The Deterministic Equivalents of Chance-Constrained Programming

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Abstract *Three concepts combine to show both the feasibility and desirability of incorporating probability within programming models First, the reliability of estimates obtained by using Chebyshev's inequality increases as variation measured by the coefficient of variation, declines Second, the coefficient of variation can be substantially reduced by the use of the mean and variance of a truncated normal distribution Third, chance-constrained programming can be converted into deterministic equivalent quadratic programming by using the parameters of a truncated normal distribution*

Keywords *Chance-constrained programming, quadratic risk programming, truncated normal distribution, Chebyshev's inequality*

The developments of the past three decades in the theory of choice under risk or uncertainty have followed the expected mean-variance approach The decisionmaker is assumed to select among alternative activities on the basis of a utility function defined in terms of the expected mean and variance of the portfolio return Accordingly, much has been written about quadratic risk programming (*5, 7, 13, 16*) [1] Researchers have not paid enough attention, however, to the burdensome data requirements and mathematical complexity associated with the risk and uncertainty pertaining to input-output coefficients(*9*)

An activity analysis model usually optimizes some objective function subject to linear constraints Coefficients for both the objective function and the constraints are assumed to be known with certainty Chance-constraint programming (CCP), originally proposed by Charnes and Cooper (*4*), makes use of individual probabilistic constraints A probability is attached to the linear constraint in such models The probabilistic constraint is subject to some predetermined critical level, and its coefficients are assumed to be randomly distributed Probabilistic constraints of this type often appear in decision analysis in the form of a safety-first rule in portfolio selection problems (*6,11,12,14,15*), or in problems associated with feed-grain mixture (*5*)

Many researchers, therefore, have attempted to convert probabilistic constraints into deterministic equivalents under various assumptions Examples include the works of Charnes (*3*), Pyle and Turnovsky (*11*), Roy (*12*), Telser (*15*), Paris and Easter (*9*), Sengupta (*14*), and Atwood (*1*) Paris and Easter, and Pyle and Turnovsky obtain their criterion based on the normality assumption with respect to the random variable Roy and Telser, in contrast, obtain deterministic equivalents of a probabilistic constraint by using Chebyshev's inequality, which does not require any knowledge about the probabilistic density function of a random variable Sengupta claims that estimates obtained from the use of Chebyshev's inequality may "sometimes" be very inefficient, in the sense that they may provide very rough approximations for the actual probability when the distribution of the random variable is known However, Sengupta failed to indicate when the use of Chebyshev's inequality provides inefficient estimates of the actual probability

Recently, Atwood and others (*1,2*) rejected the use of Chebyshev's inequality on the grounds that the estimates are too conservative They proposed the use of lower partial moments (LPM) to obtain a deterministic equivalent of a probabilistic condition The LPM approach uses the concept of a truncated distribution, which Sengupta suggested to improve reliability of estimates However, the LPM approach makes use of the parameters of a complete normal distribution Unknown is how much the reliability of estimates is improved by use of a truncated distribution using parameters of a complete normal distribution over the use of the Chebyshev inequality [2] In this paper, we first demonstrate that the reliability of estimates obtained by making use of Chebyshev's inequality increases as the coefficient of variation decreases Second, we demonstrate that, under certain conditions, use of a truncated normal distribution does not necessarily improve the reliability of estimates, and that estimates obtained from the use of Chebyshev's in-

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[1]Italicized numbers in parentheses cite sources listed in the References section at the end of this article

[2]The reliability of an estimate is defined as the probability of the absolute value of the difference between the estimate and its true value as being less than a predetermined small number

equality are generally reliable if the analysis includes the use of the mean and variance of a truncated normal distribution Third, we show how one can convert the CCP problem into a deterministic equivalent quadratic programming (DEQP) problem using Chebyshev's inequality Finally, we discuss the properties of DEQP solutions

A Complete Normal Distribution and Chebyshev's Inequality

The normality assumption has been widely used in economic literature, and its use is justified for many cases due to the Central Limit Theorem However, without any knowledge about the probability density function of a random variable x, we can consider Chebyshev's inequality for any $h > 0$, such that

$$\Pr[|x-\mu| \leq h\sigma] \geq 1 - \frac{1}{h^2} , \tag{1}$$

where μ and σ are the mean and standard deviation of the complete normal distribution

For any symmetric distribution, the quantity $Q^*(h)$, which is the probability assigned to the interval where the random variable x is defined as $\{x \ (\mu - h\sigma) < x \leq \infty\}$, may be represented as

$$Q^*(h) \geq 0\ 5 + 0\ 5\left[1 - \frac{1}{h^2}\right]$$

$$\geq 1 - 1/(2h^2) \tag{2}$$

It is clear from equations 1 and 2 that the probability of an observed value for a random variable x in the interval $(x \ (\mu - h\sigma) < x \leq \infty)$ approaches one as h increases infinitely, and that the probability increases faster for symmetric distributions than for nonsymmetric distributions

Table 1 shows probabilities estimated with equation 2 for h = 1, 2, , 5 By making use of Chebyshev's inequality, we estimated that probabilities are relatively reliable estimates for a random variable with a small coefficient of variation (σ/μ), and that the reliability increases as h increases [3] Table 1 indicates that for $h > 4$, the probabilistic constraint of a chance-constrained programming problem can be converted into a deterministic equivalent by making use of Chebyshev's inequality

[3]Parzen (*10*) showed for a two sided confidence level, $\Pr(|X-h\mu|) \geq \beta$, for a complete normal distribution, that estimates obtained using Chebyshev's inequality converge to actual probability as h increases

Table 1—Estimated probabilities $\Pr(x > \mu\text{-}h\sigma)$ for a complete normal distribution and the use of Chebyshev's inequality

h	$\phi(h)^1$	$1 - \frac{1}{2h^2}$	Reliability of estimates with Chebyshev inequality
			Percent
1	0 8413	0 5000	59 43
2	9772	8750	89 54
3	9987	9444	94 56
4	1 0000	9688	96 88
5	1 0000	9800	98 00

[1]ϕ is the CDF of N (0 1)

A Truncated Normal Distribution and Chebyshev's Inequality

There exist cases in which economic variables are not defined over the entire range of a normal distribution For instance, water applied in the production of irrigated cotton or the final mix of feed grains in livestock production cannot be negative, indicating that the random variable, say x, would have a truncated normal distribution to the left at x = 0

When we assume that x has a truncated normal distribution to the left at τ, the probability density function of x is then defined as follows (*8*)

$$f(x) = 0 \qquad \text{if } x < \tau$$

$$= \frac{K}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \qquad \text{if } x \geq \tau, \tag{3}$$

where

$$K = \frac{1}{1 - \phi(\frac{\tau-\mu}{\sigma})} ,$$

ϕ is the cumulative density function (CDF) of the normal distribution with mean zero and unity variance, N(0,1), and μ and σ are the mean and standard deviation of a complete normal distribution of x [4]

To compare estimates from a truncated normal distribution and estimates obtained by using Chebyshev's inequality, we define the quantity Q(h), for any $h > 0$,

[4]Meyer (*8*) erroneously defined K as the quantity $1 - \phi(\frac{\tau-\mu}{\sigma})$, rather than the quantity $K = 1/[1 - \phi(\frac{\tau-\mu}{\sigma})]$

as the probability assigned to the interval where the random variable x is $\{x \ (\mu - h\sigma) < x \leq \infty\}$ For the truncated normal probability density function (with mean μ and standard deviation σ of the complete normal distribution), the quantity Q(h) is given by

$$Q(h) = \frac{K}{\sigma\sqrt{2\pi}} \int_{(\mu-h\sigma)}^{+\infty} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \qquad (4)$$

Equation 4 can be compactly rewritten as

$$Q(h) = \frac{1 - \phi(-h)}{1 - \phi(\frac{\tau-\mu}{\sigma})} = \frac{\phi(h)}{\phi(\frac{\mu-\tau}{\sigma})} \qquad (5)$$

It is easy to see that Q(h) in equation 5 becomes 1 for $\tau = (\mu - h\sigma)$ In cases where $(\mu-\tau)/\sigma \geq 3\ 9$ so that $\phi[(\mu-\tau)/\sigma] = 1$, equation 5 becomes $Q(h) = \phi(h)$, which is the probability estimated from a complete normal distribution That is, use of a truncated normal distribution does not improve the reliability of estimates if $(\mu-\tau)/\sigma$ is greater than or equal to 3 9

We will show (equation 22 and footnote 5) that $(\mu-\tau)/\sigma \geq (1-\beta)^{-1/2}$, where β is the required minimum probability of success For $\beta = 0\ 95$, a traditional criterion in decision analysis, $(1-\beta)^{-1/2}$ equals 4 4721 which is greater than 3 9 Use of a truncated normal distribution, therefore, does not necessarily improve reliability of estimates

So far, the mean and standard deviation of a complete normal distribution are used for the measurement of the coefficient of variation in equation 5 However, the expected value E(x) and variance V(x) of a random variable x, which has a truncated normal distribution as given in equation 3, are expressed as follows (see appendix)

$$E(x) = \mu + D, \qquad (6)$$

and

$$V(x) = \sigma^2 + D[(\tau - \mu) - D], \qquad (7)$$

$$\text{where } D = \frac{\sigma}{\sqrt{2\pi}\left[1 - \phi\left(\frac{\tau-\mu}{\sigma}\right)\right]} \exp\left[-\tfrac{1}{2}\left(\frac{\tau-\mu}{\sigma}\right)^2\right]$$

In equations 6 and 7, $D > 0$, and therefore, $E(X) > \mu$, $V(X) < \sigma^2$, and $\sigma/\mu > [V(X)]^{1/2}/E(X)$ That is, the coefficient of variation is reduced when the mean and standardized deviation of a truncated normal distribution are substituted for those of a complete normal distribution, respectively However, we have shown that the reliability of estimates obtained by making use of Chebyshev's inequality increases as the coefficient of variation is reduced Therefore, parameter estimates of a chance-constrained problem can be improved by making use of the mean and variance of the truncated normal distribution

Deterministic Equivalent Quadratic Programming

A probabilistic constraint can be converted into a deterministic equivalent using Chebyshev's inequality Consider a chance-constrained programming problem such as the one used by Chen

$$\text{Minimize} \quad C'X, \qquad (8)$$

$$\text{subject to} \quad \Pr(P'X \geq d) \geq \beta \qquad (9)$$

$$AX \geq b \qquad (10)$$

$$X \geq 0, \qquad (11)$$

where C is an (n x 1) vector of cost coefficients, X is an (n x 1) vector of choice variables, P is an (n x 1) vector of stochastic variables, which has a truncated distribution to the left at zero, with a truncated normal mean vector $\overline{P}$ and variance-covariance matrix W, d is a prespecified constant, β is the required minimum probability of success, A is an (m x n) technical coefficient matrix, and b is an (m x 1) vector of minimum resource requirements

To formulate a deterministic equivalent constraint, the probabilistic constraint 9 can be rewritten as

$$\begin{aligned} \Pr(P'X \geq d) &= 1 - \Pr(P'X \leq d) \\ &= 1 - \Pr[(\overline{P}'X - P'X) \geq (\overline{P}'X - d)] \\ &\geq 1 - \Pr[(|P'X - \overline{P}'X|) \geq (\overline{P}'X - d)] \\ &\geq 1 - \frac{X'WX}{(\overline{P}'X - d)^2} \ \text{Chebyshev's inequality,} \\ &\geq \beta \qquad (12) \end{aligned}$$

or equivalently,

$$\Pr(P'X \geq d) \geq 1 - \frac{X'WX}{(\overline{P}'X - d)^2} \geq \beta \qquad (13)$$

The coefficient of variation clearly must be smaller for a large required probability of success

Minimum costs that satisfy the probabilistic constraint 9 are attained at $\Pr(P'X \geq d) = \beta$, or more stringently at

$$1 - \frac{X'WX}{(\overline{P}'X - d)^2} = \beta,$$

in equation 13 Consequently, the problem of finding minimum costs, $C'X = K$, subject to the probabilistic constraint 9, is equivalent to the problem of finding the minimum of

$$(1 - \beta) - \frac{X'WX}{(\overline{P}'X - d)^2} \geq 0, \tag{14}$$

at any given cost K This can be verified by comparing the Kuhn-Tucker conditions from each optimization problem The minimization of equation 14 is equivalent to the maximization of

$$\overline{P}'X + \frac{1}{2d}\,(X'MX) \leq \frac{d}{2}, \tag{15}$$

where $M = [W(1-\beta)^{-1} - \overline{P}\,\overline{P}']$ is a negative semidefinite matrix

When the equality holds in equation 15 for any level of required success, β, the probability constraint 9 is met at minimum cost, say K_β In cases where the inequality holds, the probability constraint 9 is satisfied, but at a higher cost than K_β Consequently, the following deterministic equivalent QP (DEQP) problem can be formulated, which is equivalent to the CCP problem in equations 8 through 11

$$\text{Maximize} \quad \overline{P}'X + \frac{1}{2}(X'MX) \leq \frac{d}{2} \tag{16}$$

$$\text{subject to} \quad C'X = K \qquad 0 < K \leq K_\beta \tag{17}$$

$$AX \geq b \tag{18}$$

$$X \geq 0 \tag{19}$$

The optimal choice variables X satisfying the CCP problem in equations 8 through 11 may be approximated by solving the DEQP problem in equations 16 through 19 by simply increasing cost (K) parametrically until the objective value approaches d/2

Properties of DEQP Solutions

To derive the properties of DEQP solutions, we rewrote the objective function in equation 16 as

$$\begin{aligned} Z &= \overline{P}'X + (1/2d)X'[(1-\beta)^{-1}W - \overline{P}\,\overline{P}']X \\ &= \overline{P}'X + \frac{(1-\beta)^{-1}}{2d}\;X'WX - (1/2d)\,X'\overline{P}\,\overline{P}'X \\ &\leq \frac{d}{2} \end{aligned} \tag{20}$$

Multiplying both sides of the inequality in equation 20 by 2d, we can obtain the following

$$2d\,\overline{P}'X + (1-\beta)^{-1}X'WX - X'\overline{P}\,\overline{P}'X \leq d^2 \tag{21}$$

From equation 21, we can obtain the following inequality

$$\frac{\overline{P}'X - d}{(X'WX)^{1/2}} \geq (1-\beta)^{-1/2}, \tag{22}$$

or equivalently

$$\overline{P}'X - (1-\beta)^{-1/2}(X'WX)^{1/2} \geq d \tag{22'}$$

When the condition given in equation 22′ is met, it can be said that one is β-percent confident, and that P′X will be greater than or equal to a predetermined constant, d [5]

To further investigate the properties of DEQP solutions, consider the Lagrangian equation associated with the DEQP problem in equations 16 through 19 such that

$$\begin{aligned} F(X, \lambda_1, \lambda_2) = {} & \overline{P}'X + (1/2d)X'\,[(1-\beta)^{-1}W - \overline{P}\,\overline{P}']X \\ & - \lambda_1(K - C'X) - \lambda'_2(b - AX), \end{aligned} \tag{23}$$

where λ_1 is a Lagrangian multiplier, such that $\lambda_1 \gtreqless 0$; and λ_2 is an (m x 1) vector of positive Lagrangian multipliers

Part of the Kuhn-Tucker conditions are expressed as

$$\begin{aligned} \frac{\partial F}{\partial X} = \overline{P} + {} & \frac{(1-\beta)^{-1}}{d}\,WX - (1/d)\overline{P}\,\overline{P}'X + \\ & \lambda_1 C + \lambda'_2 A \leq 0 \end{aligned} \tag{24}$$

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[5]The condition in equation 22′ is more stringent than the one imposed by negative semidefiniteness Because $M = [(1-\beta)^{-1}W - \overline{P}\,\overline{P}']$ is a negative semidefinite matrix $X'MX = (1-\beta)^{-1}X'WX - X'\overline{P}\,\overline{P}'X \leq 0$ and therefore

$$\frac{X'\overline{P}\,\overline{P}'X}{X'WX} \geq (1-\beta)^{1},$$

or equivalently $\overline{P}'X - (1-\beta)^{1/2}(X'WX)^{1/2} \geq 0$ which is less stringent than the condition expressed in the inequality 22′

$$\left(\frac{\partial F}{\partial X}\right)X = \overline{P}'X + \frac{(1-\beta)^{-1}}{d} X'WX - (1/d)X'\overline{P}\,\overline{P}'X + \lambda_1 C'X + \lambda'_2 AX = 0 \quad (25)$$

$$X \geq 0 \quad (26)$$

There exists a dual programming problem associated with the primal programming problem given in equations 16 through 19 The objective function to be minimized in a dual programming problem is obtained by simply subtracting equation 25 from the Lagrangian equation 23, represented by.

$$G(X, \lambda_1, \lambda_2) = -\lambda_1 K - \lambda'_2 b + (1/2d)X'\overline{P}\,\overline{P}'X - \frac{(1-\beta)^{-1}}{2d} X'WX \quad (27)$$

The objective value of the primal programming problem, equation 16, equals the objective value of the dual programming problem, equation 27, at optimum Equating equations 16 and 27 results in the following

$$\overline{P}'X + \frac{(1-\beta)^{-1}}{d} X'WX - (1/d)X'\overline{P}\,\overline{P}'X = -\lambda_1 K - \lambda'_2 b \quad (28)$$

By multiplying both sides of the equality in equation 28 by d, and with minor manipulation, we obtain

$$d[\overline{P}'X - \lambda_1 K - \lambda'_2 b] = 2d\overline{P}'X + (1-\beta)^{-1}X'WX - X'\overline{P}\,\overline{P}'X \quad (29)$$

$$\leq d^2 \text{ (from equation 21)},$$

or equivalently,

$$\overline{P}'X \leq d + \lambda_1 K + \lambda'_2 b \quad (30)$$

Equation 30 shows that the mean value of the random variable P′X must be less than or equal to the predetermined value d plus the sum of the opportunity costs of expenditures and the opportunity costs of other resources

Combining conditions in equations 22′ and 30 reveals that

$$d + (1-\beta)^{-1/2}(X'WX)^{1/2} \leq \overline{P}'X \leq d + \lambda_1 K + \lambda'_2 b, \quad (31)$$

or equivalently,

$$\overline{P}'X - (1-\beta)^{-1/2}(X'WX)^{1/2} \geq d \geq \overline{P}'X - \lambda_1 K - \lambda'_2 b \quad (32)$$

Both sides of the first inequality sign in equation 32 are identical with the condition given in equation 22′ However, the condition imposed by equation 32 is more restrictive than the one given in equation 22′ by requiring an additional condition given in equation 30

Conclusions

Chebyshev's inequality often has been used to convert a probabilistic constraint in a chance-constrained programming problem into a deterministic constraint Researchers, however, have criticized the use of Chebyshev's inequality which sometimes provides very rough approximations for the actual probability We have shown that the use of Chebyshev's inequality provides relatively very good approximations for the actual probability when the coefficient of variation is relatively very small We also have shown that the use of the mean and variance of a truncated normal distribution reduces the size of the coefficient of variation, compared with the case of using the mean and variation of a complete normal distribution We have also demonstrated how a chance-constrained programming problem can be converted into a deterministic equivalent quadratic programming problem

Appendix 1— Mean and Variance of a Truncated Normal Distribution

$$E(x) = \int_{T}^{+\infty} xf(x)dx = \int_{T}^{+\infty} \frac{X\left[\frac{1}{1-\phi(\frac{T-\mu}{\sigma})}\right]}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{X-\mu}{\sigma})^2}dx.$$

With $S = (\frac{X-\mu}{\sigma})$, and therefore $\sigma dS = dx$, then

$$E(x) = \frac{1}{1-\phi(\frac{T-\mu}{\sigma})} \int_{\frac{T-\mu}{\sigma}}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}}(\mu + \sigma S)e^{S^2/2}\sigma dS$$

$$= \frac{1}{1-\phi(\frac{T-\mu}{\sigma})}\left[\mu\int_{\frac{T-\mu}{\sigma}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{S^2/2}dS + \sigma\int_{\frac{T-\mu}{\sigma}}^{+\infty} \frac{S}{\sqrt{2\pi}} e^{S^2/2}dS\right]$$

$$= \frac{1}{1-\phi(\frac{T-\mu}{\sigma})}\left[\mu\left[1-\phi(\frac{T-\mu}{\sigma})\right] + \frac{\sigma}{\sqrt{2\pi}}\left(e^{S^2/2}\Big|_{\frac{T-\mu}{\sigma}}^{+\infty}\right)\right]$$

$$= \mu + \left[\frac{1}{1-\phi(\frac{T-\mu}{\sigma})}\right]\frac{\sigma}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}(\frac{T-\mu}{\sigma})^2}\right)$$

$$E(x^2) = \int_{T}^{+\infty} x^2f(x)dx = \int_{T}^{+\infty} \frac{X^2\left[\frac{1}{1-\phi(\frac{T-\mu}{\sigma})}\right]}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{X-\mu}{\sigma})^2}dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1 - \phi(\frac{T-\mu}{\sigma})} \right) \int_{\frac{T-\mu}{\sigma}}^{+\infty} (\mu + \sigma S)^2 e^{(\frac{1}{2})S^2} dS$$

$$= \frac{1}{1 - \phi(\frac{T-\mu}{\sigma})} \left[\int_{\frac{T-\mu}{\sigma}}^{+\infty} \frac{\mu 2}{\sqrt{2\pi}} e^{-(\frac{1}{2})S^2} dS + 2\mu\sigma \int_{\frac{T-\mu}{\sigma}}^{+\infty} \frac{S}{\sqrt{2\pi}} e^{-(\frac{1}{2})S^2} dS \right.$$

$$\left. + \sigma^2 \int_{\frac{T-\mu}{\sigma}}^{+\infty} \frac{S^2}{\sqrt{2\pi}} e^{-(\frac{1}{2})S^2} dS \right]$$

$$= \frac{1}{1 - \phi(\frac{T-\mu}{\sigma})} \left[\mu^2 [1 - \phi(\frac{T-\mu}{\sigma})] \right] + \frac{1}{1 - \phi(\frac{T-\mu}{\sigma})} \bullet \frac{2\mu\sigma}{\sqrt{2\pi}} \left(-e^{-S^2/2} dS \Big|_{\frac{T-\mu}{\sigma}}^{+\infty} \right)$$

$$+ \frac{1}{1 - \phi(\frac{T-\mu}{\sigma})} \left[\frac{\sigma^2}{\sqrt{2\pi}} \int_{\frac{T-\mu}{\sigma}}^{+\infty} S^2 e^{(\frac{1}{2})S^2} dS \right]$$

$$= \mu^2 + \frac{1}{1 - \phi(\frac{T-\mu}{\sigma})} \bullet \frac{2\mu\sigma}{\sqrt{2\pi}} \left(e^{-\frac{1}{2}(\frac{T-\mu}{\sigma})^2} \right) + \frac{1}{1 - \phi(\frac{T-\mu}{\sigma})} \left[\frac{\sigma^2}{\sqrt{2\pi}} \int_{\frac{T-\mu}{\sigma}}^{+\infty} S^2 e^{-(\frac{1}{2})S^2} dS \right]$$

Let $v = -e^{-S^2/2}$ $\quad dv = Se^{-S^2/2}dS$

$u = S$ $\quad du = dS$

Then the last term of the above equation can be rewritten as follows

$$\frac{1}{1 - \phi(\frac{T-\mu}{\sigma})} \left[\frac{\sigma^2}{2\pi} \int_{\frac{T-\mu}{\sigma}}^{+\infty} S^2 e^{-(\frac{1}{2})S^2} dS \right] = \frac{1}{1 - \phi(\frac{T-\mu}{\sigma})} \left[\frac{\sigma^2}{\sqrt{2\pi}} \left(-Se^{-S^2/2} \Big|_{\frac{T-\mu}{\sigma}}^{+\infty} \right) \frac{\sigma}{\sqrt{2\pi}} \int_{\frac{T-\mu}{\sigma}}^{+\infty} e^{-(\frac{1}{2})S^2} dS \right]$$

$$= \frac{1}{1-\phi(\frac{T-\mu}{\sigma})} \bullet \frac{\sigma^2}{\sqrt{2\pi}} \bullet (\frac{T-\mu}{\sigma}) \bullet \left(e^{-\frac{1}{2}(\frac{T-\mu}{\sigma})^2}\right)$$

$$+ \frac{1}{1-\phi(\frac{T-\mu}{\sigma})} \left[\sigma^2 \int_{\frac{T-\mu}{\sigma}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-S^2/2} ds\right]$$

$$= \frac{1}{1-\phi(\frac{T-\mu}{\sigma})} \bullet \frac{\sigma^2}{\sqrt{2\pi}} \bullet (\frac{T-\mu}{\sigma}) \bullet \left(e^{-\frac{1}{2}(\frac{T-\mu}{\sigma})^2}\right) + \frac{1}{1-\phi(\frac{T-\mu}{\sigma})} \bullet \sigma^2 \left[1-\phi(\frac{T-\mu}{\sigma})\right]$$

Therefore

$$E(x^2) = \mu^2 + \sigma^2 + \frac{1}{1-\phi(\frac{T-\mu}{\sigma})} \bullet \frac{2\mu\sigma}{\sqrt{2\pi}} \left(e^{-\frac{1}{2}(\frac{T-\mu}{\sigma})^2}\right) + \frac{1}{1-\phi(\frac{T-\mu}{\sigma})} \bullet \frac{\sigma^2}{\sqrt{2\pi}} \bullet (\frac{T-\mu}{\sigma}) \bullet \left(e^{-\frac{1}{2}(\frac{T-\mu}{\sigma})^2}\right)$$

Consequently, we have the following

$V(x) = E(x^2) - [E(x)]^2$

$$= \mu^2 + \sigma^2 + \frac{1}{1-\phi(\frac{T-\mu}{\sigma})} \bullet \left(e^{-\frac{1}{2}(\frac{T-\mu}{\sigma})^2}\right) \bullet \left[\frac{2\mu\sigma}{\sqrt{2\pi}} + \frac{\sigma^2}{\sqrt{2\pi}} (\frac{T-\mu}{\sigma})\right]$$

$$- \left[\mu + \left[\frac{1}{1-\phi(\frac{T-\mu}{\sigma})}\right] \frac{\sigma}{\sqrt{2\pi}} \left(e^{-\frac{1}{2}(\frac{T-\mu}{\sigma})^2}\right)\right]^2$$

$$= \mu^2 + \sigma^2 + \frac{1}{1-\phi(\frac{T-\mu}{\sigma})} \bullet \left(e^{-\frac{1}{2}(\frac{T-\mu}{\sigma})^2}\right) \bullet \left[\frac{(\mu+T)\sigma}{\sqrt{2\pi}}\right]$$

$$- \mu^2 - \left[\frac{1}{1-\phi(\frac{T-\mu}{\sigma})}\right] \frac{2\mu\sigma}{\sqrt{2\pi}} \left(e^{-\frac{1}{2}(\frac{T-\mu}{\sigma})^2}\right) - \left[\frac{1}{1-\phi(\frac{T-\mu}{\sigma})} \bullet \frac{\sigma}{\sqrt{2\pi}} \left(e^{-\frac{1}{2}(\frac{T-\mu}{\sigma})^2}\right)\right]^2$$

Therefore

$$V(x) = \sigma^2 + \frac{(\tau - \mu)}{1 - \phi(\frac{\tau-\mu}{\sigma})} \bullet \left[\frac{\sigma}{\sqrt{2\pi}}\right] \bullet \left(e^{-\frac{1}{2}(\frac{\tau-\mu}{\sigma})^2}\right) - \left[\frac{1}{1 - \phi(\frac{\tau-\mu}{\sigma})} \bullet \frac{\sigma}{\sqrt{2\pi}} \bullet \left(e^{-\frac{1}{2}(\frac{\tau-\mu}{\sigma})^2}\right)\right]^2.$$

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