Numerical Optimisation of Multiple-Phase Systems Incorporating Transition Costs[Ψ]

Graeme J. Doole[*]

School of Agricultural and Resource Economics, Faculty of Natural and Agricultural Sciences, University of Western Australia, 35 Stirling Highway, Crawley, Western Australia 6009.

Abstract. Many important economic problems concern an intertemporal choice between alternate dynamical systems. One example is determining the optimal management of alternative production technologies. This significance has motivated a substantial theoretical literature generalising the necessary conditions of Optimal Control Theory to multiple-phase problems. However, gaining detailed insight into their practical management is difficult because suitable numerical solution methods are not available. This paper resolves this deficiency through the development of a flexible and efficient computational algorithm based on a set of necessary conditions derived for finite-time multiple-phase systems. Its effectiveness is demonstrated in an application to a complex crop rotation problem.

Keywords. Crop management, multiple-phase systems, optimal control.

JEL classification codes. C61; Q24.

[Ψ] The author would like to thank Ross Dryden, Michael Harris, Greg Hertzler, Larry Karp, Jeffrey LaFrance, David Pannell, and Christian Traeger for helpful comments. The author is also grateful to the Cooperative Research Centre (CRC) for Plant-Based Management of Dryland Salinity (University of Western Australia, 35 Stirling Highway, Crawley, Western Australia 6009) and the CRC for Australian Weed Management (Waite Road, Urrbrae, PMB 1, Waite Campus, Glen Osmond, South Australia 5064) for financial support.

[*] *Telephone*: +61-8-94028162. *Fax*: +61-8-64881098. *E-mail address*: gdoole@cyllene.uwa.edu.au.

1. Introduction

The Maximum Principle of Optimal Control Theory (Pontryagin et al., 1962) has been utilised extensively in economics (Arrow and Kurz, 1970; Seierstad and Sydsaeter, 1987; Kamien and Schwartz, 1991) because of its intuitive economic interpretation (Dorfman, 1969) and the significant methodological extensions to this theory developed in other fields of study, such as engineering. However, despite this broad application, there has been limited treatment of multiple-phase systems. These consist of multiple alternate regimes, each characterised by its own dynamical system, of which only one may be active at each point in time. Selecting between individual crops to plant on a given area of land is one example (Mueller et al., 1999). Other examples are determining the optimal time to switch between alternative energy sources (Tomiyama, 1985; Tomiyama and Rossana, 1989) and identifying the optimal time for a government to abolish a policy, such as a capital control (Makris, 2001). In actual fact, many economic decisions may be studied more precisely if cast as multiple-phase problems. For example, in production theory, these are a natural means of representing choices between the alternative technologies available to a firm, such as the natural and artificial recovery of petroleum (Amit, 1986).

Switching schedules may be determined through the standard Maximum Principle if individual stages are represented by piecewise-constant control variables. However, this approach is inherently combinatorial and complicated significantly through the existence of transition costs (see Teo and Jennings, 1991, and references therein). These limitations have motivated the analysis of multiple-phase systems in which the sequence of stages is pre-assigned. This approach is, in fact, relevant to many important economic problems, such as the alternative crop, technology, or government policy examples outlined above. Such systems may be studied in a financial options framework (Dixit and Pindyck, 1994) if no control variables are exercised during the duration of a stage. In contrast, generalisation of the Maximum Principle (Pontryagin et al., 1962; Kamien and Schwartz, 1991) is required if instrument variables are defined within independent phases. Such conditions have been derived for two-stage systems with costless transition (Tomiyama, 1985; Tomiyama and Rossana, 1989) and switching costs (Amit, 1986). The latter framework has also been extended to include three stages (Mueller et al., 1999) and an infinite horizon (Makris, 2001).

Though this theory is well established, the practical management of multiple-phase problems is difficult to study given a distinct lack of suitable optimisation algorithms. Gradient-based methods (Judd, 1998) are difficult to apply to a multiple-phase system incorporating control variables in each stage because the state and costate equations are piecewise defined and the performance index has, by definition, discontinuous derivative(s) with respect to the control variable(s) within each stage (see section 2). Transition costs also introduce step discontinuities into the adjoint and Hamiltonian trajectories along an optimal path. Moreover, the efficient computation of optimal strategies for multiple-phase problems of realistic complexity through dynamic programming (Rust, 1996) is non-trivial in most instances. This is intuitive given the large state and policy spaces typically encountered within such applications.

This paper presents a novel computational algorithm for the solution of multiple-phase optimal control problems incorporating transition costs. It involves the iterative improvement of switch points utilising a root-finding procedure. This approach is inspired by the use of shooting methods to solve boundary value problems (Ascher et al., 1995; Stoer and Bulirsch, 2002). The algorithm presented here is based on a set of necessary conditions derived for a finite-time multiple-phase system with different endpoint constraints and n phases. This derivation is necessary because previous theoretical studies have ignored alternative endpoint constraints, consequently narrowing their applicability, and the prior analysis of finite-time systems has been limited to either two (Tomiyama, 1985; Amit, 1986; Tomiyama and Rossana, 1989) or three regimes (Mueller et al., 1999). The effectiveness of the algorithm is demonstrated in an application to a complex multiple crop control problem incorporating strong nonlinearities and stiff process equations. This algorithm appears to be the first in Economics to solve general multiple-phase problems and provides practitioners with the opportunity to study these systems in considerable detail, a luxury not afforded in the analytical constructs to which they have previously been restricted.

The model and necessary conditions are presented in Section 2. Section 3 describes the numerical algorithm and discusses its implementation. An application of this algorithm to a multiple crop problem is presented in Section 4. Section 5 presents conclusions and recommendations for further research. The parameter values for the numerical application are presented in an appendix.

2. Model and Necesssary Conditions

This section formally defines a model for a multiple-phase system and presents a set of necessary conditions required for its solution.

DEFINITION 2.1. *A general multiple-phase system is assumed to incorporate an m-dimensional state vector* $x(t)=\{x^1(t), x^2(t),..., x^m(t)\}$ *of continuous functions, piecewise continuous differentiable over the time interval* $t=[t_0,...,t_n]$ *and belonging to* $X \in R^m$*, and a v-dimensional vector of control functions* $u(t)=\{u^1(t), u^2(t),..., u^v(t)\}$*, piecewise continuous in* $t=[t_0,...,t_n]$ *and belonging to* $U \in R^v$*. The state variables are assumed fixed at the initial time and are denoted* x_0*. The state variables free at the terminal time are denoted* x_n^i*, for* $i=[1,2,...,d]$*. Terminal state variables* x_n^i*, for* $i=[d+1,...,m]$*, are fixed.* □

This model concerns multiple-phase systems with a given switching sequence and fixed number of stages. Relaxation of these assumptions adds significant complexity but would nevertheless be a valuable extension of this work.

DEFINITION 2.2.[1] *A* multiple-phase switching system *is defined as* $\Xi = \{T, K, \vartheta\}$ *where,*

1. *T is a set of discrete controls known as* switching times *that dictate the termination of one phase and the start of the next,*
2. $K = \{k_1, k_2, ..., k_n\}$ *is a finite, fixed, and exogenously determined sequence of discrete (integer) states that indexes individual continuous dynamical systems,* $\vartheta = \{\vartheta_k\}_{k \in K}$, *where* $\vartheta_k = [X, f_k, U]$. *The ordinal ranking of sequences is defined over the closed interval* $j = [1, 2, ..., n]$,
3. X *is a continuous state space where* $X \in R^m$,
4. f_k *is the vector of state equations for each stage k, and*
5. U *is the set of admissible controls lying in* R^v. □

DEFINITION 2.3. *A* control input *for a multiple-phase switching system* Ξ *consists of a set of vectors* $\chi_\Xi = \{t, u\}$ *where,*

1. $t = \{t_1, t_2, ..., t_{n-1}\}$ *is a sequence of real numbers denoting switching times, the moment* t_j *at which stage* k_j *is terminated and the stage* k_{j+1} *becomes active. It follows that regime* k_j *is active over the interval* $[t_{j-1^+}, t_{j^-}]$, *where* t_{j-1^+} *is the moment just after* t_{j-1} *and* t_{j^-} *is the moment just before* t_j,

[1] This definition is loosely based on the hybrid system defined in Branicky et al. (1998).

2. $t = t_n$ *is a freely determined terminal time, and*

3. $u = \{u_1, u_2, ..., u_n\}$ *is a collection of control functions defined for each stage in sequence K.*

□

It is possible for switching times to accumulate in this model. Consequently, not all regimes in the predefined sequence must be activated. For example, it may be optimal for two consecutive switching times, such as t_j and t_{j+1}, to coalesce (that is, $t_j = t_{j+1}$), in which case, movement from k_j to k_{j+2} will occur without the activation of k_{j+1}. This allows for the case where the operation of a stage or number of stages in sequence *K* is not contained in the optimal solution.

The state variable is continuous at the switching times in this model. However, jumps within the state variable (Vind, 1967) may be accommodated with manipulation of the necessary conditions (see Seierstad and Sydsaeter, 1987).

DEFINITION 2.4. *A* trajectory *(Γ) for a multiple-phase switching system Ξ and control sequence χ_Ξ is* admissable *over the interval $t = [t_0, t_1, ..., t_{n-1}, t_n]$ if it satisfies Definition 2.1 and the continuous dynamics $\dot{x} = f_j(x(t), u_j(t))$, for $[t_{j-1^+}, t_{j^-}]$ and $j \in J$, for a predefined switching sequence $K = \{k_1, k_2, ..., k_n\}$.* □

These definitions permit the classification of a general multiple-phase optimal control problem.

PROBLEM 2.1. *For a multiple-phase system Ξ identify an admissible trajectory that maximises the objective functional,*

$$J = e^{-rt_n} G(x(t_n), t_n) + \sum_{j=1}^{n-1} e^{-rt_j} C_j(x(t_j)) + \sum_{j=1}^{n} \left[\int_{t_{j-1}^+}^{t_j^-} \left[e^{-rt} F_j(x(t), u_j(t)) \right] dt \right], \tag{1}$$

subject to,

$$\dot{x} = f_j(x(t), u_j(t)),\ for\ [t_{j-1^+}, t_{j^-}]\ and\ j = [1,2,...,n]\ given\ K = \{k_1, k_2, ..., k_n\}, \tag{2}$$

$$t_j\ free\ for\ j = [1,2,...,n], \tag{3}$$

$$x(t_j)\ free\ for\ j = [1,2,...,n-1], \tag{4}$$

$$x_0\ fixed, \tag{5}$$

$$x_n^i(t_n)\ free,\ for\ i = [1,...,d],\ and \tag{6}$$

$$x_n^i(t_n)\ fixed\ for\ i = [d+1,...,m], \tag{7}$$

where r is a discount rate, $G(x(t_n), t_n)$ *is a terminal reward function,* $C_j(x(t_j))$ *is a switching cost function for the jth phase, and* $F_j(x(t), u_j(t))$ *is a single-valued reward function on* $X^m \times U^v$ *for the jth phase. Functions* $G(\cdot)$, $C(\cdot)$, *and* $F(\cdot)$ *are all real-valued functions that are once continuously differentiable in the relevant arguments. The terminal value function G is defined for* $x_n^i(t_n)$, *where* $i = [1,...,d]$. □

The terminal reward function $G(x(t_n), t_n)$ is defined as a salvage value in economic applications of optimal control. The switching cost function is a cost accruing to the termination of one stage and the start of another. (These can be understood as terminal value functions for individual regimes.) They are a pertinent feature of many multiple-phase systems. For example, it can be costly to remove one crop and establish another (Mueller et al., 1999) or invest in the productive capacity required for the artificial recovery of petroleum (Amit, 1986). Both the terminal value function $G(\cdot)$ and the switching cost function $C(\cdot)$ are dependent on the state variable ($x(t_j)$).

The latter is included because it is likely to exist in a number of important multiple-phase problems. For example, the herbicide dose required for the establishment or removal of a crop may be dependent on weed density. Or, investing in a new production technology may require an initial outlay that is dependent on the current capacity of the existing firm.

THEOREM 2.1. *Consider a multiple-phase system* Ξ *described by Definitions 2.1-2.4. For* $j=[1,2,...,n]$ *and switching sequence* $K=\{k_1,k_2,...,k_n\}$*, let* $(x^*(t),u_j^*(t),t_j^*)$ *denote the admissible trajectory that maximises the value of J in Problem 2.1. This is the optimal trajectory* Γ^*.

Define a Hamiltonian function for each regime k_j *as,*

$$H_j(x(t),u_j(t),\lambda_j(t),t)=e^{-rt}F_j(x(t),u_j(t))+\lambda_j(t)f_j(x(t),u_j(t),t), \tag{8}$$

across the interval $[t_{j-1^+},t_{j^-}]$.

An optimal trajectory Γ^* *requires,*

i) *initial condition* $x_0=x(t_0)$ *for fixed initial state variable(s)* x_0, (9)

ii) *n m-dimensional vectors of real-valued, piecewise continuous adjoint functions* $\lambda_j(t)=\{\lambda_j^1(t),\lambda_j^2(t),...,\lambda_j^m(t)\}$*, defined across* $j=[1,2,...,n]$ *and piecewise continuously differentiable over the interval* $[t_{j-1^+},t_{j^-}]$*, that satisfy,*

$$\dot{\lambda}_j^T(t)=-\frac{\partial H_j(x(t),u_j(t),\lambda_j(t),t)}{\partial x(t)}, \tag{10}$$

where $\lambda_j^T(t)$ *denotes the transpose of the n adjoint vectors,*

iii) *optimal control function(s) that satisfy,*

$$\underset{u_j(t)}{\text{Max}}\, H_j(x(t), u_j(t), \lambda_j(t), t) \quad \text{for all } t \in [t_{j-1^+}, t_{j^-}], \tag{11}$$

iv) *an adjoint vector* $\lambda_n(t_n)$ *that satisfies,*

$$\lambda_n^T(t_n) = \frac{\partial e^{-rt_n} G(x(t_n), t_n)}{\partial x(t_n)}, \tag{12a}$$

for state variables $x_n^i(t_n)$*, where* $i = [1,...,d]$*, free at the terminal time and defined in G,*

NOTE: $\lambda_n^T(t_n) = 0$ *replaces (12a) for those state variables* $x_n^i(t_n)$*, where* $i = [1,...,d]$*, that are not defined in G,* (12b)

NOTE: $x_n^i(t_n) = x(t_n)$ *replaces (12a) and (12b) for fixed terminal state variables* $x_n^i(t_n)$*, where* $i = [d+1,...,m]$, (12c)

v) *a terminal time that satisfies,*

$$H_n(x(t), u_n(t), \lambda_n(t), t)\Big|_{t_n} + \frac{\partial e^{-rt_n} G(x(t_n), t_n)}{\partial t_n} = 0, \tag{13a}$$

if no terminal value function is defined, then the equivalent of (13a) is,

$$H_n(x(t), u_n(t), \lambda_n(t), t)\Big|_{t_n} = 0, \tag{13b}$$

if, instead, the terminal time is fixed, then no additional necessary condition is required, as $t = t_n$, (13c)

vi) *adjoint vectors that satisfy the boundary conditions,*

$$\lambda_j^T(t_{j^-}) + \frac{\partial e^{-rt_j} C_j(x(t_j))}{\partial x(t_j)} = \lambda_{j+1}^T(t_{j^+}), \tag{14}$$

at switching times $t = \{t_1, t_2, ..., t_{n-1}\}$ *and* $j = [1, 2, ..., n-1]$,

vii) $$H_j(x(t), u_j(t), \lambda_j(t), t)\Big|_{t_{j^-}} - \frac{\partial e^{-rt_j} C_j(x(t_j))}{\partial t_j} = H_{j+1}(x(t), u_{j+1}(t), \lambda_{j+1}(t), t)\Big|_{t_{j^+}}, \tag{15}$$

for those switching times in $t = \{t_1, t_2, ..., t_{n-1}\}$ *for which* $t_{j-1} < t_j < t_{j+1}$ *holds,*

viii) $$H_j(x(t), u_j(t), \lambda_j(t), t)\Big|_{t_{j^-}} - \frac{\partial e^{-rt_j} C_j(x(t_j))}{\partial t_j} \leq H_{j+1}(x(t), u_{j+1}(t), \lambda_{j+1}(t), t)\Big|_{t_{j^+}}, \tag{16}$$

for those switching times in $t = \{t_1, t_2, ..., t_{n-1}\}$ *for which* $t_{j-1} = t_j < t_{j+1}$ *holds , and*

ix) $$H_j(x(t), u_j(t), \lambda_j(t), t)\Big|_{t_{j^-}} - \frac{\partial e^{-rt_j} C_j(x(t_j))}{\partial t_j} \geq H_{j+1}(x(t), u_{j+1}(t), \lambda_{j+1}(t), t)\Big|_{t_{j^+}}, \tag{17}$$

for those switching times in $t = \{t_1, t_2, ..., t_{n-1}\}$ *for which* $t_{j-1} < t_j = t_{j+1}$ *holds.* □

PROOF. An extensive proof is provided in a mathematical appendix available at www.are.uwa.edu.au/home/derivation.

Necessary conditions (8)-(13) are analogous to the standard Maximum Principle (Seierstad and Sydsaeter, 1987). This follows the definition of a multiple-phase problem as a set of *n* dynamical systems. In contrast, switching conditions (14)-(17) are not found in standard control problems. These describe how individual systems are linked over time under optimal management. These conditions appear in similar form in the models of Amit (1986), Mueller et al. (1999), and Makris (2001). It is demonstrated here that they generalise to a finite-time multiple-phase model with n regimes, positive switching costs, and alternative endpoint constraints. Equations (14) and (15) are also equivalent to the smooth pasting and value-matching conditions found in applications of stochastic control in finance (Brekke and Oksendal, 1994; Dixit and Pindyck, 1994).

Equation (14) determines the optimal level of the state variable(s) at each switching time ($x(t_j)$) (these are referred to as transition states in the following). The shadow price variables, $\lambda_j^T(t_j)$ and $\lambda_{j+1}^T(t_j)$, represent the marginal adjustment in optimal value accruing to a change in the state variable within the corresponding stage when switching time t_j is approached from below or above respectively. The second term in (14) represents the marginal transition cost for the active regime. Equation (14) states that it is optimal to switch when the marginal value of a change in the state variable is equivalent between stages.

Switching conditions (15)-(17) describe the management of optimal switching times given the relative value of alternate stages. The value of a Hamiltonian function $H_j(x(t),u_j(t),\lambda_j(t),t)$ evaluated at a given time represents the shadow price of altering the length of this phase. The second term in each of conditions (15)-(17) is the rate at which transition costs change over time within regime *j*. Equation (15) states that it is optimal to switch to the subsequent regime at time t_j if the rate at which the capital value of each stage changes over time is equal at that point. Regime *j* should not be activated if its total value, reflected through its Hamiltonian and switching cost functions, is dominated at each potential switching time by that of the successive regime. This is described in (16). Moreover, the successive regime should not be adopted if there is no time t_j at which its capital value matches that earned within the active phase. This is stated in equation (17).

Necessary conditions (14)-(17) are not required if T is empty. In this instance, Theorem 2.1 collapses to the standard Maximum Principle. The state variable(s) could be fixed for a given switching time t_j. In this instance, equation (14) is no longer required for the determination of $x(t_j)$. Alternatively, the control input χ_Ξ may contain fixed switching times. Necessary conditions (15)-(17) are not required in this case.

The boundary conditions are obviously affected if switching cost functions $e^{-rt_j}C_j(x(t_j))$ and/or their relevant derivatives are not defined. If switching costs do not exist or are independent of the state vector, condition (14) requires equality between the adjoint variables of stages *j* and *j+1*. That is, $\lambda_j^T(t_j) = \lambda_{j+1}^T(t_j)$. Likewise, equation (15) simplifies to a requirement of equality between the total capital value of each regime at the switching time; that is, $H_j(\cdot)\big|_{t_j} = H_{j+1}(\cdot)\big|_{t_j}$; if switching costs are not defined or are independent of time. (Switching costs will be a function of time in most economic problems because of discounting.) These results are analogous to the Weierstrass-Erdmann corner conditions (Seierstad and Sydsaeter, 1987) from variational calculus, which are also required when state and/or control variables are subject to inequality constraints (Pontryagin et al., 1962). This equivalency highlights the close symmetry between multiple-phase problems with fixed and free stage sequencing, if the latter is incorporated utilising piecewise constant controls and transition costs do not exist.

3. Algorithm

Theorem 2.1 may be used to identify analytical solutions to multiple-phase problems of low dimension. However, such solutions are extremely difficult to obtain, even in systems incorporating only weakly non-linear differential equations. This section consequently describes an optimisation algorithm suited to the study of more complex problems.

The following algorithm is motivated by the structure of Theorem 2.1, which infers decomposition into two distinct stages. The first concerns the solution of each phase as an independent control problem at each iteration. The second concerns the updating of the switch points using the switching conditions (14) and (15) and a bisection technique (Stoer and Bulirsch, 2002). Bisection successively reduces the size of an interval where a root is bound between function values that are opposite in sign. Bisection is utilised here as other root-finding methods, such as the Newton, Broyden and secant methods (Ortega and Rheinboldt, 1970; Judd, 1998), require continuity of the switching conditions. Newton's method also requires derivative information that is not available in this instance. The existence of a solution to an interval bisection technique is guaranteed for a continuous function through the intermediate value theorem, provided the initial function evaluations are opposite in sign. The step discontinuity that occurs at each switch point (given the presence of transition costs) does not void this condition in computational application given its equivalence to a continuous function whose root is located between two floating-point numbers (Press et al., 1992).

ALGORITHM 3.1

PURPOSE: Identify an optimal control sequence χ_{Ξ} for the multiple-phase system Ξ.

INITIALISATION:

a) Determine a fixed stage sequence *K*. Define the maximum number of permissible iterations ($\hat{i}$). Define the stopping tolerance ε. Define a set of initial conditions $\Lambda = \{t_0, x_0\}$. Provide estimates for the optimal switching times (t_j for $j = [1,2,...,n-1]$) and the transition states ($x(t_j)$ for $j = [1,2,...,n-1]$) for $i = \{1,2\}$.

b) Optimise each phase k_j, for $j = [1,2,...,n-1]$, as a fixed point control problem utilising conditions (8)-(11) and (12c) and (13c). (12c) and (13c) are determined by the estimates of t_j and $x(t_j)$. Optimise the terminal stage utilising conditions (8)-(11) and the relevant terminal conditions from (12)-(13).

c) Obtain $\lambda_j^T(t_j)$ and compute $H_j(t_j)$ for all *j*. Ensure that $\left(\lambda_j^1(t_j) + \left(e^{-rt_j}C_j(x(t_j))\right)^1_{x(t_j)} - \lambda_{j+1}^1(t_j)\right)\left(\lambda_j^2(t_j) + \left(e^{-rt_j}C_j(x(t_j))\right)^2_{x(t_j)} - \lambda_{j+1}^2(t_j)\right) < 0$ and $\left(H_{t_j}^1(t_j) - \left(e^{-rt_j}C_j(x(t_j))\right)^1_{t_j} - H_{j+1}^1(t_j)\right)\left(H_{t_j}^2(t_j) - \left(e^{-rt_j}C_j(x(t_j))\right)^2_{t_j} - H_{j+1}^2(t_j)\right) < 0$ where numeric superscripts denote the iteration number, $(\cdot)_x$ denotes the derivative of the term enclosed in brackets with respect to the subscripted variable (x in this example), and $t_j^1 < t_j^2$ and $x(t_j^1) < x(t_j^2)$.

MAIN COMPUTATION:

For *i*=3: $\hat{i}$

1. Form switch points for the current iteration using the midpoint formulas $t_j^i = (t_j^{i-1} - t_j^{i-2})/2$ and $x(t_j^i) = (x(t_j^{i-1}) - x(t_j^{i-2}))/2$.
2. Optimise each phase k_j for $j = [1,2,...,n-1]$ as a fixed point control problem utilising conditions (8)-(11) and (12c) and (13c). Optimise the terminal stage utilising conditions (8)-(11) and the relevant terminal conditions in (12)-(13). Obtain $\lambda_j^T(t_j)$ and compute $H_j(t_j)$ for all *j*.
3. If $\left(\lambda_j^i(t_j) + \left(e^{-rt_j}C_j(x(t_j))\right)^i_{x(t_j)} - \lambda_{j+1}^i(t_j)\right)\left(\lambda_j^{i-2}(t_j) + \left(e^{-rt_j}C_j(x(t_j))\right)^{i-2}_{x(t_j)} - \lambda_{j+1}^{i-2}(t_j)\right) > 0$ then $x(t_j^i) = x(t_j^{i-2})$ and $x(t_j^{i-1}) = x(t_j^{i-1})$. Else, $x(t_j^i) = x(t_j^{i-1})$ and $x(t_j^{i-2}) = x(t_j^{i-2})$.

4. If $\left(H^i_{t_j}(t_j) - \left(e^{-rt_j}C_j(x(t_j))\right)^i_{t_j} - H^i_{j+1}(t_j)\right)\left(H^{i-2}_{t_j}(t_j) - \left(e^{-rt_j}C_j(x(t_j))\right)^{i-2}_{t_j} - H^{i-2}_{j+1}(t_j)\right) > 0$ then $t^i_j = t^{i-2}_j$ and $t^{i-1}_j = t^{i-1}_j$. Else, $t^i_j = t^{i-1}_j$ and $t^{i-2}_j = t^{i-2}_j$.
5. Stop and print output if $t^i_j - t^{i-1}_j < \varepsilon$ and $x(t^i_j) - x(t^{i-1}_j) < \varepsilon$ or $\left(\lambda^i_j(t_j) + \left(e^{-rt_j}C_j(x(t_j))\right)^i_{x(t_j)} - \lambda^i_{j+1}(t_j)\right) < \varepsilon$ and $\left(H^i_j(t_j) - \left(e^{-rt_j}C_j(x(t_j))\right)^i_{t_j} - H^i_{j+1}(t_j)\right) < \varepsilon$ for all j.
6. If $i = \hat{i}$, then stop and report progress; else go to Step 1.

The boundary conditions for each individual control problem in step (b) in the initialisation and step (2) in the main computation are well-defined following the prior definition of the switching times and the transition states. It is natural to question whether the designation of these fixed points will affect satisfaction of the optimality condition (11) for interior solutions, $\left(H_j(\cdot)\right)_u = 0$ for $j = [1,2,...,n]$, as the weak variation δu in equation (A.14) in the accompanying mathematical appendix (available at www.are.uwa.edu.au/home/derivation) is no longer entirely arbitrary but must now satisfy these endpoint constraints. However, it may be shown that (11) holds despite this induced restriction (see Kamien and Schwartz, 1991, Section II.6).

The approach taken in Algorithm 3.1 resembles the single shooting algorithm used for the solution of two-point boundary value problems commonly defined by the necessary conditions of the standard Maximum Principle. The single shooting algorithm involves integration of the state and costate equations using an Initial Value Problem method and updating of the unspecified initial condition(s) through use of a root-finding method until the given endpoint condition(s) are satisfied to sufficient accuracy (Keller, 1968; Osborne, 1969; Ascher et al., 1995). Their stability is significantly increased through division of the problem into multiple intervals that reduce the length of each integration (Lipton et al., 1982; Stoer and Bulirsch, 2002). This method, known as multiple shooting, may be adapted to analyse multiple-phase problems (see Bulirsch and Chudej 1995 for a two-phase example). However, in contrast to Algorithm 3.1, an expensive approximation of the Jacobian matrix is typically required for the non-linear equation solver at each iteration and this solver is also required to enforce the continuity of each state variable at the switching time (Pesch, 1994; Stoer and Bulirsch, 2002).

Phases are bypassed if equation (15) is satisfied for consecutive switching times at a single moment. However, this algorithm does not cater for the situation where (16) and (17) hold as inequalities. These may be incorporated in simple problems utilising mathematical programming (see Mueller et al., 1999). However, this requires that the differential equations governing the dynamic behaviour of the state and costate variables are explicitly solvable. Algorithm 3.1 does not face such restrictions and is therefore capable of solving problems of much greater complexity.

The following application is programmed in MATLAB version 7.1 (Miranda and Fackler, 2002). Each sub-problem (phase) is solved utilising a variant of the MISER parameterisation algorithm of Teo et al. (1991), which is engineered to operate more efficiently in an iterative scheme. This algorithm involves an approximation of control functions within each phase through interpolation with sets of linear basis functions and solution of the discretised problem using non-linear programming (NLP). Adjoint and state equations are integrated explicitly over the length of a stage using a differential algebraic equation method (Ascher et al., 1995) following the definition of an initial guess of the optimal control. These control histories are subsequently iteratively improved using NLP, with the integration of the process equations repeated at each step to calculate the required gradients, until an optimal solution is obtained. A sequential quadratic programming (SQP) NLP algorithm (Gill et al., 1981) is used because it is the most robust and efficient method presently available for this form of optimisation (Betts et al., 1993; Betts and Gablonsky, 2002). Control parameterisation is adopted for the solution of each phase due to its efficiency and improved convergence relative to other methods. Approximation of control variables utilising basis functions introduces some degree of suboptimality but this is significantly reduced as the number of such functions in each phase is increased, with an optimum of around twenty knot points (Teo et al., 1991), which is subsequently adopted in the following application.

It is well known that the bisection technique employed in Algorithm 3.1 will converge linearly to a root in $\log(\mu_0 / \varepsilon)/\log(2)$ iterations, where μ_0 is the size of the initial interval and ε is the stopping tolerance (Press et al., 1992). A loose stopping criterion ($\varepsilon = 0.0001$) is utilised in the outer iteration in the following application so that numerical errors generated in the optimisation phase do not detrimentally affect convergence (Judd, 1998).

There are a number of ways to improve the efficiency of Algorithm 3.1. First, solution time is often significantly decreased through using an optimal trajectory from the previous iteration as an initial guess for the next. Solution time may be reduced by up to 80 percent. However, this strategy must be carefully implemented to prevent poor results from affecting convergence. Second, parallel processing may be used to solve each independent phase.

4. Application

This section describes the application of Algorithm 3.1 to a complex multiple-phase control problem.

Annual ryegrass (*Lolium rigidum*) is the most economically important weed constraining crop production in Western Australia (Pannell et al., 2004). Moreover, nearly half of the annual ryegrass populations in the primary grain-growing region of this state (the West Australian wheat belt) are estimated to be resistant to regular selective herbicides (Llewellyn and Powles, 2001). This reduces producer profit through forcing substitution towards less cost-effective substitutes, such as the mechanical collection of weed seeds at harvest. The adoption of grain legumes and the greater profitability of cereals, relative to livestock activities, in many farming systems in this region motivate continuous cropping (Pannell, 1995; Poole et al., 2002). However, the inclusion of regular pasture phases has the potential to delay or help to minimise the effects of herbicide resistance through permitting the use of a wide range of weed control strategies (Powles et al., 1997), such as grazing, the use of non-selective herbicides, or green-manuring. The economics of herbicide resistance and the utilisation of non-chemical treatments have been investigated previously (Gorddard et al., 1995, 1996; Pannell et al., 2004). Yet, the optimal management of multiple phases and pasture treatments has not been studied because significant methodological difficulties have been predicted (see, for example, Gorddard et al., 1995, p. 73). These may be overcome, however, through the use of Algorithm 3.1.

It is assumed that a producer wishes to determine the optimal management of a single field in the eastern wheat belt of Western Australia. The goal of the producer is to determine the optimal management of two phases in a steady-state field rotation. The initial phase involves lucerne (*Medicago sativa*) pasture and the second phase involves wheat (*Triticum aestivum*) cropping. Stationarity of the steady-state cycle is imposed through requiring equality between the initial ($x(t_0)$) and terminal ($x(t_2)$) state vectors. Algorithm 3.1 is not limited to the solution of this type of problem, however, and may easily be extended to deal with any feasible problem defined by Problem 2.1.

It is assumed that crop yield is detrimentally affected by the population of a single weed, annual ryegrass. There is one switching time (t_1) and the terminal time (t_2) is free. The latter determines the length of the second phase in the rotation. Two state variables are required to represent the weed seed population because of herbicide resistance (Gorddard et al., 1995, 1996). First, $x^s(t)$ is the population of annual ryegrass seeds that following germination is susceptible to the selective Group A herbicide (diclofop-methyl) (Preston, 2000).[2] Second, $x^h(t)$ is the population of seeds that following germination are resistant to this herbicide. Time notation is omitted where not required in the following discussion for notational parsimony.

4.1 Pasture phase dynamics

The producer's problem in the lucerne phase is,

$$\max_{u_i^{v1}} F_1 = \int_{t_0}^{t_1^-} e^{-rt}\left(au_1^1\left(1-\frac{u_1^1}{b}\right)-c_{np}\left(\frac{u_1^2}{1-u_1^2}\right)\right)dt\,, \tag{18}$$

subject to,

$$\dot{x}^s = x^s\left(v_1 + v_2\left(1-\frac{u_1^1}{u_1^1 d + l}\right)\left(1-u_1^2\right)R\right), \tag{19}$$

$$\dot{x}^h = x^h\left(v_1 + v_2\left(1-\frac{u_1^1}{u_1^1 d + l}\right)\left(1-u_1^2\right)R\right), \tag{20}$$

$$x_0 = \{x^s(t_0), x^h(t_0)\}\,, \tag{21}$$

$$x_1 = \{x^s(t_1), x^h(t_1)\}\,, \tag{22}$$

t_1 fixed, (23)

---

[2] Resistance to a single herbicide is studied to focus attention on the intertemporal management of herbicide resistance.

where v_j denotes the size of the control vector for phase j, r is the discount rate, u_1^1 is the sheep stocking rate, a and b are parameters describing the relationship between stocking rate and profit, c_{np} is the cost of achieving 50 percent weed control utilising alternative weed control treatments available during the pasture phase (u_1^2) (Gorddard et al., 1995), $v_1 = -g - (1-g)M_{seed}$ where g is the germination rate and M_{seed} is the natural mortality rate of ungerminated seeds, $v_2 = g(1 - M_{plant})$ where M_{plant} is the natural mortality rate of germinated seeds, d and l are parameters describing the strength of the relationship between grazing rate and weed control, and R is the number of seeds produced by a single weed. Equation (21) is the set of initial conditions and terminal conditions (22)-(23) will be determined by the estimated switch points in Algorithm 3.1.

4.2 Cereal phase dynamics

The producer's problem for the cereal phase is,

$$\max_{u_2^{v_2}} F_2 = \int_{t_{1^+}}^{t_2} e^{-rt}\left(py_0(1-\eta u_2^1)\left((1-z) + z\left(\frac{b}{s+gW(t)}\right)\right) - c_h u_2^1(t) - c_{nc}\left(\frac{u_2^2}{(1-u_2^2)}\right) - c_{cest}\right)dt - e^{-rt_2}c_{lest}, \tag{24}$$

subject to,

$$\dot{x}^s = x^s\left(v_1 + v_2 e^{-qu_2^1}(1-u_2^2)R\right), \tag{25}$$

$$\dot{x}^h = x^h\left(v_1 + v_2(1-u_2^2)R\right), \tag{26}$$

$$x_1 = \{x^s(t_1), x^h(t_1)\}, \tag{27}$$

$$t_1 \text{ fixed}, \tag{28}$$

$$x_2 = \{x_0^s(t_0), x_0^h(t_0)\}, \tag{29}$$

$$t_2 \text{ free}, \tag{30}$$

where p is a constant price, y_0 is weed-free yield, η is the proportion of yield lost to phytotoxic damage for a given dosage (measured in kilograms of active ingredient per hectare) of selective herbicide (u_2^1), z is the maximum proportion of grain yield lost at high weed density, s is a crop-dependent density parameter, g is a constant representing the competitiveness between the weed population and the wheat crop, *W(t)* represents the total weed population, c_h is the cost of the selective herbicide dose, c_{nc} is the cost of achieving 50 percent weed control utilising alternative weed control treatments available during a cropping phase (u_2^2) (Gorddard et al., 1995), c_{cest} is a fixed cost representing the establishment costs of wheat, c_{lest} is a fixed cost representing the establishment costs of lucerne, and q is a parameter designating the strength of the relationship between ryegrass mortality and selective herbicide dosage. The weed population is defined as $W(t) = W^s(t) + W^h(t)$, where W^s is the susceptible weed population and W^h is the herbicide resistant weed population. These are related to the susceptible and resistant seed populations through $W^s = x^s g(1 - M_{plant})e^{-qu_2^1}(1 - u_2^2)$ and $W^h = x^h g(1 - M_{plant})(1 - u_2^2)$.

The initial conditions (27)-(28) for the second phase will be determined by the estimated switch points in Algorithm 3.1. The terminal condition (29) is required given the cyclical nature of this problem discussed above. A terminal value function ($e^{-rt_2} c_{lest}$) is required in (24) to reflect establishment costs for the subsequent lucerne phase in the cycle.

The effective removal of lucerne requires careful grazing management and the application of non-selective herbicides (Bee and Laslett, 2002). A switching cost function for t_1 is therefore defined as $e^{-rt_1}c_{lrem}$, where c_{lrem} is the fixed cost of lucerne removal. This is obviously not a function of the state variables so condition (14) will hold as $\lambda_1^T(t_1) = \lambda_2^T(t_1)$ at $x(t_1)$ in this example. An interesting extension of this work would be the inclusion of a relationship between herbicide application when lucerne is removed and the density of annual ryegrass plants. This would require better information and the inclusion of plants, rather than seeds, as state variables. Moreover, this would require manipulation of Theorem 2.1 as a jump in the state variables would occur at the switching time. This extension may provide little additional insight, however, as two ryegrass plants or less are present at the switching time under optimal management of both scenarios in the following application.

The parameter values for this application and a brief description of their estimation is provided in Table 1 in Appendix 1. All values are expressed in 2004 Australian dollars. More detailed information on the estimation of parameters may be obtained from the author on request.

4.3 Model output

The first scenario represents an established resistance problem, with an initial susceptible seed ($x_0^s(t_0)$) population of 70 seeds m^{-2} and an initial herbicide resistant seed ($x_0^h(t_0)$) population of 35 seeds m^{-2}. The model solves after fifteen iterations. The optimal trajectories for both seed populations are shown in Figure 1. Here, the optimal switching time is denoted with a vertical line labeled t_1.

Insert Figure 1 near here

Figure 1 displays that both seed populations decline significantly over the duration of the lucerne pasture phase (phase one). This follows a combined use of grazing, at a constant rate of around 7.64 Dry Sheep Equivalents (DSE) per hectare, and alternative treatments that are utilised at around 70 percent intensity over this stage. This demonstrates the value of an integrated weed management strategy for reducing weed burdens before a subsequent cropping phase begins. The pasture phase is utilised for just over three years ($t_1 = 3.3$). This is less than the length of the cropping phase (phase two) that continues for four years in the cycle. This finding is intuitive because of the higher profitability of cereal cropping relative to grazing systems at low weed densities in this dryland environment.

The continuity of the state variable at the switching time (t_1) is observable in Figure 1. Discontinuity in the time derivatives of the state variables is also obvious given that the state trajectories experience a point of non-differentiability (corner) at t_1. This, of course, follows naturally from the piecewise definition of the constituent phases.

The second scenario involves an initial susceptible seed population of 70 seeds m^{-2} and no herbicide resistance. The optimal trajectories for the susceptible seed population for the "with resistance" and "without resistance" scenarios are shown in Figure 2. The switching times for these cases are labeled $t1$ and $t1'$ respectively. These trajectories demonstrate a number of important concepts. First, the length of the cropping phase increases significantly, from around four years to six years, when herbicide resistance is not present. This underlies a moderate increase in the optimal terminal time, from $t_2 = 7.326$ years to $t_2 = 8.33$ years. This extension in the length of the cereal phase follows an increase in its value, as wheat yield is not constrained by resistant weeds and the ineffectiveness of the efficient selective herbicide. Second, the lucerne phase finishes a year earlier ($t_1 > t_1'$) when there is no herbicide resistance. This follows a reduction in the relative profitability of the pasture phase. Last, the susceptible weed population at the switching time ($x^s(t_1)$) is significantly lower given an increase in the marginal value of in-pasture weed control. This greater level of control is achieved through a 2 percent increase in the mean stocking rate (from 7.6407 DSE/ha to 7.797 DSE/ha) and a 19.6

percent increase in the mean intensity of alternative treatments (from 69.86 percent to 83.54 percent).

Insert Figure 2 near here

Alternative weed control treatments are used at a significant intensity over the cereal phase when herbicide resistance constrains production (Figure 3a) because of the ineffectiveness of the selective herbicide against resistant weeds. This decreases producer profit given the increasing marginal cost associated with the utilisation of alternative treatments. The main cost of herbicide resistance consequently appears to arise from higher weed control costs and not significantly higher weed burdens (see Figure 2). This reinforces survey evidence (Llewellyn and Powles, 2001) and output from simulation modelling (Pannell et al., 2004) that identifies little difference in weed density between fields under standard management practice both with and without herbicide resistance. In contrast, the higher profitability of the cereal phase when herbicide resistance is not present arises from the steady application of both selective herbicide (Figure 3b) and alternative treatments (Figure 3c) at moderate intensities.

Insert Figure 3 near here

V. Conclusions

There appears to be no general framework for the numerical optimisation of multiple-phase systems in which control variables are defined in each stage. This is a significant limitation because such systems arise in many important situations, such as determining the optimal time to switch between production technologies, energy sources, and land uses. The computational algorithm presented in this paper offers a flexible and efficient platform for the solution of multiple-phase problems in which the number and sequence of phases is pre-assigned. However, this framework does not explicitly permit switching times to accumulate. Removing this limitation would increase its flexibility and is consequently a valuable area for further work.

Appendix 1

Table 1: Parameter values for the two-phase herbicide resistance model.

Parameter	Value	Source
r	r=.05	Pannell et al. (2004)
a, b	a=25.316, b=14.879	Non-linear least squares estimates from a simulated relationship between stocking rate and lucerne profitability, based on Mott (1960).
c_{np}, c_{nc}	$c_{np} = c_{nc} = \$5$	These are estimates of the cost of 50 percent control from unpublished estimates in the Resistance and Integrated Management simulation model (Pannell et al., 2004).
c_h	$c_h = \$40$	DAWA (2004)
g	$g = 0.8$	Gill (1996)
M_{seed}, M_{plant}	$M_{seed} = 0.55$, $M_{plant} = .05$	Unpublished data in RIM model (Pannell et al., 2004)
d, l	$d = 1.1111$, $l = 0.5$	The parameter d is determined from the maximum level of annual ryegrass control reported for grazing sheep (90 percent) (Pearce and Holmes, 1976; unpublished RIM estimates) using d=(1/ceiling). The parameter l is selected to fit the functional form to available data (Reeves and Smith, 1975; Pearce and Holmes, 1976; unpublished RIM estimates), with d fixed.
q	$q = 7.451$	Gorddard et al. (1995, 1996)
p	$p = \$185$	Estimate for wheat price after legume pasture, taken from RIM model (Pannell et al., 2004).
y_0	$y_0 = 1.82$ tonnes	Weed-free yield for continuous wheat crop (1.3 tonnes) (Pannell et al., 2004) is increased by 40 percent because of higher nitrogen, decreased disease, and improved soil structure after lucerne phase (Latta and Devenish, 2002).
η	$\eta = 0.1448$	Gorddard et al. (1995,1996)
z, s, k	$z = 0.6, s = 105, k = 0.33$	Pannell et al. (2004)
c_{cest}	$c_{cest} = \$82$	Pannell et al. (2004)
c_{lest}, c_{lrem}	$c_{lest} = \$58.50$, $c_{lrem} = \$21.25$	Calculated from information in DAWA (2004)

References

Amit, R. (1986), 'Petroleum reservoir exploitation: switching from primary to secondary recovery', *Operations Research* 34(July-August), pp. 534-49.

Arrow, K.J. and Kurz, M. (1970), *Public investment, the rate of return, and optimal fiscal policy*, John Hopkins University Press, Baltimore.

Ascher, U.M., Mattheij, R.M.M. and Russell, R.D. (1995), *Numerical solution of boundary value problems for ordinary differential equations*, SIAM, Philadelphia.

Bee, G.A. and Laslett, G. (2002), 'Development of a rainfed lucerne-based farming system in the Mediterranean climatic region of southwestern Australia', *Agricultural Water Management* 53, pp. 111-16.

Betts, J.T., Eldersveld, S.K. and Huffman, W.P. (1993), 'A performance comparison of non-linear programming algorithms for large sparse problems', in *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, Washington DC, AAIA, Paper 93-3751.

Betts, J.T. and Gablonsky, J.M. (2002), *A comparison of interior point and SQP methods on optimal control problems*, in *Mathematics and Computing Technology Technical Document Series*, Boeing, Seattle, Paper 02-0004.

Branicky, M., Borkar, V. and Mitter, S. (1998), 'A unified framework for hybrid control: model and optimal control theory', *IEEE Transactions on Automatic Control* 43(1), pp. 31-45.

Brekke, K.A. and Oksendal, B. (1994), 'Optimal switching in an economic activity under uncertainty', *SIAM Journal on Control and Optimisation* 32, pp. 1021-36.

Bulirsch, R. and Chudej, K. (1995), 'Combined optimisation of trajectory and stage separation of a hypersonic two-stage space vehicle', *Zeitschrift fur Flugwissenschaften und Weltraumforschung* 19, pp. 55-60.

DAWA (Department of Agriculture Western Australia) (2004), *2004 Farm Budget Guide*, Western Australian Primary Industry Press, Perth.

Dixit, A.K. and Pindyck, R.S. (1994), *Investment under uncertainty*, Princeton University Press, Princeton.

Dorfman, R. (1969), 'An economic interpretation of optimal control theory', *American Economic Review* 59, pp. 817-31.

Gill, P.E., Murray, W. and Wright, M.H. (1981), *Practical optimisation*, Academic Press, New York.

Gill, G.S. (1996), 'Ecology of annual ryegrass', *Plant Protection Quarterly* 11, pp. 195-98.

Gorddard, R.J., Pannell, D.J. and Hertzler, G. (1995), 'An optimal control model for integrated weed management under herbicide resistance', *Australian Journal of Agricultural Economics* 39, pp. 71-87.

Gorddard, R.J., Pannell, D.J. and Hertzler, G. (1996), 'Economic evaluation of strategies for management of herbicide resistance', *Agricultural Systems* 51, pp. 281-89.

Judd, K.L. (1998), *Numerical methods in economics*, MIT, Cambridge.

Kamien, M.I. and Schwartz, N.L. (1991), *Dynamic optimisation: the calculus of variations and optimal control in economics and management*, Bliss, C.J. and Intriligator, M.D. (eds), Advanced textbooks in economics, Elsevier, New York.

Keller, H.B. (1968), *Numerical methods for two-point boundary value problems*, Blaisdell, London.

Latta, R.A. and Devenish, K. (2002), 'Perennial pastures in annual cropping systems: lucerne and beyond', in *Proceedings of the 2002 Crop Updates*, Perth, Department of Agriculture Western Australia.

Lipton, D., Poterba, J., Sachs, J. and Summers, L. (1982), 'Multiple shooting in rational expectations models', *Econometrica* 50, pp. 1329-34.

Llewellyn, R.S. and Powles, S.B. (2001), 'High levels of herbicide resistance in rigid ryegrass (*Lolium rigidum*) in the wheatbelt of Western Australia', *Weed Technology* 15, pp. 242-48.

Makris, M. (2001), 'Necessary conditions for infinite-horizon discounted two-stage optimal control problems', *Journal of Economic Dynamics and Control* 25, pp. 1935-50.

Miranda, M.J. and Fackler, P.L. (2002), *Applied computational economics and finance*, MIT, Cambridge.

Mott, G.O. (1960), 'Grazing pressure and the measurement of pasture production', *Proceedings of the International Grasslands Congress* 8, pp. 606-11.

Mueller, U., Schilizzi, S. and Tran, T. (1999), 'The dynamics of phase farming in dryland salinity abatement', *Australian Journal of Agricultural and Resource Economics* 43(1), pp. 51-67.

Ortega, J.M. and Rheinboldt, W.C. (1970), *Iterative solution of non-linear equations in several variables*, Academic Press, New York.

Osborne, M.R. (1969), 'On shooting methods for boundary value problems', *Journal of Mathematical Analysis and Applications* 27, pp. 417-33.

Pannell, D.J. (1995), 'Economic aspects of legume management and legume research in dryland farming systems of southern Australia', *Agricultural Systems* 49, pp. 217-36.

Pannell, D.J., Stewart, V., Bennett, A., Monjardino, M., Schmidt, C. and Powles, S.B. (2004), 'RIM: a bioeconomic model for integrated weed management of *Lolium rigidum* in Western Australia', *Agricultural systems* 79, pp. 305-25.

Pearce, G.A. and Holmes, J.E. (1976), 'The control of annual ryegrass', *Journal of Agriculture Western Australia* 17(3).

Pesch, H.J. (1994), 'A practical guide to the solution of real-life optimal control problems', *Control and Cybernetics* 23, pp. 7-60.

Pontryagin, L., Boltyanskii, V., Gamkrelidze, R. and Mishchenko, E. (1962), *The mathematical theory of mathematical processes*, Wiley, New York.

Poole, M.L., Turner, N.C. and Young, J.M. (2002), 'Sustainable cropping systems for high rainfall areas of south-western Australia', *Agricultural Water Management* 53, pp. 201-12.

Powles, S.B., Preston, C., Bryan, I.B. and Jutsum, A.R. (1997), 'Herbicide resistance: impact and management', *Advances in Agronomy* 58, pp. 57-93.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. and Flannery, B.D. (1992), *Numerical recipes in C: the art of scientific computing*, 2nd edn, Cambridge University Press, Cambridge.

Preston, C. (2000), 'Herbicide mode of action and herbicide resistance', in Sindel, B.M. (Ed), Australian weed management systems, Richardson, Melbourne, pp. 209-26.

Reeves, T.G. and Smith, I.S. (1975), 'Pasture management and cultural methods for the control of annual ryegrass (*Lolium rigidum*) in wheat', *Australian Journal of Experimental Agriculture and Animal Husbandry* 15, pp. 527-30.

Rust, J. (1996), 'Numerical dynamic programming in economics', in Amman, H., Kendrick, D. and Rust, J. (Eds), *Handbook of Computational Economics*, North Holland, New York.

Seierstad, A. and Sydsaeter, K. (1987), *Optimal control theory with economic applications*, North Holland, Amsterdam.

Stoer, J. and Bulirsch, R. (2002), *Intoduction to numerical analysis*, Springer, New York.

Teo, K.L., Goh, C.J. and Wong, K.H. (1991), *A unified computational approach for optimal control problems*, Longman Scientific and Technical, New York.

Teo, K.L. and Jennings, L.S. (1991), 'Optimal control with a cost on changing control', *Journal of Optimisation Theory and Applications* 68(2), pp. 335-57.

Tomiyama, K. (1985), 'Two-stage optimal control problems and optimality conditions', *Journal of Economic Dynamics and Control* 9(3), pp. 317-87.

Tomiyama, K. and Rossana, R. (1989), 'Two-stage optimal control problems with an explicit switch point dependence', *Journal of Economic Dynamics and Control* 13(3), pp. 319-37.

Vind, K. (1967), 'Control systems with jumps in the state variables', *Econometrica* 35, pp. 273-77.

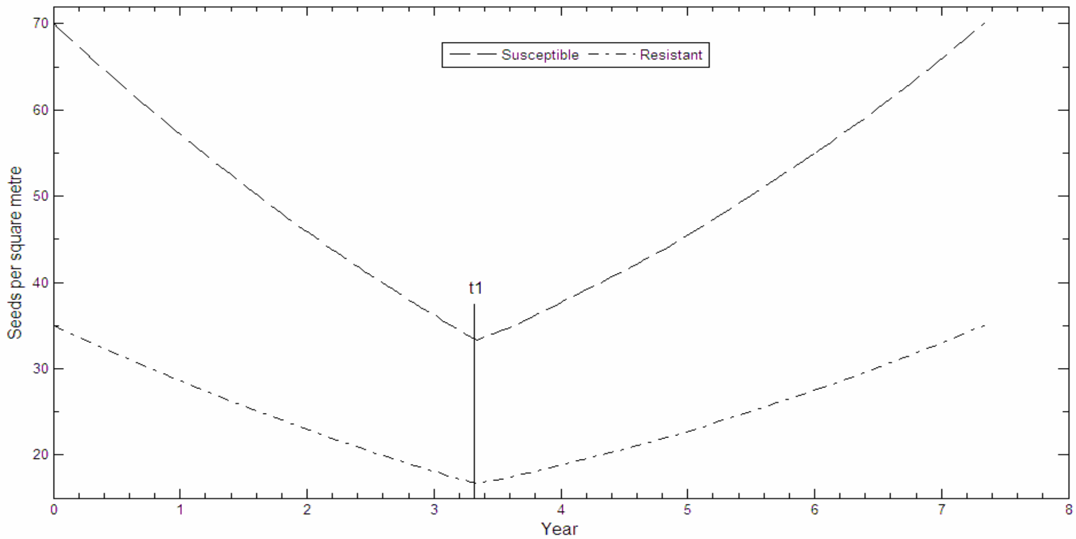

Figure 1. Optimal trajectories for susceptible and resistant annual ryegrass seeds over a lucerne pasture – wheat crop rotation.

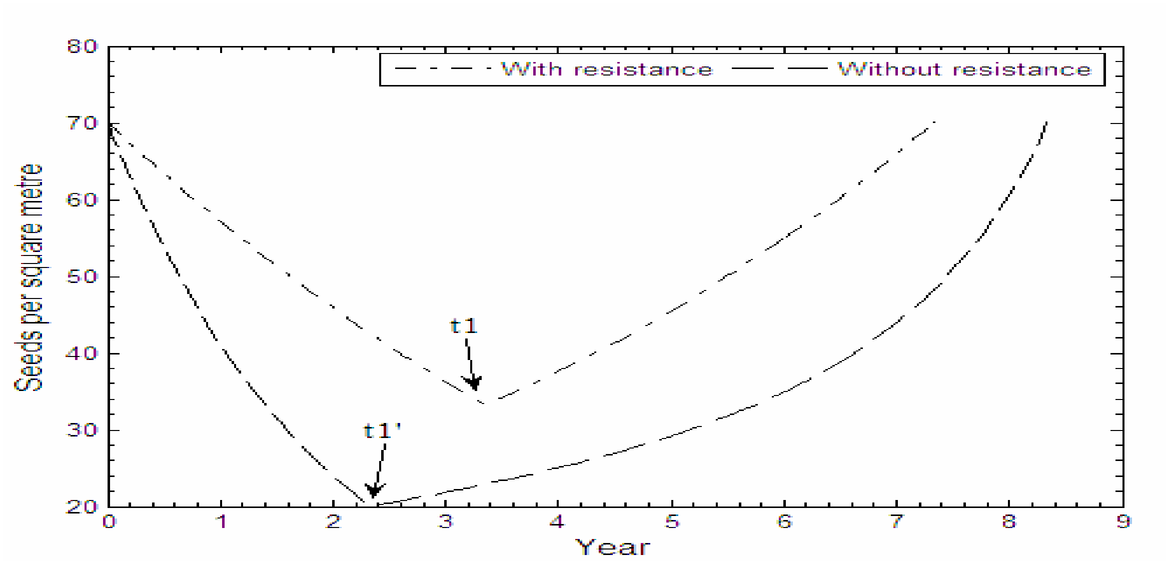

Figure 2. Optimal trajectories for susceptible annual ryegrass seeds with and without herbicide resistance over a lucerne pasture – wheat crop rotation.

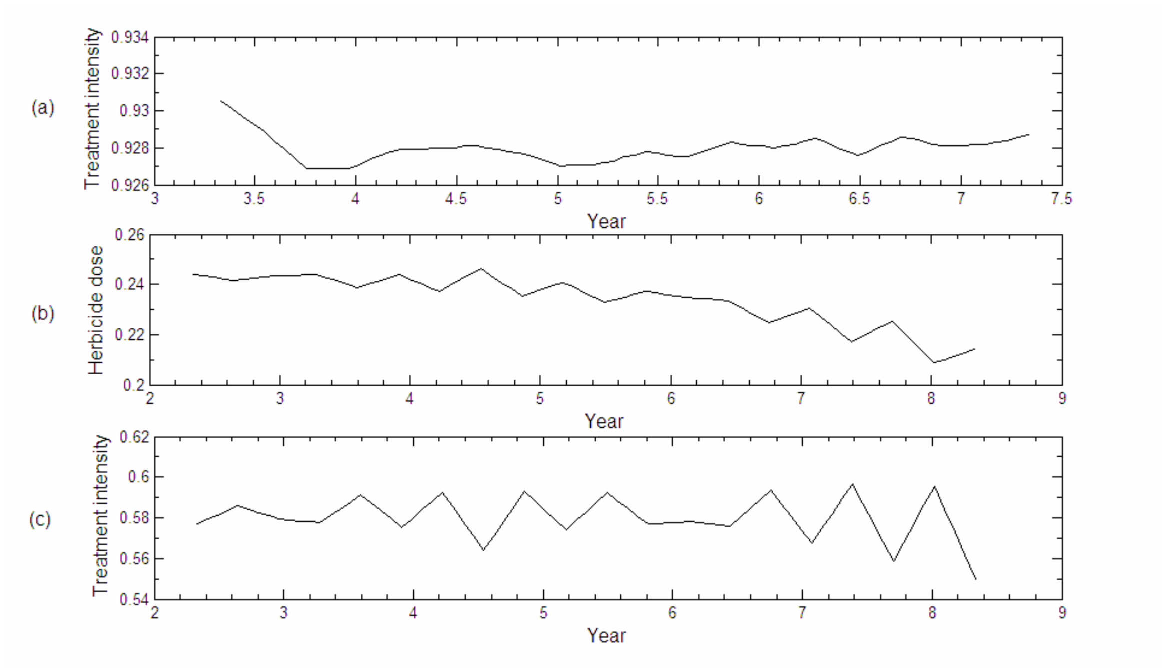

Figure 3. (a) Intensity of alternative treatments with herbicide resistance, (b) selective herbicide dose (kilograms of active ingredient applied per hectare) without herbicide resistance, and the (c) intensity of alternative treatments without herbicide resistance. All of these treatments are applied in the second (crop) phase.