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# Sustained Development of a Population and a Resource

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## Sustained Development of a Population and a Resource

**Abstract.** The dynamics of a population and a resource are investigated in a maximin model based on Brander and Taylor's stylization of Easter Island, in order to consider the sustainability of the society represented. There are continua of both regular and non-regular maximin solutions, the type depending on the initial conditions. A non-regular maximin steady state corresponds with the steady state in Brander and Taylor's model. All solutions are time consistent and Pareto optimal. For the regular paths, a partial analytic characterization and a simulation are provided. The non-regular paths involve two regular sections and one degenerate solution in which the maximin constraint is not effective. The high degree of mathematical subtlety of the solution to this ostensibly simple problem calls into question the likelihood of a planner's being able to devise and follow a program of efficient, sustained development.

**JEL Classifications:** Q20, D63, C62

**Keywords:** Sustainability, Intergenerational Equity, Maximin, Regular Path, Population.

## INTRODUCTION

The literature on the simultaneous determination of optimal population and optimal stock levels for renewable natural resources has been dominated by discounted-utility maximization, in which social welfare is expressed as a weighted sum of utility levels. Adding utility levels across individuals and generations, however, whether with equal or unequal weights, is objectionable to some thinkers.

According to Rawls (1971), social welfare should be the utility or consumption level of the least advantaged individual. His *maximin principle* suggests trying to level the consumption of all individuals in society. Extending it to an intertemporal program, Solow (1974) shows that, if manufactured capital and an exhaustible resource are sufficiently substitutable, it is possible to sustain a constant level of consumption by depleting the resource stock toward zero and accumulating capital indefinitely. Although it is derived from an unconventional economic objective, a maximin path is suited to expressing what Solow called intergenerational equity and what has come to be known as economic or weak sustainability. (Compare Pezzey and Toman 2002-03: 168, 176-7.) Still, Solow's model constitutes the only example of a completely solved maximin problem.

The present paper characterizes the maximin path of another simple economy. Much of what is known from a sparse anthropological record of Easter Island is distilled by Brander and Taylor (1998) into a model of an economy with a renewable resource and endogenous population growth. Easter Island is of more than theoretic interest, as it is a frequently cited prototype of a society that was not sustained because of a human failure to live within environmental constraints. One's conclusions about human failings may be different depending on whether the island was, like some other Pacific islands mentioned by Brander and Taylor, unsustainable or whether it was sustainable but not sustained. It is of some interest, therefore, to study conditions

relating to sustainability in their model, even though it is very abstract, on the assumption that a planner strove consistently to attain it.

We find that, for certain initial values of the resource stock and population, there exist so-called *regular* maximin paths. Unlike in discounted-utility models where typically there is a unique steady state, there is a continuum of maximin steady states; which is approached depends on the initial conditions. We characterize the solutions for regular paths and simulate one of them. For other initial values there are non-regular approach paths to a steady state that involve an intervening period when the maximin constraint is not effective. Our analysis is indicative of the subtle issues that would have to be confronted by even a simple society attempting to sustain its economic well being.

## THE MAXIMUM PRINCIPLE IN A MAXIMIN PROBLEM

Consider a dynamic system with  $n$  state variables  $s_i$  ( $i = 1, 2, \dots, n$ ) and  $m$  control variables  $c_j$  ( $j = 1, 2, \dots, m$ ), where  $m$  can be greater than, equal to, or less than  $n$ . The state variables represent stocks such as natural resources, manufactured capital and population. The control variables represent the rates of harvesting, of investment, of consumption, etc. A set of  $n$  differential equations describes the dynamics of the system:

$$\dot{s}_i = f_i(s, c) \text{ for } i = 1, 2, \dots, n. \quad (1)$$

Let  $s = (s_1, s_2, \dots, s_n)$  and  $c = (c_1, c_2, \dots, c_m)$ .

An individual's utility at time  $t$  is represented by

$$u(t) = U(s(t), c(t)).$$

It is convenient to imagine that the individual is born at time  $t$ , enjoys the utility level  $U(s(t), c(t))$ , and immediately dies. We assume that all functions are continuously differentiable.

The planner's objective is to achieve the highest level of utility that can be sustained for all time, or to

$$\max_{[c(t)]_{t=0}^{\infty}} \bar{u} \quad (2)$$

subject to the initial conditions  $s_i(0) = s_{i0}$ , the differential equations (1), and the maximin constraint,

$$U(s(\tau), c(\tau)) \geq \bar{u} \text{ for all } \tau \geq t. \quad (3)$$

We follow Cairns and Long (2006) in treating  $\bar{u}$  as a control parameter. The Hamiltonian is

$$H(s, c, \pi) = \sum_i \pi_i f_i(s, c), \quad (4)$$

and  $\pi_i$  is the shadow price of  $s_i$ .<sup>1</sup> The corresponding Lagrangian is

$$\mathcal{L}(s, c, \pi, \mu, \bar{u}) = H + \mu [U(s, c) - \bar{u}].$$

We denote optimal levels by an asterisk. The following conditions are necessary:

- The vector  $c^*$  maximizes  $H(s^*, c, \pi^*)$  subject to constraint (3):

$$\frac{\partial \mathcal{L}^*}{\partial c_j} = \sum_i \pi_i^* \frac{\partial f_i^*}{\partial c_j} + \mu^* \frac{\partial U^*}{\partial c_j} = 0, j = 1, \dots, m. \quad (5)$$

- The multiplier  $\mu^*$  satisfies the complementary slackness conditions:

$$\mu^*(t) \geq 0, U(s^*(t), c^*(t)) - \bar{u} \geq 0, \text{ and } \mu^*(t) [U(s^*(t), c^*(t)) - \bar{u}] = 0.$$

- The vector  $\pi^*$  satisfies the differential equations

$$\dot{\pi}_k^* = -\frac{\partial \mathcal{L}^*}{\partial s_k} = -\sum_i \pi_i^* \frac{\partial f_i^*}{\partial s_k} - \mu^* \frac{\partial U^*}{\partial s_k}, k = 1, \dots, n. \quad (6)$$

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<sup>1</sup>See also Leonard and Long (1993, esp.Theorem 7.11.1) and Cairns and Long (2005). The objective function is not an integral; in a loose sense, the integrand is identically zero.

- Along the optimal path,

$$\frac{d}{dt}\mathcal{L}(s^*(t), c^*(t), \pi^*(t), \mu^*(t)) = \frac{\partial}{\partial t}\mathcal{L}(s^*(t), c^*(t), \pi^*(t), \mu^*(t)) = 0. \quad (7)$$

Thus,  $\mathcal{L}(s^*(t), c^*(t), \pi^*(t))$  and *a fortiori*  $H(s^*(t), c^*(t), \pi^*(t))$  are constant.

- The optimal choice of the control parameter  $\bar{u}$  maximizes the expression

$$\pi_0 \bar{u} + \int_0^\infty \mathcal{L}(s^*, c^*, \pi^*, \mu^*, \bar{u}) dt,$$

where  $\pi_0$  is a non-negative constant. Hence,

$$\pi_0 - \int_0^\infty \mu^*(t) dt = 0. \quad (8)$$

- The following transversality condition, first proved by Michel (1982), has been extended to this type of problem by Cairns and Long (2006):

$$\lim_{t \rightarrow \infty} H(s^*(t), c^*(t), \pi^*(t)) = 0. \quad (9)$$

- *Along the optimal path,*

$$\text{the vector } (\pi_0, \pi_1, \dots, \pi_n, \mu) \text{ is not identically zero.} \quad (10)$$

Conditions (7) and (9) imply that

$$\sum_i \pi_i^*(t) f_i(s^*(t), c^*(t)) = 0. \quad (11)$$

At all times on an efficient maximin path the value of net investment is zero. Condition (11) is the converse of Hartwick's rule (Hartwick 1977).

Consider the following *regularity condition*:

$$\mu^*(t) > 0 \text{ for all } t.$$

If this condition holds then the shadow prices can be normalized by setting  $\pi_0$  to unity. In a *regular problem*, condition (8) implies that, at least for large enough

$t$ ,  $\dot{\mu}(t) < 0$ . The shadow value  $\mu(t)$  is a measure of the tightness of the maximin constraint (3). If  $\mu(t)$  decreases over time, a marginal relaxation of the constraint at time  $t$  adds more to the objective if  $t$  is nearer to the present.

It is also possible to normalize  $\pi_0$  to unity if  $\mu^*(t) = 0$  for some values of  $t$ , provided that  $\mu^*(t) > 0$  over a non-degenerate interval. In this case, we call the problem *non-regular*.

We define a steady state of a regular maximin problem as a point  $(s_{ss}, c_{ss})$  such that, if the system ever attains that point, it will remain there forever. Let  $\rho(t) = -\dot{\mu}(t)/\mu(t)$  and  $\psi_i(t) = \pi_i(t)/\mu(t)$ . For a steady state, it is consistent with all of the necessary conditions to let the  $\psi_i(t)$  be constants. Division of condition (6) by  $\mu$  yields that

$$-\rho\psi_k + \sum_i \psi_i \partial f_i / \partial s_k + \partial U / \partial s_k = 0, \quad (12)$$

so that  $\rho$  is also constant. By condition (8),  $\dot{\mu} < 0$  and hence  $\rho > 0$  in a steady state. Also, by equation (12),

$$\rho_{ss}\psi_{kss} - \sum_i \psi_{iss} \frac{\partial f_i}{\partial s_{kss}} - \frac{\partial U}{\partial s_{kss}} = 0, \quad \text{for } k = 1, 2, \dots, n. \quad (13)$$

A steady state is a point  $(s_{ss}, c_{ss}, \psi_{ss}, \rho_{ss})$  in  $R^{2n+m+1}$  that satisfies equations (13) and the following:

$$\begin{aligned} \frac{\partial U}{\partial c_j} + \sum_{i=1}^n \psi_{iss} \frac{\partial f_i}{\partial c_j} &= 0, \quad \text{for } j = 1, 2, \dots, m; \\ f_k(s_{ss}, c_{ss}) &= 0, \quad \text{for } k = 1, 2, \dots, n. \end{aligned}$$

Since there are  $2n + m$  equations and  $2n + m + 1$  unknowns, we expect a continuum of steady states.

## A MAXIMIN MODEL OF EASTER ISLAND

In this section we adapt Brander and Taylor's (1998) model of Easter Island. There are two state variables, population or labor force  $L(t)$  and a stock of a renewable



resource,  $S(t)$ . A part of the labor force harvests the resource and the remainder produces a composite good. The average product of labor in harvesting is  $\alpha S$  and in producing the composite good is unity. Let consumption per capita of the two goods be  $h$  and  $m$ . Then

$$m(t) = 1 - \frac{h(t)}{\alpha S(t)} \geq 0.$$

Each individual has the utility function,

$$U(h, m) = h^\beta m^{1-\beta} = h^\beta \left(1 - \frac{h}{\alpha S}\right)^{1-\beta}, \text{ where } 0 < \beta < 1. \quad (14)$$

If  $h(t) = \alpha S(t)$  then  $U(h(t), m(t)) = 0$ , a level which is clearly not optimal. For any  $S(t) > 0$ , the planner must ensure that

$$h(t) < \alpha S(t). \quad (15)$$

Let the natural birth rate, the natural death rate and the effect of consumption of the resource good (food) on the net rate of reproduction all be constant and be represented by  $b$ ,  $d$  and  $\phi$ , respectively, and let  $d > b$ . Then the growth rate of population is

$$\dot{L}(t) = [b - d + \phi h(t)] L(t). \quad (16)$$

The growth rate of the resource stock is

$$\dot{S}(t) = rS(t) \left[1 - \frac{S(t)}{K}\right] - h(t)L(t). \quad (17)$$

Following Brander and Taylor, we assume that

$$0 < \frac{d-b}{\alpha\phi} < \frac{K}{2} < \frac{d-b}{\alpha\beta\phi} < K.$$

Below, we show that the assumption that  $(d-b)/(\alpha\beta\phi) < K$  is necessary for the existence of steady states with  $\dot{\mu} < 0$ , as is required by condition (8).

In a steady state, since  $\dot{L} = 0$ , equation (16) requires that  $h = (d - b)/\phi = h_{ss}$ . We obtain from equation (17) with  $\dot{S} = 0$  that

$$L = \frac{r\phi}{d - b} S \left(1 - \frac{S}{K}\right). \quad (18)$$

Equation (18) traces a concave, quadratic curve in the space  $(S, L)$ , with  $L$  tending to 0 as  $S$  tends to  $K$ . Let the set of *feasible stationary points* with strictly positive population be represented by  $\mathcal{X}$ . Since  $\alpha S_{ss} > h_{ss} = (d - b)/\phi$ ,

$$\mathcal{X} = \left\{ (S, L) : \frac{d - b}{\alpha\phi} < S < K \text{ and } L = \frac{r\phi}{d - b} S \left(1 - \frac{S}{K}\right) \right\}.$$

Below, we show that the steady state approached in Brander and Taylor's model is not a regular maximin point. The set of steady-state, regular, maximin points is, then, a proper subset of  $\mathcal{X}$ .

We now modify the model by introducing a social planner whose objective is to maximize the minimum level of utility,  $\bar{u}$ , over all times  $t \geq 0$ , subject to conditions (16) and (17) with initial values

$$L(0) = L_0 \text{ and } S(0) = S_0, \quad (19)$$

and to the maximin constraint, that for all  $t \geq 0$ ,

$$[h(t)]^\beta \left[1 - \frac{h(t)}{\alpha S(t)}\right]^{1-\beta} - \bar{u} = U(h(t), m(t)) - \bar{u} \geq 0. \quad (20)$$

The Lagrangian for this problem is

$$\mathcal{L}^E = \pi_1 \left[ rS \left(1 - \frac{S}{K}\right) - hL \right] + \pi_2 [b - d + \phi h] L + \mu \left[ h^\beta \left(1 - \frac{h}{\alpha S}\right)^{1-\beta} - \bar{u} \right].$$

The necessary conditions are:

$$(\pi_1 - \phi\pi_2)L = \mu \left( \frac{\beta\alpha S - h}{\alpha Sh} \right) \left( \frac{\alpha Sh}{\alpha S - h} \right)^\beta; \quad (21)$$

$$\dot{\pi}_1 = -r\pi_1 \left[1 - \frac{2S}{K}\right] - (1 - \beta)\mu \frac{h}{\alpha S^2} \left( \frac{\alpha Sh}{\alpha S - h} \right)^\beta; \quad (22)$$

$$\dot{\pi}_2 = \pi_1 h - \pi_2 [b - d + \phi h]; \quad (23)$$

$$\int_0^\infty \mu(t) dt = 1; \quad (24)$$

$$(\pi_0, \pi_1, \pi_2, \mu) \neq (0, 0, 0, 0); \quad (25)$$

$$\mu \geq 0, h^\beta \left[ 1 - \frac{h}{\alpha S} \right]^{1-\beta} - \bar{u} \geq 0, \mu \left[ h^\beta \left( 1 - \frac{h}{\alpha S} \right)^{1-\beta} - \bar{u} \right] = 0;$$

$$\pi_1 \left[ rS \left( 1 - \frac{S}{K} \right) - hL \right] + \pi_2 [b - d + \phi h] L = 0. \quad (26)$$

Furthermore, conditions (16), (17) and (19) are satisfied.

### REGULAR MAXIMIN PATHS

Since  $\mu > 0$  in a regular problem, let  $\psi_1 = \pi_1/\mu$  and  $\psi_2 = \pi_2/\mu$ . Conditions (21), (22), (23), and (26) can be re-expressed as:

$$(\psi_1 - \phi\psi_2)L = \left( \frac{\beta\alpha S - h}{\alpha Sh} \right) \left( \frac{\alpha Sh}{\alpha S - h} \right)^\beta; \quad (27)$$

$$\dot{\psi}_1 = \rho\psi_1 - r\psi_1 \left[ 1 - \frac{2S}{K} \right] - (1 - \beta) \frac{h}{\alpha S^2} \left( \frac{\alpha Sh}{\alpha S - h} \right)^\beta; \quad (28)$$

$$\dot{\psi}_2 = \rho\psi_2 + \psi_1 h - \psi_2 [b - d + \phi h]; \quad (29)$$

$$\psi_1 \left[ rS \left( 1 - \frac{S}{K} \right) - hL \right] + \psi_2 [b - d + \phi h] L = 0. \quad (30)$$

This set of necessary conditions can be used interchangeably with the more general set of conditions from which they are derived whenever  $\mu(t) > 0$ , even in a non-regular program. The necessary conditions imply the following.

**Proposition 1** (i) *There is a continuum of steady-state, regular maximin points  $(S_{ss}, L_{ss})$  with*

$$\frac{d-b}{\alpha\beta\phi} < S_{ss} < K.$$

*Corresponding to each value of  $S_{ss}$  there is a unique value of  $L_{ss}$ . As  $S_{ss}$  varies from  $\frac{d-b}{\alpha\beta\phi}$  to  $K$ , the value of  $\rho_{ss}$  decreases and the maximin utility level increases.*

- (ii) For each regular steady-state pair  $(S_{ss}, L_{ss})$ , where  $S_{ss} \in (\frac{d-b}{\alpha\beta\phi}, K)$ , there exists a neighborhood of  $(S_{ss}, L_{ss})$  in which there is a regular maximin path converging to  $(S_{ss}, L_{ss})$ . The path is monotone, with both  $S$  and  $L$  either increasing or decreasing.
- (iii) The line  $h = \beta\alpha S$  forms a barrier to regular paths in the space  $(h, S)$  and the segment of the line  $L = \frac{r}{\alpha\beta} (1 - S/K)$  for  $S \in (\frac{d-b}{\alpha\beta\phi}, K)$  is a barrier to regular paths in the space  $(S, L)$ .
- (iv) A regular maximin path is stable in the saddle-point sense.
- (v) The limit point of the maximin paths,  $(L, S) = (\frac{r}{\alpha\beta} (1 - \frac{d-b}{\alpha\beta\phi K}), \frac{d-b}{\alpha\beta\phi})$ , is not regular.

Proposition 1 is proved in the Appendix. It tells us that the continuum of steady-state regular paths corresponds to the subset of  $\mathcal{X}$  in which  $S_{ss} > (d-b)/(\alpha\beta\phi)$ . Greater values of  $S_{ss}$  (and lower values of  $L_{ss}$ ) correspond to higher utility levels  $\bar{u}$  and to lower values of  $\rho_{ss}$ . The value of  $\rho_{ss}$  increases as we move along the arc of  $\mathcal{X}$  to approach the limit of the regular paths, where  $S = (d-b)/(\alpha\beta\phi)$ . There is a dense set of maximin paths that converge to these steady-state paths. Each path is truncated in the interior of the space  $(S, L)$  at the line  $L = \frac{r}{\alpha\beta} (1 - \frac{S}{K})$ . A representative path is shown in Figure 1 as (depending on the initial point)  $DE$  or  $FE$ .

PLEASE PLACE FIGURE 1 HERE

We can interpret Proposition 1 by re-writing equation (14), with  $U(h, m) = \bar{u}$  and  $m = 1 - h/(\alpha S)$ , as

$$S = \frac{h/\alpha}{1 - (\bar{u}h^{-\beta})^{1/(1-\beta)}} \quad (31)$$

PLEASE PLACE FIGURE 2 HERE

Equation (31), at a given level of utility  $\bar{u}$ , is depicted in Figure 2. By equation (37) in the Appendix, when  $h = \beta\alpha S$ ,  $dS = 0$  and hence  $\dot{S} = 0$ . At point **d**, on the line  $S = h/(\alpha\beta)$ , the slope  $dS/dh$  is zero. The curve **def** corresponds to **DEF** in

Figure 1. (The curve for a higher utility level is upward from and to the right of **def**.) The steady-state point is at **e**, where  $h = (d - b)/\phi$  and  $dS/dh < 0$ . Near point **e** there is an iso-utility path leading to the steady state. (a) If  $S_0 < S_{ss}$  the path is **de**. The corresponding path in Figure 1 is **DE**. Since  $h > (d - b)/\phi$ , the population is increasing. (b) If  $S_0 > S_{ss}$  the path is along **fe**; because  $h < (d - b)/\phi$ , the population is decreasing. The corresponding path in Figure 1 is **FE**. Although the path **FE** reaches the line  $S = K$ , it is not possible to extend the path **DE** to lower resource-stock sizes. In order to achieve the utility level  $\bar{u}$ , the resource stock must not be less than it is at point **d** in Figure 2 or at **D** in Figure 1.

Regular maximin trajectories **DEF** (**def**) are bounded by the curve **YW** (**yw**). The region in which there are regular paths is the semi-open region **WYDZF** (**wydzf**), which is closed when  $S = K$  except at **W** (**w**) and **Z** (**z**).

We are unable to obtain an analytic solution to the problem, giving the path to the steady state as a function  $v(S, L) = 0$ . Numerical solutions using the parameters assumed by Brander and Taylor (1998: 128) confirm our theoretical conclusions. Along the approach path to the steady state,  $dL/dS > 0$  when  $S_{ss} \in \left(\frac{d-b}{\alpha\beta\phi}, K\right)$ . Among the regular steady states, a higher stock level entails a higher utility level and a lower value of  $\rho$ . Since  $\mu(t) = \rho e^{-\rho t}$  in a steady state, the latter result implies that, for higher stock levels (and the corresponding lower population levels) the tightness of the maximin constraint (2.3) is initially lower and declines more gently. In a sense, achieving a maximin solution is less onerous if there is a higher stock. Along the approach path to the steady state, the shadow price of the population is negative and  $\rho$  varies in the opposite direction to the population size. For example, along the path **DE**, the value of  $\rho(t)$  decreases toward its steady-state value.

We summarize the simulation of the approach to a particular maximin steady state in Table 1. Price variables show substantial variation and are reported to two significant digits; real variables adjust comparatively slowly and are reported to four

significant digits. The level of accuracy is not spurious as this is a simulation exercise.

$t$	$S(t)$	$L(t)$	$h(t)$	$\psi_1(t)$	$-\psi_2(t)$	$\rho(t)$
	$(10^3)$	$(10^3)$	$(10^{-2})$	$(10^{-4})$	$(10^{-3})$	$(10^{-2})$
-1	7.997	4.259	2.500	0.65	3.2	2.1
-6	7.996	4.257	2.501	1.5	3.0	2.7
-11	7.995	4.254	2.501	2.2	2.8	3.0
-16	7.994	4.251	2.502	2.7	2.7	3.6
-21	7.992	4.247	2.503	3.2	2.6	4.0
-26	7.990	4.242	2.505	3.6	2.5	4.3

Table 1: Simulation of a Path Approaching a Steady State

## NON-REGULAR MAXIMIN PATHS

Non-regular maximin paths have not been studied extensively but are an important feature of the present model.

PLEASE PLACE FIGURE 3 HERE

Consider an initial point **a** to the west of region **wydzf** in Figure 2. Corresponding to it is point **A** outside the region of regular paths **WYDZF** in Figure 3. We propose paths **ab** (**AB**) on which  $\dot{L} < 0$  (so that  $h < (d - b)/\phi$ ) and  $\dot{S} < 0$ : with the high population, even the limited harvest per capita causes the resource stock to fall. Since  $h < \beta\alpha S$ , by equation (37) in the proof of Proposition 1,  $\dot{S}/\dot{h} = dS/dh < 0$ , and so the (per-capita) harvest is increasing ( $\dot{h} > 0$ ).

Our reasoning is as follows. By a theorem due to Leonard (1981), the resource stock is a ‘good’ stock ( $\pi_1 > 0$ ) and the population is a ‘bad’ stock ( $\pi_2 < 0$ ). Therefore,  $(\pi_1 - \phi\pi_2)L > 0$  and hence by condition (21)  $\mu > 0$ . Except at point **b**, equation (14) gives two values of  $h$  which, for a given value of  $S$ , yield the same value of utility  $\bar{u}$ . One value of  $h$ , for which  $h > \alpha\beta S$ , is inefficient, as it needlessly uses up more of the resource stock. Optimal points must lie in the region to the northwest of the line

$S = h/(\alpha\beta)$  in Figure 2. Along the iso-utility curve **ab**,  $h < \alpha\beta S$  and

$$\frac{\partial u}{\partial h} = \frac{\alpha\beta S - h}{\alpha S h} \left( \frac{\alpha S h}{\alpha S - h} \right)^\beta > 0.$$

Current utility would be higher with  $h = \alpha\beta S$ ; sustaining utility has a current cost ( $\mu > 0$ ).

At point **b**,  $h = \alpha\beta S$  and  $\partial u/\partial h = 0$ . By equation (37) in the proof of Proposition 1,  $\dot{S} = 0$ . Since  $h = \alpha\beta S$ , equation (16) gives that  $L = \frac{r}{\alpha\beta} \left(1 - \frac{S}{K}\right)$ , a point on the line segment **VY** in Figure 3. Therefore,  $\dot{S} = 0$  at point **B**. Still,  $h < (d - b)/\phi$  and  $\dot{L} < 0$ . At **B**,  $dL/dS$  is infinite. The path then passes into the region below the line **VY**.

At point **B**, since utility is being maximized myopically, the maximin constraint is no longer effective. If after point **B** the harvest remains at the myopic maximum,  $h = \alpha\beta S$ , then by equation (14),

$$U|_t = h^\beta \left(1 - \frac{h}{\alpha S}\right)^{1-\beta} = (\alpha\beta S)^\beta (1 - \beta)^{1-\beta}. \quad (32)$$

Again by equation (16),  $\dot{S} > 0$  and so  $\dot{U} > 0$ . Since current utility is now greater than the maximin level (attained along **AB**), the maximin constraint does not bind:  $\mu = 0$ . In Figure 2, the path passes upward along the line  $S = h/(\alpha\beta)$  from point **b**.

Along the path  $h = \alpha\beta S$ , both  $h$  and  $S$  are increasing but  $h < (d - b)/\phi$  and hence  $L$  is decreasing. In Figure 3, if the path reaches one of the non-regular steady states in the interior of  $\chi$ , it is not absorbed by that steady state but passes through. It is unusual to see a dynamic path pass through a steady state. Stopping at such a steady state, however, would entail lowering current utility, which is maximized when  $h = \alpha\beta S$  and is increasing, and hence would violate the maximin criterion. Eventually the harvest rises to  $(d - b)/\phi$ , so that  $S = (d - b)/(\alpha\beta\phi)$ . At this point  $\dot{L} = 0$ . But the population is lower than at the non-regular steady state **Y**; therefore,

$$\dot{S} = rS \left(1 - \frac{S}{K}\right) - Lh = \alpha\beta S \left[ \frac{r}{\alpha\beta} \left(1 - \frac{S}{K}\right) - L \right] > 0,$$

so that  $dL/dS = \dot{L}/\dot{S} = 0$ . The planner can continue to allow the harvest, the resource stock and the utility level to grow (to let  $h = \alpha\beta S$ ) without violating the maximin constraint. Since now  $h > (d - b)/\phi$ , the population grows as well. All variables continue to grow, but equation (17) implies that the rate of growth of the stock is curtailed as the population and harvest grow. The path eventually reaches the line  $L = \frac{r}{\alpha\beta} \left(1 - \frac{S}{K}\right)$  at point **D**, where  $\dot{S} = 0$  but  $\dot{L} > 0$ , so that  $dL/dS$  is infinite.

From point **D**, the program crosses into the regular region. Utility does not jump with the change of regime, and so at point **D** exceeds that at any previous point on the non-regular path. *At a reference point before point D*, at all future times, even in the regular region, the shadow value of equity,  $\mu$ , is zero and hence  $\pi_0 = 0$ . After the change in regime (after point **D**), however, when society looks forward, all the shadow values are as described in Proposition 1.

This argument above does not use the necessary conditions from the maximum principle to derive the solution along **BD**. Indeed, those conditions tell us nothing about the solution. By condition (21), along the line  $S = h/(\alpha\beta)$  the maximum principle would hold that  $(\pi_1 - \phi\pi_2)L = 0$ , and hence that  $\pi_1 - \phi\pi_2 = 0$ . Since  $\pi_1 \geq 0 \geq \pi_2$ , we have  $\pi_1 = \pi_2 = 0$ . (The values make intuitive sense since the resource stock is greater than and the population is less than the levels required to maintain the maximin utility, attained along **AB**.) Furthermore,  $\mu = 0$  along the path so that  $\pi_0 = \int_0^\infty \mu(t) dt = 0$ . Therefore, condition (25) is violated. The path is degenerate, in the sense that having maximin as objective has no effect on the choice of harvest level; the harvest is equal to the level that maximizes current utility without regard for the future. The path **BD** corresponds to a part of a path in Brander and Taylor's model.

Since  $\mu = 0$ , it is not possible to relax the maximin constraint (20) at or after point **B** in order to increase utility before point **B**, when the constraint is effective (Cairns and Long 2006). Along **BD**, the shadow value  $\mu$ , being zero, cannot be interpreted



as a discount factor and  $\rho(t)$  is not defined.

There is also a path leading to the steady state at **Y**. Along **WY**,  $\pi_1 > 0 > \pi_2$ ,  $\mu > 0$  and  $\pi_0 > 0$ . (It would be costly in terms of the attainable level of  $\bar{u}$  to deviate from **WY** to follow the myopic path for a short time before adopting a maximin policy.) Point **Y**, where  $h = \frac{d-b}{\phi}$ ,  $S = \frac{h}{\alpha\beta} = \frac{d-b}{\alpha\beta\phi}$  and  $L = \frac{r}{\alpha\beta} \left(1 - \frac{d-b}{\alpha\beta\phi K}\right)$  and

$$\bar{u} = (\alpha\beta S)^\beta (1 - \beta)^{1-\beta} = \left(\frac{d-b}{\phi}\right)^\beta (1 - \beta)^{1-\beta} > 0,$$

is the steady state of Brander and Taylor's decentralized, myopic economy. It is also the limit of the set of regular maximin steady states. In this steady state, utility is positive since access is not completely open; the population is limited by the demographic constraint (16). Since Easter Island consisted of a cohesive, limited society with given institutions, it is more accurate to view the resource as common property than open access. Utility is sustained at point **Y**. By Proposition 1 (iv), however, this steady state is not regular: looking forward from any time  $s$  at which the economy is in this steady state, the planner sets  $\mu(t) = 0$  for all  $t \geq s$ . It is clear that a small perturbation in the resource stock would lead to a change in the path and a change in utility in the same direction, so that, at **Y**,  $\pi_1 > 0$ ; the solution is not degenerate.

In another paper on this model, Pezzey and Anderies (2003) consider the effect on a decentralized economy of specifying a subsistence food requirement per capita,

$$h \geq h_0 > 0.$$

In a maximin model, the implications of the subsistence level of harvest can easily be seen. In Figure 2, the region to the left of  $h_0$  is eliminated, and with it some non-regular paths. But “man does not live by bread alone.” Suppose also that there is also a minimal level of non-food goods:  $1 - h/(\alpha S) = m \geq m_0$ , or  $h \leq (1 - m_0)\alpha S$ . This condition would affect our results if  $1 - m_0 < \beta$ : the solutions would have to

stay northwest of the line  $S = h / [(1 - m_0) \alpha]$  rather than of  $S = h / (\beta \alpha)$ . There would be no non-regular maximin paths, and the regular paths would be curtailed near the line  $L = \frac{r}{\alpha \beta} \left(1 - \frac{S}{K}\right)$ .

## CONCLUSION

Depending on initial conditions, this stylized economy has two dense families of maximin paths. One is a continuum of regular paths converging to a steady state. For the pairs  $(S, L)$  lying on a maximin trajectory and some function  $v$ , utility is given by  $\bar{u} = v(S, L)$ , where  $\partial v / \partial S > 0 > \partial v / \partial L$ . The steady states are conditionally stable in the saddle-point sense. Regular maximin paths are confined to the semi-open region **WYZ** in Figure 3.

Brander and Taylor's stylization saw about forty people arriving at Easter Island when the resource stock was equal to the environmental carrying capacity,  $K$ . A (regular) maximin path beginning at the date of arrival would have had the population and resource decrease to reach the stationary state along a path like **FE** in Figure 3. That the population increased some 250-fold indicates the divergence of *laissez faire* from sustaining per-capita utility. The steady state to which Brander and Taylor's economy was tending was the limit of the set of maximin steady states and is a non-regular maximin steady state. If it had been attained, the economy would have continued in this state forever, irrespective of whether its myopic, common-property regime was maintained or a new, maximin objective was adopted.

Along the other set of paths, utility remains constant until it passes into a second regime, consisting of a path with increasing utility, not constrained by the maximin objective, until passing into the regular region with constant utility and eventually attaining a regular steady state. A representative of this family is **ADE**. One path, **WY**, is the boundary of these two families; utility is constant throughout the path, with the maximin constraint effective at all points up to a non-regular steady state

at  $\mathbf{Y}$ , where it is not effective.

All paths are time consistent. Furthermore, they are Pareto optimal. By the second theorem of welfare economics, they can be decentralized using competitive prices. Some authors have perceived the decentralization by viewing the shadow value of equity,  $\mu$ , as tantamount to a discount factor in a virtual utilitarian maximization. Such a decentralization is possible for a regular maximin problem. Looking forward from a point on either type of non-regular path, however, the planner assigns a shadow value of  $\mu = 0$  when the maximin constraint is not effective, so that discount factor is zero and the force of interest,  $\rho = -\dot{\mu}/\mu$ , is not defined. The appropriate discount factor for the virtual utilitarian problem in the non-regular cases is not  $\mu$ .

Brander and Taylor show, and Pezzey and Anderies confirm, that a common-property regime may lead ultimately to a sustained level of utility. But if sustainment is the objective, a common-property regime in even a primitive society may be far from efficient. Pezzey and Anderies find that implementing a policy of sustainment in a decentralized economy can be extremely difficult. Our examination suggests that one reason for this difficulty may be the high degree of subtlety, even in this ostensibly simple problem, of the maximin solution itself.

## REFERENCES

- [1] Brander, James and Scott Taylor (1998) "The Simple Economics of Easter Island: A Ricardo-Malthus Model of Renewable Resource Use," *American Economic Review* 88: 119-138.
- [2] Burmeister, Edwin and Peter Hammond (1978) "Maximin Paths of Heterogenous Capital Accumulation and the Instability of Paradoxical Steady States," *Econometrica* 45: 853-870.

- [3] Cairns, Robert D. and Ngo Van Long (2006), “Maximin: A New Approach,” forthcoming, *Environment and Development Economics*.
- [4] Hartwick, John M. (1977), “Intergenerational Equity and the Investing of Rents From Exhaustible Resources,” *American Economic Review* 66: 972-4.
- [5] Leonard, Daniel (1981), “The Signs of the Co-State Variables and Sufficiency Conditions in a Class of Optimal Control Problems”, *Economics Letters* 8, 321-325.
- [6] Leonard, Daniel and Ngo Van Long, (1992), *Optimal Control Theory and Static Optimization in Economics*, Cambridge University Press, New York.
- [7] Pezzey, John C.V. and John M. Anderies (2003), “The Effect of Subsistence on Collapse and Institutional Adaptation in Population–Resource Economies,” *Journal of Development Economics* 72, 299-320.
- [8] Pezzey, John C.V. and Michael A. Toman (2002-3), “Progress and Problems in the Economics of Sustainability,” in *International Yearbook of Environmental and Resource Economics 2002-3*, eds. T. Tietenberg and H. Folmer, Edgar Elgar, Cheltenham UK, 165-232.
- [9] Rawls, John, (1971), *A Theory of Justice*, Harvard University Press, Cambridge MA.
- [10] Solow, Robert M. (1974), “Intergenerational Equity and Exhaustible Resources,” *Review of Economic Studies*, Symposium, 29-45.

## APPENDIX: PROOF OF PROPOSITION 1.

- (i) Substituting the steady-state value  $L_{ss} = rS_{ss}(1 - S_{ss}/K)/h_{ss}$  into equation (27) gives that

$$\left(\frac{\beta\alpha S_{ss} - h_{ss}}{\alpha S_{ss}}\right) \left(\frac{\alpha S_{ss} h_{ss}}{\alpha S_{ss} - h_{ss}}\right)^{\beta} = (\psi_{1ss} - \phi\psi_{2ss}) \left[rS_{ss} \left(1 - \frac{S_{ss}}{K}\right)\right]. \quad (33)$$

Setting  $\dot{\psi}_i = 0$  ( $i = 1, 2$ ) in equations (28) and (29) yields that

$$\psi_{1ss} \left[ \rho_{ss} - r + \frac{2rS_{ss}}{K} \right] = (1 - \beta) \left( \frac{h_{ss}}{\alpha S_{ss}^2} \right) \left( \frac{\alpha S_{ss} h_{ss}}{\alpha S_{ss} - h_{ss}} \right)^\beta > 0 \quad (34)$$

and

$$\psi_{2ss} = -\frac{h_{ss}}{\rho_{ss}} \psi_{1ss}. \quad (35)$$

Equations (33), (34) and (35) determine  $(\psi_{1ss}, \psi_{2ss}, \rho_{ss})$ . Since the resource is a ‘good’ stock in the sense of Léonard (1980), and the population is a ‘bad’ stock,  $\psi_{1ss} > 0 > \psi_{2ss}$ . There is a function  $v(S, L)$  for which  $\bar{u} = v(S, L)$ , and our argument implies that  $\psi_1 = \partial v / \partial S > 0$  throughout the program.

Since  $\psi_{1ss} > 0$ , equation (35) implies that  $\psi_{1ss} - \phi \psi_{2ss} > 0$ . Since all the other factors in equation (27) are positive,  $\beta \alpha S_{ss} - (d - b) / \phi = \beta \alpha S_{ss} - h_{ss} > 0$ , or

$$S_{ss} > \frac{d - b}{\alpha \beta \phi}.$$

Differentiation of equation (34) yields that  $d\rho_{ss} / dS_{ss} < 0$ .

(ii) Since  $\psi_{1ss} > 0 > \psi_{2ss}$  and the shadow values are continuous, in a neighborhood of the steady state by equation (30),

$$\frac{dL}{dS} = \frac{\dot{L}}{\dot{S}} = -\frac{\psi_1}{\psi_2} > 0.$$

(iii) Along a regular path,

$$[h(t)]^\beta \left[ 1 - \frac{h(t)}{\alpha S(t)} \right]^{1-\beta} = \bar{u}, \text{ or} \quad (36)$$

$$(\beta \alpha S - h) dh + (1 - \beta) \frac{h^2}{S} dS = 0. \quad (37)$$

Now we are going to show that, at any point that is not a steady state (so that  $\dot{h} \neq 0$ ) and where  $\dot{S} = 0$  it must be that  $L = \frac{r}{\alpha \beta} \left( 1 - \frac{S}{K} \right)$  and that  $\dot{L} > 0$ , so that the slope of the trajectory at that point is infinite. The argument is as follows. By equation (37),

$$\dot{S} = -\frac{S(\beta \alpha S - h)}{(1 - \beta) h^2} \dot{h} = 0,$$

so that, given that  $\dot{h} \neq 0$ ,  $\dot{S} = 0$  iff  $h = \beta\alpha S$ . Also,

$$\dot{S} = rS \left(1 - \frac{S}{K}\right) - hL = S \left[ r \left(1 - \frac{S}{K}\right) - \alpha\beta L \right].$$

Therefore,  $L = \frac{r}{\alpha\beta} \left(1 - \frac{S}{K}\right)$  whenever  $\dot{S} = 0$ . For  $S \in \left(\frac{d-b}{\alpha\beta\phi}, K\right)$ ,

$$h = \beta\alpha S > \beta\alpha \frac{d-b}{\alpha\beta\phi} = \frac{d-b}{\phi},$$

and hence  $\dot{L} = (b - d + \phi h) L > 0$ . Therefore,

$$\frac{dL}{dS} = \frac{\dot{L}}{\dot{S}} \rightarrow \infty \text{ as } L \rightarrow \frac{r}{\alpha\beta} \left(1 - \frac{S}{K}\right).$$

For  $L > \frac{r}{\alpha\beta} (1 - S/K)$ ,  $\dot{S}/\dot{L} = -\psi_2/\psi_1 > 0$  by equation (30).

By equations (23) and (26),

$$\dot{\pi}_2 = \pi_1 h - \pi_2 (b - d + \phi h) = \pi_1 h - \pi_2 \frac{\dot{L}}{L} = \frac{\pi_1}{L} (Lh + \dot{S}) = \pi_1 \frac{rS}{L} \left(1 - \frac{S}{K}\right), \quad (38)$$

so that  $\text{sgn}\dot{\pi}_2 = \text{sgn}\pi_1$ .

Now suppose that we move backward along a maximin path in Figure 1, from a point near a steady state such as **E** to a point on the line  $L = \frac{r}{\alpha\beta} (1 - S/K)$ , such as point **D**, where  $\dot{S} = 0$  and  $\dot{L} > 0$ . We shall argue that at **D**,  $\pi_1(t)$  and  $\mu(t)$  are both infinite. First, along the path **ED** (excluding point **D**), condition (25) holds that

$$(\pi_0, \pi_1(t), \pi_2(t), \mu(t)) \neq 0.$$

Second, we show that, as we move backward along **ED**,  $\pi_2(t)$  must approach a limiting value that is less than zero. For, suppose that  $\pi_2$  does approach zero. Since  $\pi_1 > 0$  and  $\dot{S} > 0$  on **ED** except at **D**, the right-hand side of equation (38) is positive, and thus  $\dot{\pi}_2 > 0$ . This, together with the supposition that  $\pi_2$  approaches zero near **D**, implies that  $\pi_2 > 0$  near **D**. Bearing in mind that  $\dot{S} > 0$  and  $\dot{L} > 0$  along **ED** (though not at **D**), having  $\pi_2 > 0$  implies that

$$\pi_1 \dot{S} + \pi_2 \dot{L} > 0,$$

contrary to condition (26). Furthermore, since all of  $\pi_1$ ,  $\dot{S}$  and  $\dot{L}$  are positive,  $\pi_2 < 0$ .

From equation (26),

$$\pi_1 = -\pi_2 \frac{\dot{L}}{\dot{S}}.$$

Since  $\dot{L} > 0$  and  $\pi_2 \neq 0$ , as  $\dot{S} \rightarrow 0$  (approaching point **D**) it must be that  $\pi_1 \rightarrow \infty$ . Since  $\psi_1(t) = \pi_1(t)/\mu(t)$ , it must be that  $\mu(t) \rightarrow \infty$  as we approach point **D**.

(iv) Along the stable branch of the saddle point associated with the steady-state pair  $(S_{ss}, L_{ss})$  and the maximin utility level  $\bar{u}$ , let  $h = \theta(S; \bar{u})$ . To find the slope of the stable branch at a point such as **E** in Figure 1, we first divide equation (16) by equation (17) to obtain

$$\frac{dL}{dS} = \frac{\dot{L}}{\dot{S}} = \frac{(b - d + \phi\theta(S; \bar{u})) L}{rS(1 - S/K) - \theta(S; \bar{u}) L}.$$

At the steady state  $(S_{ss}, L_{ss})$ , we use L'Hôpital's rule and the fact that  $b - d + \phi\theta(S; \bar{u}) = 0$  at  $(S_{ss}, L_{ss})$  to get

$$\begin{aligned} \frac{dL}{dS} &= \frac{\frac{d}{dS} [(b - d + \phi\theta(S; \bar{u})) L]}{\frac{d}{dS} [rS(1 - S/K) - \theta(S; \bar{u}) L]} \\ &= \frac{L\phi \frac{d\theta}{dS}}{r(1 - 2S/K) - \theta \frac{dL}{dS} - L \frac{d\theta}{dS}}, \text{ or} \\ \left(\frac{dL}{dS}\right)^2 + \left(\frac{L}{h} \frac{d\theta}{dS} - \frac{r}{h} \left(1 - 2\frac{S}{K}\right)\right) \left(\frac{dL}{dS}\right) + \frac{L\phi}{h} \frac{d\theta}{dS}, \end{aligned} \quad (39)$$

a quadratic equation in  $dL/dS$ . Since  $\frac{L\phi}{h} \frac{d\theta}{dS} < 0$  by equation (37), the two solutions are of opposite signs. We take  $dL/dS > 0$  because Figure 1 indicates that the slope of the stable branch of the saddle point is positive.

Now let  $\varepsilon$  be a small, positive number and consider  $S = S_{ss} + \varepsilon$ . The corresponding value of  $L$  is  $L_{ss} + \varepsilon \frac{dL}{dS}$ . Since we do not have an explicit functional form for  $\theta(S; \bar{u})$ , we replace the function  $\theta(S; \bar{u})$  by its linear approximation,

$$\theta(S; \bar{u}) = \theta(S_{ss}; \bar{u}) + (S - S_{ss}) \frac{d\theta(S_{ss}; \bar{u})}{dS}.$$

Having solved for  $\theta(S; \bar{u})$ , we have a system of two differential equations conditional on  $h = \theta(S; \bar{u})$ . We analyze the stability of this system by linearizing the system (16) and (17):

$$\begin{bmatrix} \dot{L} \\ \dot{S} \end{bmatrix} = \begin{bmatrix} 0 & \phi L_{ss} \theta_S \\ -h^* & -L_{ss} \theta_S + r(1 - 2S_{ss}/K) \end{bmatrix} \begin{bmatrix} L - L_{ss} \\ S - S_{ss} \end{bmatrix}, \quad (40)$$

By equation (37),

$$\theta_S = -\frac{(1 - \beta) h^{*2}}{(\alpha \beta S_{ss} - h^*) S_{ss}}.$$

Let the matrix on the RHS of equation (40) be represented by  $J$ , its determinant,  $h^* \theta_S \phi L_{ss}$ , by  $\det J$  and its trace,  $-L_{ss} \theta_S + r(1 - 2S_{ss}/K)$ , by  $\text{tr} J$ . Its two characteristic roots are

$$\lambda_{1,2} = \frac{1}{2} \left[ \text{tr} J \pm \left( (\text{tr} J)^2 - 4 \det J \right)^{\frac{1}{2}} \right].$$

The product of the roots is  $\det J$  and the sum of the roots is  $\text{tr} J$ . If  $\det J < 0$ , there are a negative and a positive root, and the steady state has the saddle point property: there is a path converging to it. Since for all  $S_{ss} \in \left( \frac{d-b}{\alpha \beta \phi}, K \right)$ ,  $\theta_S < 0$ , it follows that  $\det J < 0$  in that interval. This proves stability in the saddle-point sense.

(v) At the point  $\left( \frac{r}{\alpha \beta} \left( 1 - \frac{d-b}{\alpha \beta \phi K} \right), \frac{d-b}{\alpha \beta \phi} \right)$ ,  $S = (d - b) / (\alpha \beta \phi)$ . This is not a regular steady-state point by the proof of part (i) above. ■