepsilon^i_{t,t+\tau}$$

$$\forall\ t+\tau \le T,\ E^i_{t-1}(P) \le P_t - \varepsilon^i_{t,t+\tau}\}.$$

The union of (IV.2.0) and (IV.2.1) is the supplier set ($\hat{I}_{st}$):

$$\text{(IV.2.2)} \qquad \hat{I}_{st} = I_{SL_t} \cup I_{SS_t}.$$

Lastly, the nonparticipant set is derived as the union of those individuals who hold (either long (IV.3.0) or short (IV.3.1) and the nonparticipant, nontrader set (IV.3.2).

$$\text{(IV.3.0)} \qquad I_{HL_t} = \{i\varepsilon\tilde{I}/A^i_{t-1}(P_{t+\tau}) \ge P_t + \varepsilon^i_{t,t+\tau},\ \text{for some } t+\tau \le T,$$

$$E^i_{t-1}(P) > P_t + \varepsilon^i_{t,t+\tau}\}.$$

$$\text{(IV.3.1)} \qquad I_{HS_t} = \{i\varepsilon\tilde{I}/A^i_{t-1}(P_{t+\tau}) \le P_t + \varepsilon^i_{t,t+\tau},\ \text{for some } t+\tau \le T,$$

$$E^i_{t-1}(P) < P_t - \varepsilon^i_{t,t+\tau}\}$$

$$\text{(IV.3.2)} \qquad I_{NT_t} = \{i\varepsilon\tilde{I}/P_t - \varepsilon^i_{t,t+\tau} < E^i_{t-1}(P) < P_t + \varepsilon^i_{t,t+\tau},$$

$$P_t - \varepsilon^i_{t,t+\tau} \le A^i_{t-1}(P_{t,t+\tau}) \le P_t + \varepsilon^i_{t,t+\tau},\ \forall\ t+\tau \le T\}.$$

The intersection of (IV.3.0) - (IV.3.2) is the nonparticipant set ($\overline{I}_{\emptyset t}$):

$$\text{(IV.3.3)} \qquad \overline{I}_{\emptyset t} = I_{HL_t} \cup I_{HS_t} \cup I_{NT_t}.$$

It should be clear to the reader that the union of the demander, supplier and nonparticipant set is the entire index set:

$$\tilde{I} = \hat{I}_{dt} + \hat{I}_{st} + \bar{I}_{\emptyset t}, \tag{IV.4.0}$$

and that the intersection of the above sets is the null set:

$$\emptyset = I_{dt} \cap I_{st} \cap I_{\emptyset t}. \tag{IV.4.1}$$

With the index sets properly specified, we will now derive the market supply and demand, excess demand and adjustment mechanism.

V. Market Process

In the previous section, the relationships between the rules for investor behavior were presented. In this section, these relationships are used to derive a market supply and demand, excess demand and an adjustment function. These functions will provide the basis for the existence and stability theorems derived in the following section.

One point should be noted here. In traditional economic analysis, where a tatonnement adjustment process is assumed, all trades occur at equilibrium. This is an unrealistic assumption in this case and will not be made here. The tatonnement process implies market clearing conditions on a daily or even shorter market period basis. The problem with this, is that it provides little insight into how an actual organized market operates such as a security market. In this section, we will postulate such an adjustment mechanism and demonstrate its implications on price determining behavior.

V.A. Demand and Supply Function

Based on the conditions of investor behavior presented in the previous section, a market demand function will be derived. This will begin with individual demand and then aggregated to form a market demand function.

Individual Demand Function--Buy Long and Short: It is assumed here that the behavioral rules (III.1.0), (III.1.1) and (III.4.0), (III.4.1) can be quantified into the following demand relationships for each $i\varepsilon\hat{I}_{dt}$:

$$d_t^i = d_t^i[\overset{-}{P}_t;\ E_{t-1}^i(\overset{+}{P}),\ \overset{\pm}{\varepsilon}{}^i_{t,t+\tau}], \tag{V.1.0}$$

where d_t^i denotes the desired quantity of securities demanded in time t. The values $E_{t-1}^i(P)$ and $\varepsilon_{t,t+\tau}^i$ are exogenous at time t since, as stated previously, it is assumed that the decision and transaction activities are separated.

It has been assumed that the value $A_{t-1}^i(P_{t,t+\tau})$ converges towards $E_{t-1}^i(P)$. Therefore, rather than having both $A_{t-1}^i(P_{t,t+\tau})$ and $E_{t-1}^i(P)$ only $E_{t-1}^i(P)$ is presented here for convenience.

Letting $E_{t-1}^i(P) \pm \varepsilon_{t,t+\tau}^i = E_{dt+\tau}^i$ for all $i\varepsilon\hat{I}_{dt}$, the value $E_{dt+\tau}^i$ becomes a shift parameter of the demand function. This gives:

$$d_t^i = d_t^i(-P_t,\ E_{dt+\tau}^i)\ \forall i\varepsilon\hat{I}_{dt}, \tag{V.1.1}$$

where it is assumed that

$$d_t^i \geq 0\ \forall P_t.$$

The desired quantity of a security demand is an inverse relationship to the price of the security in both the buy long and buy short positions for a given instant of time. This follows since a change in price produces the opposite change in expected profits.

Individual Supply Function--Sell Long and Short: Using the same approach as above, by quantifying the individual decision rules, the individual supply function becomes:

$$S_t^i = S_t^i(\overset{+}{P}_t;\ E^i_{s,t+\tau}) \quad \forall i \varepsilon \hat{I}_{st}, \tag{V.2.0}$$

where S_t^i denotes the desired quantity of the security supplied in time t and where

$$S_t^i \geq 0, \ \forall \ P_t.$$

The desired quantity of securities supplied in t is positive relationship to the price of the security in both the sell long and sell short positions. This follows since a positive (negative) change in price produces a positive (negative) change in expected profits.

V.B. Market Supply and Demand

The market supply and demand functions are derived in the traditional way by horisontally summing the individual functions. This gives us:

$$d_t = \sum_i d_t^i, \ \forall i \varepsilon \hat{I}_{dt}. \tag{V.3.0}$$

$$s_t = \sum_i s_t^i, \ \forall i \varepsilon \hat{I}_{st}. \tag{V.4.0}$$

The market supply and demand functions are thus dependent on the market price and the vector of expectations (E_{st}, E_{dt}), i.e.

(V.5.0) $$d_t = d_t(P_t; E_{dt})$$

(V.6.0) $$s_t = s_t(P_t; E_{st})$$

where

$$E_{dt} = (E^1_{dt+\tau}, E^2_{dt+\tau}, \ldots, E^i_{dt+\tau}, \ldots) \quad \forall i \epsilon \hat{I}_{dt}$$

$$E_{st} = (E^1_{st+\tau}, E^2_{st+\tau}, \ldots, E^i_{st+\tau}, \ldots) \quad \forall i \epsilon \hat{I}_{st}.$$

V.B. Excess Demand Function

It is convenient for our later purposes to consider the excess demand function rather than the supply and demand functions separately. The excess demand function (X_t) is the difference between (V.5.0) and (V.6.0), i.e.

(V.7.0) $$X_t = d_t - s_t.$$

So that we may consider X_t as a "simple" function of price and expectations, it is convenient to define the aggregate vector of expectations over both the supply and demand functions (E_t) to be the joint vector (E_{dt}, E_{st}), i.e.

(V.8.0) $$E_t = (E_{dt}, E_{st}).$$

Thus the excess demand equation becomes:

(V.8.1) $$X_t = (P_t; E_t),$$

where $X_t = 0$ if $d_t = s_t$.

V.C. Exchange Function

Since equilibrium trading is not assumed, it is necessary to distinquish between the desired quantity demanded, the desired quantity supplied and the quantity traded. This exchange function which defines the quantity traded (q_t) is specified by the following conditions:

(V.9.0) (a) $q_t = d_t \; \forall P_t$ where

$$X_t < 0,$$

(b) $q_t = s_t \; \forall P_t$ where

$$X_t > 0,$$

(c) $q_t = d_t = s_t \; \forall P_t$ where

$$X_t = 0.$$

<u>Remark</u>: Certain ranges of either the supply or demand function or both can be zero in which case no trades will take place. The conditions for trading or not trading are:

(a) A trade will occur if P_t exists such that

$$d_t^i > 0 \text{ and } s_i^j > 0 \text{ for some } i \varepsilon \hat{I}_{dt}$$
$$\text{for some } j \varepsilon \hat{I}_{st}$$

(b) No trades will occur if P_t exists such that

(i) $d_t^i = 0$ and $s_t^2 > 0$, for all $i \varepsilon \hat{I}_{dt}$

for some $j \varepsilon \hat{I}_{st}$,

(ii) $d_t^i > 0$ and $s_t^i = 0$, for some $i \epsilon \hat{I}_{dt}$

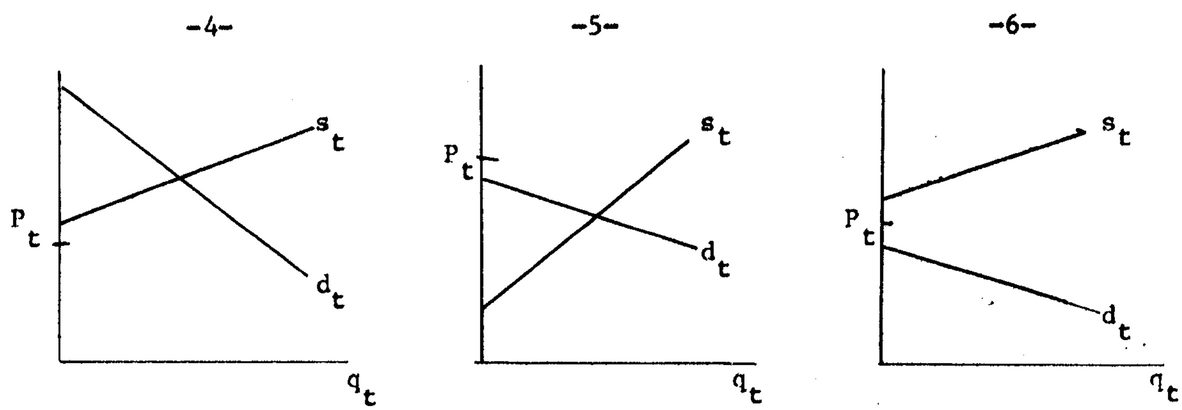
for all $j \epsilon \hat{I}_{st}$,

(iii) $d_t^i = 0$ and $s_t^i = 0$, for all $i \epsilon \hat{I}_{dt}$

for all $j \epsilon \hat{I}_{st}$.

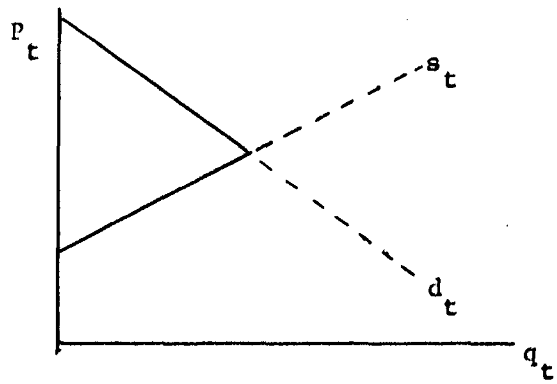
Figure 4-6 below represents the three cases under which no trades will take place.

Figure 4-6: The Three Cases of No Trading



Remark 2: The exchange function is not a one-to-one mapping. Thus, although for each price there is only one quantity traded, more than one price may be associated with a particular quantity traded. Figure 7 below pictures the exchange function for the normal case.

Figure 7: The Exchange Function



To complete the presentation of the model, the adjustment function must be derived. Although the adjustment function is obviously non-tatonnement, the adjustment process is assumed to be Walrasian in nature. The exchange function is deliberately designed as stock "specialists" are said to act. The adjustment process is of the traditional form, i.e. the rate of price adjustment, $\dot{P}_t$, is assumed directly dependent on the size of excess demand.

(V.10.0) $$\dot{P}_t = \dot{P}_t\ (X_t)$$

where $\dfrac{d(\dot{P}_t)}{dX_t} > 0.$

With the system thus presented, we will now continue on to a presentation of some of the important properties which can be derived.

VI. Theorems and Derivations

In this section, the basic implications of the model will be derived. The implications will be of two types: (1) existence of a stationary and growth equilibrium, (2) system stability of these equilibriums given a change in "expectations" (E_t^i). These results have definite implications to understanding stock price changes over time and the conditions for dependence and independence of these price series over time.

VI.A. Existence of a Stationary Equilibrium

The problem of the existence of a stationary equilibrium reduces to the problem of demonstrating that for given expectations, a non-negative price can be found which equates aggregate supply and demand. Since we are dealing with a rather elementary situation, only the outline of the existence proof will be presented.

Theorem 1: For every given vector of expectations and costs (E_{dt}, E_{st}), there exists at least one nonnegative price ($\overline{P}_t$) such that market supply equals market demand.

Proof: The proof relies on demonstrating that a function can be defined that satisfies Brouwer's fixed point theorem. This involves defining a continuous function from a closed bounded convex set of enclidean space into itself.[5] To define the closed bounded convex set, choose

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[5] Brouwer's fixed point theorem states: "if f defines a continuous point to point mapping of a closed bounded convex set C into itself, then there exists $\hat{X} \varepsilon C$, such that $f(\hat{X}) = \hat{X}$." This theorem is, of course, a sufficient condition, amapping may give a point $\hat{X}$ such that $\hat{X} = f(\hat{X})$ without satisfying the conditions of the theorem.

an arbitrarily bound for the supply and demand functions such that the average slopes over the resulting sets are equal in absolute value, i.e. such that the endpoints of the functions go through the verticies of the resultant box. This is possible since both the supply and demand function are bounded below by the origin.

As defined, both the supply and demand function are continuous over the region although not necessarily continuously differentiable. The demand function as defined within the set, has domain $[0, q_t']$ and range $[0, P_t']$, whereas the supply function has range $[0, q_t']$ and domain $[0, P_t']$. Thus we define the continuous mapping over the interval of a set $\emptyset$ such that $\emptyset\ (q_t, p_t) = d_t(P_t),\ s_t^{-1}(q_t)$. From Brouwer's theorem this implies an equilibrium. Q.E.D.

Next we will determine whether or not constant expectations and cost imply an adjustment process which converge to equilibrium, i.e. stability.

VI.B. Stationary Stability

Two types of stationary stability will be investigated. The simplest involves constant E_t^i and $\tilde{I}$ over time. The more general case of constancy of expectations involves offsetting changes in expectations. In both of these cases, given the model as presented, stability is guaranteed. As the first example of constancy is implied by the second, the proof of stationary stability will be presented in terms of the more general case.

<u>Theorem 2</u>: Let $\dot{E}_t = 0$ for all $t > t_o$, where $P_{t_o} \gtrless \overline{P}_{t_o}$, the equilibrium price. Then, there exists a $\overline{t} > t_o$ such that for all $t \geq \overline{t}$, $|P_t - \overline{P}_t| < \delta$, where δ is arbitrarily small.

Proof: The proof will be by contradiction. Suppose not, i.e. suppose that

$$|P_t - \overline{P}_t| > \delta \ \forall\, t > t_o$$

By definition (V.8.1), this implies

$$X_t \neq 0 \quad \text{for all } t > t_o.$$

However, by the properties of X_t, we get that

$$X_t > 0 \quad \text{if } P_t < \overline{P}_t,$$

$$X_t < 0 \quad \text{if } P_t > \overline{P}_t.$$

But from the definition of the adjustment function (V.10.0):

$$X_t > 0 \Rightarrow \quad \dot{P}_t > 0,$$

$$X_t < 0 \Rightarrow \quad \dot{P}_t < 0 \qquad \forall\, t > t_o$$

$$\therefore P_t \searrow \overline{P}_t \quad \text{for all } t > t_o,$$

and a point $t \geq \overline{t}$ must exist where

$$|P_t - \overline{P}_t| < \delta.$$

Contradiction. Q.E.D.

Remark: This theorem must hold either if $\dot{E}_t$ is a vector or if $\dot{E}_t$ is defined as that separable part of the excess demand function which is dependent on the vector of expectation, i.e. the weighted functional value of the vector E_t on f.

Growth Problem

In this section, we will prove a theorem generalizing the results of the above existence and stability section to the case where expectations are continually changing.

Theorem 3: Let $\left|\frac{\dot{E}_t}{E_t}\right| = \delta_1$ for all $t > t_o$. Then there exists a $\bar{t} > t_o$ such that

$$\left|\frac{\dot{P}_t}{P_t}\right| = \delta_2,$$

i.e. that for any given δ_1 we can find a $\bar{t}$ such that δ_2 exists.

Proof: The proof will be completed in two parts:

(1) solving for δ_1 if the rate of price change is on the equilibrium path, i.e.

$$\frac{\dot{P}_t}{P_t} = \frac{\dot{\bar{P}}_t}{\bar{P}_t},$$

where excess demand is maintained to be zero, and (2) the demonstration that the system must approach that point.

The excess demand function provides the link between the rate of inflation and the rate of change in expectations. In this first section, it will be demonstrated that the equilibrium path satisfies the condition of the theorem.

Assuming that the excess demand function is linear homogeneous, we can derive the following reduced function:

$$\text{(Th.3.1)} \qquad \frac{X_t}{E_t} = f\left(\frac{P_t}{E_t}\right) \; \underline{6/}$$

Taking the time derivative of (Th.3.1) gives:

$$\text{(Th.3.2)} \qquad \frac{\dot{X}_t E_t - \dot{E}_t X_t}{E_t^2} = f'\left[\frac{\dot{P}_t}{P_t} - \frac{\dot{E}_t}{E_t}\right] \frac{P_t}{E_t} \, .$$

With $P_t = \bar{P}_t$ for all $t \quad t_o$ implies that

$$\text{(Th.3.3)} \qquad \dot{X}_t = X_t = 0$$

for all $t > \bar{t}_o$. From this condition, it follows directly that

$$\text{(Th.3.4)} \qquad \frac{\dot{\bar{P}}_t}{\bar{P}_t} = \delta_1 .$$

Thus the equilibrium growth path of prices is a constant equal to that of the growth in expectations. 7/

(2) It will now be shown that given an arbitrary initial price that the rate of change in prices must approach the equilibrium rate of change in prices.

Assuming that the adjustment equation (V.10.0) is linear homogeneous, we get:

$$\text{(Th.3.5)} \qquad \dot{P}_t/P_t = P_t(X_t/P_t) .$$

6/ In the case that the excess demand function is homogeneous of degree r, we get the following reduced expression

$$\frac{X_t}{E_t^r} = f\left(\frac{P_t}{E_t}\right).$$

7/ Notice that if prices adjust instantaneously to a change in expectations, then

$$\frac{\dot{P}_t}{P_t} = \frac{\dot{\bar{P}}_t}{\bar{P}_t} = \delta_1 .$$

Thus it follows from (Th.3.5) that

$$\text{(Th.3.6)} \qquad \frac{\dot{P}}{P} = \delta_2 \quad \text{iff} \quad \frac{X_t}{P_t} = k$$

where k is an arbitrary constant.[8/] The proof of the second part of the theorem involves demonstrating that such a k exists.

Suppose not. Then either

$$\text{(a)} \quad \frac{\dot{X}_t}{X_t} - \frac{\dot{P}_t}{P_t} > 0 \qquad \text{for all } t > t_o$$

or

$$\text{(b)} \quad \frac{\dot{X}_t}{X_t} - \frac{\dot{P}_t}{P_t} < 0 \qquad \text{for all } t > t_o.$$

Suppose (a). Then

$$\text{as } t \searrow \infty, \ \frac{X_t}{P_t} \searrow \infty.$$

But

$$\frac{X_t}{P_t} \searrow \infty \Rightarrow \frac{\dot{P}}{P} \searrow \infty .$$

But this implies $\dfrac{X_t}{P_t} \searrow 0$, i.e.

$$P_t \searrow \infty . \quad \text{Contradiction.}$$

Suppose (b). Then by a similar argument we get that

$\dfrac{X_t}{P_t} \searrow 0$, but if

8/ Notice that $k = 0$ satisfies this condition. That was the implication of the first part of the proof.

$$\frac{X_t}{P_t} \searrow 0 \Rightarrow X_t \searrow 0$$

$$\Rightarrow P_t \searrow \overline{P}_t, \text{ i.e.}$$

$$\frac{\dot{P}}{P} \Longrightarrow \delta_1. \quad \text{Contradiction}$$

Q.E.D.

This completes the section on proofs. We will now continue on to a discussion of the implication and conclusions of the paper.

VIII. Implications and Conclusions

Most of the work examining security markets concludes that the random walk hypothesis is a "good" explanation of observed price behavior. In this paper, we have described price behavior generated by rationality of the individual participant and by the formation of individual expectations and anticipations. In what circumstances would the results of this model be consistent with the random walk hypothesis? For most securities, it is probably correct to assume that changes in expectations and cost are rather small. In this case, price will approximate equilibrium and therefore the price changes will be approximately random. As long as a non-tatonnement process is postured, i.e., the adjustment in price to nonequilibrium is not necessarily instantaneous, then there exist at least certain periods of price dependence--those periods when price adjustments are taking place either because of a one-time or a continuous change in expectations.

If we consider securities only when there is a large change in expectations, then according to this analysis there would be a significant period of dependence. What we must conclude therefore is that the majority of research on security markets did not separate out those securities which were going through periods of adjustment from those which were not. Since it is reasonable to expect only a small number of securities to be rapidly adjusting to expectational changes at any one time, most stock price changes would conform to the random walk hypothesis. Those that are rapidly adjusting would appear as extreme variates from the average case and the total probability distribution would demonstrate leptokurtopic properties.

In another paper,[9] we investigate the empirical evidence in support of the conclusion that at least a small subset of securities over limited periods of time demonstrate dependence. It is shown there that existing evidence is consistent with a non-tatonnement adjustment hypothesis.

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