Optimal Feed Mill Blending

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Abstract
Commercial feed blending is a complex process consisting of many potential raw ingredients and final products. The sheer number of daily orders and final products at a typical feed mill means that raw ingredients cannot be mixed to directly produce final products in an economical fashion. As a result, the intermediate production of pellets with pre-specified nutritional content is a necessity that makes the feed blending problem highly nonlinear. A nonlinear approach to feed blending is discussed and results from an empirical application are compared to the results from a sequential linear programming approach common to most feed mills.

Keywords: agribusiness, feed blending, linear programming, non-linear programming

Introduction and Background

Modern commercial feed mills in the U.S. that blend feed for sale to agricultural operations are engaged in a complex production process consisting of many potential raw ingredients and final products. The typical feed mill likely produces and delivers feed for poultry (turkeys and broilers), swine, beef cattle, and dairy operations.[2] Each of these operations requires several types of feed rations since rations are typically designed around the nutritional requirements of animals in various stages of growth or production. For example, a lactating dairy cow requires a different feed ration than a bred heifer and a just weaned pig requires a different feed ration than a hog approaching market weight.

Although the feed ration balancing problem is a prototype blending problem found in many introductory linear programming (LP) textbooks (for example, Paris 1991, or Taylor, 1999), there are many reasons why a strict application of the model is not used at feed mills. First, the sheer number of final products produced by a mill in conjunction with multiple daily orders of varying size means that raw ingredients cannot be mixed to directly produce final products in an economical or practical manner. Also, storage and other production capacity constraints limit the usefulness of such a basic approach. As a result, feed mills use raw ingredients to produce a set of intermediate products that can then be used in a variety of ways to produce the needed final products possessing the appropriate nutritional composition. These intermediate products are called *pellets* and they typically possess nutritional characteristics (or at least tolerances) specified by management.[3] Once pellets are produced, their nutritional content is known, and they are blended to meet the nutritional characteristics of final products.

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[2] In addition, the typical feed mill likely also produces feed for noncommercial animals such as dogs, cats, chicks, ducks, etc.
[3] Management in this context typically includes a procurement specialist and/or a nutritionist.

Management's objective is to meet the demand for final products with specific nutritional characteristics at least cost, given the availability and cost of raw ingredients. The intermediate step of producing pellets is a necessity, which makes the feed blending problem nonlinear (NL). To see this, consider a single nutritional requirement for a pellet, say crude protein. The amount of crude protein in a pellet is essentially a decision variable since it is the choice over a set of raw ingredients with varying amounts of crude protein that determines the crude protein composition of the pellet. However, this information becomes technical coefficient data when pellets are mixed to produce final products and it is in this sense that the feed blending problem is nonlinear. This nonlinearity is demonstrated mathematically in the first three sets of constraints in the NL feed blending model presented in the Appendix.

One way around the nonlinearity is to model the blending problem as a two-stage LP framework. In this setting, raw ingredients are turned into pellets at minimum cost with the recipe for a given pellet determined by an LP model. Subsequently, another LP model determines the minimum cost recipe for a given final product using the pellets with nutrient content determined by the first LP model. Obviously the recipes for the final products are only truly optimal if the recipes for the pellets (including which pellets to make) were also optimal.

Another problem encountered by feed mills is that some raw ingredients do not need to be or cannot be turned into pellets. Molasses, for example, is a raw ingredient added to many feeds to improve palatability. However, molasses is not incorporated directly into pellets, but rather, added along with pellets to make a final product. Thus, some raw ingredients enter the feed blending process through the pellet stage of production while others enter when the final products are blended from pellets. Despite these drawbacks, feed mills continue to use the LP approach. In most cases, the reason cited is the lack of commercially available software for handling the more complex nonlinear problem.

One alternative to the two-stage LP that deals with the issue of nonlinearity as well as the issue of when certain raw products enter production is to construct a nonlinear pooling model, rather than a blending model, that simultaneously determines the composition of the pellets and final products from a set of raw ingredients. An example of this sort of problem is described by Fieldhouse (1993). The feed blending problem becomes a pooling problem rather than a blending problem as the nutrient content of the pellets is unknown prior to solution of the model, and is determined endogenously by the constraints of both the pellets and the final products. The pooling problem has been examined in many different industries, such as oil and gas, chemical, and food ingredient industries (see for example Amos *et al.*, 1997; Karmarkar and Rajaram, 2001). In the oil and gas pooling problem, typically the ingredients available to the pooler are different types of crude oil. These oils are fractionated and pooled to achieve quality standards specified by the oil refinery (Amos *et al.*, 1997).

No studies have been undertaken for the feed mill blending problem in the context of a nonlinear pooling problem. In this paper, a nonlinear feed blending model is presented and the results from an empirical application of the model are compared to those suggested by the more common sequential LP approach.

Empirical Application

The essence of the feed-mill's blending problem can be captured through a relatively simple example. In this example, the feed mill wishes to determine a least cost means of producing four final products using at most, six different pellets and four raw ingredients that enter production in the second stage (i.e. when the pellets are combined to make the final products). The objective is to determine how much of each raw ingredient to procure and how to blend the ingredients to make a set of pellets. Pellets are then combined with other raw ingredients to make the set of final products, within the limits specified, at minimum cost. A standard set of raw ingredients is assumed[4], as is a standard set of nutritional constraints.[5]

The first four of the six different pellets are differentiated by crude protein or fiber content. For example pellet 1 has a low crude protein percentage, but high fiber content, and pellet 4 has high crude protein and low fiber. Pellets 5 and 6 differ from the first four due to the addition of supplemental minerals and vitamins. These two pellets differ amongst themselves by crude protein and phosphorus levels. The final products vary by the level of crude protein and levels of raw ingredients added after the intermediate production (pellet) stage.

At current prices[6], a standard sequential LP approach to the feed blending problem results in the solutions presented in Tables 1 and 2. In this setting, the feed-mill managers would procure the ingredients listed and combine them in the manner suggested to form each of the six pellets from which final products may be produced. At this point, the exact nutritional content of each pellet can be determined since the nutritional content of each raw ingredient is known and the LP provides the recipe for the pellets. These outputs can then be used in a second stage LP model along with other raw ingredients to form the final products.

An LP approach such as this circumvents the nonlinearity by breaking the problem into a sequence of two problems where the output of the first LP (the pellet solution) is imposed on the second LP (the final product problem). An example of the results from such a model is presented in Table 2 which shows how each of the final products should be produced using the available pellets and other raw ingredients. Notice that two pellets (pellet 2 and pellet 5) are not used in the set of final products. *A priori*, the feed mill cannot afford to ignore these two pellets when deciding on a production run as relative prices may mean that one week they should be made, and the next otherwise. However, it is clear that another drawback of the sequential LP approach is that recipes need to be determined for all potential pellets irrespective of whether they will be used or not.

A nonlinear model of the feed mill's problem is presented in the Appendix. In the model, the appropriate pellets to produce are optimally determined at the same time that the recipes for pellets and final products are determined. As shown in Table 3, the optimal recipes suggested by the nonlinear model are somewhat different than those suggested by the sequential LP. Most notable is the fact that it is not optimal, in the non-linear approach to the problem to produce pellet 3.

Presented in Table 4 are the recipes for the final products suggested by the NL model. Also shown in all the tables is the cost of pellets and final products produced in dollars per ton. Somewhat counter intuitively, the cost per ton of pellet produced suggested by the sequential LP solution is always less than the cost per ton of pellet produced suggested by the nonlinear program solution. It is important to point out, however, that the two models suggest different recipes for a given pellet so that the

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[4] [4] See, for example, those raw ingredients listed in Tables 1 or 3. Raw ingredients listed in Tables 2 or 4 are not pelletized.

[5] Nutritional constraints (maximums and minimums) modeled include those relating to crude protein, fat, fiber, Undigestable Protein (UDP), Net Energy Lactation (NEL), Non-fiber Carbohydrate (NFC), Phosphorus, Calcium, Potassium, Selenium, Magnesium, Manganese, Iodine, Iron, Zinc, Copper, Cobalt, Salt, Vitamins A, D, and E. Additionally, feed texture and product palatability constraints were also modeled. These nutritional requirements represent a comprehensive set of constraints actually used by the feed mill in question.

[6] Prices are as of December 2003.

nutritional content of a given pellet can vary depending on the method (i.e. linear vs. nonlinear) chosen to formulate the pellet.

Table 1. Optimal Linear Programming Solutions for Pellet Ration Formulation.

	Intermediate Products					
Ingredient	**Pellet 1**	**Pellet 2**	**Pellet 3**	**Pellet 4**	**Pellet 5**	**Pellet 6**
Alfalfa Meal	-	-	-	-	-	-
Broiler Vitamin Premix	-	-	-	-	-	-
Calcium Sulfate	-	-	-	-	-	-
Copper Sulfate	-	-	-	-	-	-
Cottonseed Meal	-	-	-	-	-	10.8%
Dicalcium Phosphate	-	0.4%	-	1.1%	6.6%	4.6%
Distiller's Grain	-	-	-	-	6.2%	7.7%
Dynamate	-	0.1%	-	-	-	-
Fat	-	-	-	-	-	-
Gluten Feed	-	-	-	-	-	-
Ground Corn	-	8.3%	-	-	5.8%	-
Limestone	5.4%	2.3%	3.7%	7.4%	-	-
Magnesium Oxide	-	-	-	0.3%	0.6%	0.6%
Monosodium Phosphate	-	-	-	-	0.8%	1.1%
Salt	-	1.5%	1.5%	3.2%	2.5%	2.5%
Selenium Premix I	-	-	-	-	-	0
Selenium Premix II	-	0.1%	0.1%	0.6%	1.1%	1.0%
Soybean Hulls	-	-	-	-	-	-
Soybean Meal	-	1.2%	15.2%	81.3%	2.6%	7.6%
Trace Mineral Premix	-	0.1%	-	-	-	1.2%
Vitamin A, D, E	-	-	-	-	-	-
Vitamin E	-	-	-	-	0.2%	0.2%
Vitamin Premix	-	0.1%	0.1%	0.2%	0.7%	0.7%
Wheat Middlings	94.6%	85.9%	79.5%	2.9%	72.7%	61.7%
ZinPro	-	-	-	-	-	-
Cost ($/ton)	**$78.75**	**$88.45**	**$96.36**	**$169.29**	**$137.59**	**$157.13**

Table 2. Optimal Linear Programming Solutions for Final Product Ration Formulation

	Final Products			
Ingredient	**Product A**	**Product B**	**Product C**	**Product D**
Pellet 1	-	0.7%	0.7%	-
Pellet 2	-	-	-	-
Pellet 3	35.6%	3.3%	38.8%	27.7%
Pellet 4	23.1%	80.9%	23.3%	34.8%
Pellet 5	-	-	-	-
Pellet 6	18.6%	14.3%	34.1%	3.5%
Molasses	2.5%	0.8%	3.2%	14.0%
Soybean Oil	-	-	-	-
Steam Crimped Barley	20.2%	-	-	20.0%
Steam Flaked Corn	-	-	-	-
Cost ($/ton)	**$127.58**	**$163.89**	**$133.69**	**$125.91**

Even so, the cost per ton of final product produced is always in favor of the nonlinear program solution. Notice when comparing the last two rows of Tables 2 and 4 that the cost per ton differences are $8.37, $8.18, $19.08, and $29.80 for Products A, B, C, and D. These differences amount to savings per ton of 6.6%, 5.0%, 14.3%, and 23.7% for the four final products. This result is not unanticipated given the potential superiority of the NL approach over the sequential LP approach. For example, using the sequential LP solution, Product A should be mixed using pellets 3, 4, and 6. The nonlinear program suggests using pellets 1, 4, and 5 is optimal. It is tempting to conclude that since the LP solution suggests recipes for pellets 3, 4, and 6, which are less expensive than Pellets 1, 4, and 5 that the nonlinear solution is less desirable. However, the recipes for pellets 1, 4, and 5 in the nonlinear program differ from the LP recipes for the same pellets. This is because the nonlinear program has more information from which to determine the optimal composition of the pellets. In short, the recipes for pellets 3, 4, and 6 (from the LP solution) are not optimal from the perspective of the final products.

Table 3. Optimal Nonlinear Programming Solutions for Pellet Ration Formulation

	Intermediate Products					
Ingredient	**Pellet 1**	**Pellet 2**	**Pellet 3**	**Pellet 4**	**Pellet 5**	**Pellet 6**
Alfalfa Meal	-	-	-	-	-	-
Broiler Vitamin Premix	-	-	-	-	-	-
Calcium Sulfate	-	-	-	-	-	-
Copper Sulfate	-	-	-	-	-	-
Cottonseed Meal	-	-	-	-	-	-
Dicalcium Phosphate	-	2.7%	-	1.2%	6.7%	4.7%
Distiller's Grain	-	-	-	-	-	-
Dynamate	-	21.8%	-	-	-	-
Fat	-	6.9%	-	-	-	-
Gluten Feed	-	-	-	-	-	-
Ground Corn	-	40.9%	-	-	5.0%	-
Limestone	2.0%	1.5%	-	7.1%	-	-
Magnesium Oxide	-	-	-	0.5%	0.7%	0.7%
Monosodium Phosphate	-	-	-	-	0.9%	1.5%
Salt	0.7%	1.5%	-	3.2%	2.5%	2.5%
Selenium Premix I	0.2%	-	-	0.3%	-	-
Selenium Premix II	-	0.1%	-	-	1.3%	1.0%
Soybean Hulls	-	-	-	-	-	-
Soybean Meal	0.5%	24.2%	-	87.2%	8.7%	26.8%
Trace Mineral Premix	0.2%	0.1%	-	0.5%	-	7.4%
Vitamin A, D, E	-	-	-	-	-	-
Vitamin E	-	-	-	-	0.5%	0.2%
Vitamin Premix	-	0.3%	-	0.2%	2.3%	0.7%
Wheat Middlings	96.4%	-	-	-	71.5%	54.4%
ZinPro	-	-	-	-	-	-
Cost ($/ton)	**$82.15**	**$173.08**	**$0.00**	**$176.33**	**$178.39**	**$193.81**

It is also important to point out that another potential advantage of the NL model, as formulated in the Appendix, is that final product demands drive the pooling process. While the results have been presented in percentage terms the model shows how much (in tons) of each raw ingredient to procure. This allows for better management of a feed mill because final products with higher demands are allowed to have a greater influence over the amount and type of pellets produced.

Practical Implementation Issues

While it would appear that the NL approach to the feed blending problem is a far better approach, a few caveats are in order. First, nonlinear programs such as the one presented in this paper do not satisfy the convexity requirements that insure a globally optimal solution. Therefore, it is conceivable that in some settings the NL model may produce multiple local optima some of which may actually be inferior to the LP solution. Also, for even relatively small applications such as the one presented in this paper, model runtimes can be considerably longer than LP. For a problem such as the one presented, runtime is inconsequential for modern LP solvers while the nonlinear model takes on average about a minute to solve.

Table 4. Optimal Nonlinear Programming Solutions for Final Product Ration Formulation

	Final Products			
Ingredient	**Product A**	**Product B**	**Product C**	**Product D**
Pellet 1	44.8%	21.2%	64.9%	58.6%
Pellet 2	-	-	-	7.4%
Pellet 3	-	-	-	-
Pellet 4	30.3%	78.1%	32.0%	-
Pellet 5	2.4%	-	-	-
Pellet 6	-	-	2.1%	-
Molasses	2.5%	0.8%	1.1%	14.0%
Soybean Oil	-	-	-	-
Steam Crimped Barley	20.0%	-	-	10.0%
Steam Flaked Corn	-	-	-	10.0%
Cost ($/ton)	**$119.21**	**$155.71**	**$114.61**	**$96.11**

Presented in Table 5 are results designed to examine the possibility of multiple local optima, solution stability, and the impact of nonlinearities on solver runtime. The dollar cost per ton of each final product from the LP solution is presented for reference. Also shown are five additional solutions from the nonlinear model generated from various starting points (initial solutions) and methods. For example, solution one uses the standard NL approach to the problem and zeros for starting values for all decision variables. The second solution uses the previous solution as starting values and shows only improvement in the number of iterations required to solve the model. In the third solution, the LP solution was used as the starting values and there is noticeable improvement in the solution in terms of cost with virtually no increase in runtime. The fourth solution uses the previous solution's starting values and shows no improvement. Notice that in comparing the two sets of solutions, the primary difference beyond cost is the inclusion of pellet five in the optimal solution when starting from the LP solution. The second set of solutions resulted in an overall cost reduction of about ½ percent when compared to the first set of solutions.

Table 5. Comparison and Stability of Linear and Nonlinear Feed Blending Solutions.

	Final Product Cost ($/ton)						
Model Run[a]	**Product A**	**Product B**	**Product C**	**Product D**	**Pellets**	**Iterations**	**Time**[b]
LP Solution	$127.58	$163.89	$133.69	$125.91	-	-	-
Nonlinear Solutions:							
1	$121.08	$156.66	$118.13	$92.34	1,2,4,6	724	1:05
2	$121.08	$156.66	$118.13	$92.34	1,2,4,6	289	1:06
3	$119.21	$155.71	$114.61	$96.11	1,2,4,5,6	326	1:06
4	$119.21	$155.71	$114.61	$96.11	1,2,4,5,6	333	1:07
5	$119.53	$155.42	$113.43	$92.07	1,2,3,4,6	238	1:10

[a] The five nonlinear solutions correspond to (1) zero initial values for all decision variables, (2) initial values for all decision variables equal to those from solution 1, (3) initial values for all decision variables equal to the LP solution, (4) initial values for all decision variables equal to those from solution 3, and (5) initial values for all decision variables determined via a heuristic and steepest edge strategy employed..

[b] All model solutions were generated on a Pentium 4 operating at 3.2 GHz with 1 GHz of RAM memory

The fifth solution presented in Table 5 is different in two respects. First, a heuristic was used to determine starting values (referred to as crashing the initial solution). Second, a steepest-edge algorithm was employed so that the nonlinear solver would spend more time in selecting variables (to improve the objective function) by looking at the rate that the objective function would improve relative to movements in the other nonzero variables. While both these options have the potential to increase runtime dramatically, neither did in any of the cases studied. The optimal solution suggests that pellet 3 should be substituted for pellet 5 resulting in another 1% reduction in total cost.

From this analysis, it is clear that the nonlinearities inherent in the feed blending problem are potentially problematic in that local solutions are clearly being found by the solver. However, in all cases examined, the local solutions generated via NL programming all outperform the LP solutions from a cost perspective. For example, the total cost of all four final products from the LP solution is $551.07 compared to the best NL solution which was $480.45. Costs savings such as this can translate into many thousands of dollars for even a moderate-sized feed mill over the course of a year. Even so, the NL approach is likely most useful as a complementary model to help management make better decisions about ration formulation. This is because runtimes likely increase dramatically as the number of raw ingredients and/or pellets and/or final products considered by the model increases.

Summary and Conclusions

In this paper, a nonlinear approach to feed blending is described and an empirical application of the model is presented. The results are compared to feed rations suggested by a more common approach of applying linear programming in a two stage manner. The final products formulated in the nonlinear model were cheaper - 5 to 24 percent per ton less costly - than those determined by a two-stage linear programming method, even though the intermediate pellet costs were higher. The reason for this is that the NL model utilizes more complete information regarding the specification of the final and intermediate products than does the two-stage model. As a result, there is the potential for significant cost savings when formulating rations according to the nonlinear model. Future research should be directed at expanding the capability of the nonlinear model to find timely solutions when faced with a larger set of intermediate (i.e. pellets) and final products.

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Appendix
Nonlinear Feed Blending Model

Notation:

Constants:
C = minimum cost of producing all products (endogenous)

Subscripts:
i = final products
j = ingredients used in making pellets
k = nutrients required for final products and/or pellets
n = pellets to consider making
m = ingredients used in making final products

Parameters:
w_j = unit cost of raw ingredient j (input data)
v_m = unit cost of raw ingredient m (input data)
a_{kj} = amount of nutrient k in raw ingredient j (input data)
e_{km} = amount of nutrient k in raw ingredient m (input data)
l^n_{kn} = minimum amount of nutrient k required in pellet n (input data)
u^n_{kn} = maximum amount of nutrient k required in pellet n (input data)
l^i_{ki} = minimum amount of nutrient k required in final product i (input data)
u^i_{ki} = maximum amount of nutrient k required in final product i (input data)
l^Z_{mi} = minimum amount of raw ingredient m required in final product i (input data)
u^Z_{mi} = maximum amount of raw ingredient m required in final product i (input data)
l^X_{jn} = minimum amount of raw ingredient j required in pellet n (input data)
u^X_{jn} = maximum amount of raw ingredient j required in pellet n (input data)
d_i = demand for (or scheduled production of) final product i (input data)
s^X_j = supply of ingredient j available
s^Z_m = supply of ingredient m available

Variables:
X_{jn} = physical amount of ingredient j to use in pellet n (endogenous)
P_{ni} = fraction of the nth pellet to use in final product i (endogenous)
Z_{mi} = physical amount of ingredient m to use in final product i (endogenous)
B_{kn} = physical amount of nutrient k in pellet n (endogenous)

Model:

The objective is to minimize cost:

(1) $$\min \quad C = \sum_j w_j \left(\sum_n X_{jn} \right) + \sum_m v_m \left(\sum_i Z_{mi} \right)$$

subject to:

Minimum and maximum nutritional requirements of the pellets:

(2) $$\sum_j a_{kj} X_{jn} \geq l^n_{kn} \left(\sum_j X_{jn} \right) \text{ and } \sum_j a_{kj} X_{jn} \leq u^n_{kn} \left(\sum_j X_{jn} \right) \quad \forall\, k, n$$

Equations to transfer the nutritional content of the pellets for use in final products:

(3) $$\sum_j a_{kj} X_{jn} = B_{kn} \quad \forall\, k, n$$

Minimum and maximum nutritional requirements on the final products (these are nonlinear):

(4) $$\sum_n B_{kn} P_{ni} + \sum_m e_{km} Z_{mi} \geq l^i_{ki} \sum_n P_{ni} \left(\sum_j X_{jn} \right) \text{ and}$$

$$\sum_n B_{kn} P_{ni} + \sum_m e_{km} Z_{mi} \leq u^i_{ki} \sum_n P_{ni} \left(\sum_j X_{jn} \right) \quad \forall\, k, i$$

Pellet balance equations:

(5) $$\sum_j X_{jn} \geq \sum_i P_{ni} \left(\sum_j X_{jn} \right) \quad \forall\, n$$

Final product balance equations:

(6) $$\sum_n P_{ni} \left(\sum_j X_{jn} \right) + \sum_m Z_{mi} \geq d_i \quad \forall\, i$$

Maximum and minimum amounts of raw ingredients that can be used in making pellets:

(7) $$X_{jn} \leq u^X_{mi} \left(\sum_j X_{jn} \right) \quad \forall\, j, n \text{ and } X_{jn} \geq l^X_{mi} \left(\sum_j X_{jn} \right) \quad \forall\, j, n$$

Maximum and minimum amounts of ingredients that can be used in final products:

(8) $$Z_{mi} \le u_{mi}^{Z}\left[\sum_{n} P_{ni}\left(\sum_{j} X_{jn}\right)\right] \quad \forall\ m,i \ \text{ and } \ Z_{mi} \ge l_{mi}^{Z}\left[\sum_{n} P_{ni}\left(\sum_{j} X_{jn}\right)\right] \quad \forall\ m,i$$

Ingredient supply (availability) constraints:

(9) $$\sum_{n} X_{jn} \le s_{j}^{X} \ \text{ and } \ \sum_{i} Z_{mi} \le s_{m}^{Z}$$

Variable Bounds (to aid in finding a solution faster):

(10) $$0 \le P_{ni} \le 1 \quad \forall\ n,i$$

Non-negativity:

(11) $$X_{jn}, P_{ni}, Z_{mi}, B_{kn} \ge 0 \quad \forall\ j,n,i,m, \textit{and } k$$