Beyond Expected Utility: Risk Concepts for Agriculture from a Contemporary Mathematical Perspective

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Abstract. *Expected utility theory, the most prominent economic model of how individuals choose among alternative risks, exhibits serious deficiencies in describing empirically observed behavior Consequently, economists are actively searching for a new paradigm to describe behavior under risk Their mathematical tools, such as functional analysis and measure theory, reflect a new, more sophisticated approach to risk This article describes the new approach, explains several of the mathematical concepts used, and indicates some of their connections to agricultural modeling*

Keywords. *Individual choice under risk, expected utility theory, risk preference ordering, utility function on a lottery space, Fréchet differentiability, random function, random field*

In their attempts to model individual behavior under risk, agricultural economists have relied heavily on the expected utility hypothesis This hypothesis stipulates that individuals presented with a choice among various risky options will choose one that maximizes the mathematical expectation of their personal "utility" An accumulation of evidence reported in the literatures of both economics and psychology, however, has by now clearly demonstrated that expected utility theory exhibits serious deficiencies in describing empirically observed behavior (For reviews, see Schoemaker, 1982, Machina, 1983, 1987, Fishburn, 1988)[1] As a result, economists and psychologists have been formulating and testing new theories to describe behavior under risk These theories do not so much deny classical expected utility theory as generalize it By imposing weaker restrictions on the functional forms used in risk models, they allow empirical behavior more scope in telling its own story

To a significant extent, this search for a new paradigm of behavior under risk is being conceived and conducted in the spirit and language of contemporary mathematics The concepts being employed, such as the derivative of a functional with respect to a probability distribution or vector spaces whose "points" are functions, cannot be reduced to the graphical analysis traditionally favored by applied economists Rather, they involve a genuinely new approach, a way of thinking that is at the same time more precise and more abstract

This article is intended to provide agricultural and applied economists with an introduction to these newer ways of thinking about behavior under risk Designed to be largely self-contained, the article first sketches some prerequisites from set theory and measure theory, then defines and discusses several key risk concepts from a modern perspective

On the surface, the mathematical ideas we describe may appear distant from direct practical application Yet, they already play an important role in various theories on which practical applications have been or can be based Some examples

- **Commodity futures and options.** A revolution in the theory of finance, begun in the 1970's and continuing today, has been brought about by the adoption of advanced mathematical tools, such as continuous stochastic processes, the Black-Scholes option pricing formula, and stochastic integrals (used to represent the gains from trade) The insights afforded by these methods have had a substantial practical impact on securities trading Understanding commodity futures and options trading in this new environment requires greater familiarity with the new mathematical machinery This machinery, in turn, draws heavily on measure theory, which is now a prerequisite for advanced finance theory (Dothan, 1990, Duffie, 1988)

- **Commodity price stabilization.** In recent years, economists increasingly have drawn on the techniques of stochastic dynamics to analyze the behavior of economic processes over time Applications of stochastic dynamics range from the optimal management of renewable resources, such as timber, to optimal firm investment strategies For agricultural

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[1]Sources are listed in the References section at the end of this article

economists, a particularly important application is the construction of policy models of commodity price stabilization (Newbery and Stiglitz, 1981) Such models often portray a stochastic sequence of choices by both producers and policymakers In every time period, each side must confront not only uncertain future prices and yields, but the uncertainties of the other's future actions An understanding of this subject requires concepts of dynamics, probability, and functional analysis

Modern treatments of stochastic dynamics (Stokey and Lucas, 1989) couch their explanations in the language of sets and functions We describe and use this language in this article We also describe Fréchet differentiability, a generalization of ordinary differentiability that allows consideration of the rate of change of one function with respect to another Frechet differentiability not only is important in risk theory (Machina, 1982) but has been invoked in the field of dynamic analysis (Lyon and Bosworth, 1991) to argue for a reassessment of some of the received dynamic theory (Treadway, 1970) cited by agricultural economists in interpreting empirical results (Vasavada and Chambers, 1986, Howard and Shumway, 1988)

• **The modeling of information.** The information available to individuals plays a pivotal role in their economic behavior Thus, in analyzing such subjects as food safety, crop insurance, and the purchase of commodities of uncertain quality, economists must somehow incorporate this intangible entity, information, into their models We will describe two approaches to dealing with this problem First, we will introduce the notion of a Borel field of sets This seemingly abstruse tool is now fundamental to finance theory, where increasing families of Borel fields are used to represent the flow of information available to a trader over time Second, we will discuss how the choice set of an economic agent's risk preference ordering can be used to distinguish between situations of certainty and uncertainty

• **The measurement of individuals' risk attitudes.** A question of both theoretical and empirical interest in the risk literature, one whose answer is important for the practical elicitation of risk preferences, is whether individuals' utility functions for risky choices are (a) determined by, or (b) essentially separate from, their utility functions for riskless choices It has been widely assumed that case (a) prevails within expected utility theory We will show, however, that within this theory, the utility function for continuous probability distributions can be constructed independently of the utility function for riskless choices Thus, expected utility theory permits more flexible functional forms than perhaps generally realized If an individual uses distinct rules for choosing among certainties and among continuous probability distributions, the expected utility paradigm may still be applicable

Mathematical Preliminaries

The starting point for a clear understanding of risk is a clear understanding of the basic mathematical objects (random variables, probability spaces, and so forth) in terms of which risk is discussed and modeled Since much of contemporary risk theory is described in the language of set theory, we first review some basic terminology from that subject

The notation "$s \in S$" indicates that s is an element of the set S, while the brace notation "$\{2,5,3\}$" defines $\{2,5,3\}$ as a set whose elements are 2, 5, and 3 Two sets are equal if and only if they contain the same elements Thus, $\{5,2,3,3\}$ is equal to $\{2,5,3\}$, the order of listing is immaterial as is the appearance of an element more than once The set of all x such that x satisfies a property P is denoted $\{x \mid P(x)\}$ Thus, within the realm of real numbers, $\{x \mid x^2 = 1\}$ is the set $\{-1,1\}$ There is a unique set, called the *empty* set and denoted $\varnothing$, that contains no elements

For any sets A_1 and A_2, their *intersection*, $A_1 \cap A_2$, is the set $\{x \mid \text{for each } i,\ x \in A_i\}$, their *union*, $A_1 \cup A_2$, is $\{x \mid \text{for at least one } i,\ x \in A_i\}$, and their *difference*, $A_1 \setminus A_2$, is $\{x \mid x \in A_1 \text{ and not } x \in A_2\}$ The definitions of union and intersection extend straightforwardly to any finite or infinite collection of sets A set A_1 is a *subset* of a set A_2 if each element of A_1 is an element of A_2

A set of the form $\{\{a\},\{a,b\}\}$ is called an *ordered pair* and denoted (a,b) The essential feature of ordered pairs, that (a,b) = (c,d) if and only if a = c and b = d, is easily demonstrated If A and B are sets, their *Cartesian product*, $A \times B$, is the set of all ordered pairs (a,b) for which $a \in A$ and $b \in B$ The extension to ordered n-tuples $(a_1, \ \ ,a_n)$ and n-fold Cartesian products $A_1 \times \ \ \times A_n$ is straightforward

A *relation* is a set of ordered pairs If R is a relation, the set $\{x \mid \text{for some } y\ (x,y) \in R\}$ is called the *domain* of R (denoted D_R), and the set $\{y \mid \text{for}$

some x (x,y) $\in$ R} is called the *range* of R (denoted R_R) A *function* (or *mapping*) is a relation for which no two distinct ordered pairs have the same first coordinate When f is a function and (x,y) $\in$ f, y is denoted f(x) and called the *value* of f at x Symbolism like f A $\rightarrow$ B (read "f maps A into B") indicates that f is a function whose domain is A and whose range is a subset of B

Finally, if c is a number and f,g are real-valued functions having a common domain D, then cf and f + g are functions defined on D by [cf](x) = cf(x) and [f+g](x) = f(x) + g(x) for each x $\in$ D If f and g are *any* functions, then f$\circ$g, the *composition* of f and g (in that order), is the function defined by $[f\circ g](x) = f(g(x))$ for each x in the domain $D_{f\circ g} \equiv \{x \mid x \in D_g \text{ and } g(x) \in D_f\}$

Representations of Risk

As Stokey and Lucas (1989) point out, measure theory, which has served as the mathematical foundation of the theory of probability since the 1930's, is rapidly becoming the standard language of the economics of uncertainty We sketch a few of the basic ideas of this subject

Borel Fields of Events

In probability theory, the events to which probabilities are assigned are represented as subsets of a sample space of possible outcomes Thus, in the toss of a standard, six-sided die, the event "an even number comes up" would be represented as the subset {2,4,6} of the sample space {1,2,3,4,5,6} However, it is not logically possible, in general, to assign a probability to every subset of a sample space To see why, imagine an ideal mathematical dart thrown randomly, according to a uniform probability distribution, into the interval [0,1] The probability of hitting the subinterval [3/5,4/5] would be 1/5 Likewise, the probability of hitting any other subset of [0,1] would seem to be its length But, there are subsets of [0,1], called nonmeasurable, that *have* no length To construct an example, define any two numbers in [0,1] as "equivalent" if their difference is rational This equivalence relation partitions [0,1] into a union of disjoint equivalence sets analogous to the indifference sets of demand theory Choose one number from each equivalence set Then, the set of these choices is nonmeasurable (see Natanson, 1955, pp 76-78)

Thus, some subsets cannot be assigned a probability in the situation we have described One cannot assume, therefore, that every subset of an arbitrary sample space can be assigned a probability Rather, in every risk model, the question of which subsets of the sample space are admissible must be addressed individually

A set of admissible subsets of a sample space is characterized axiomatically as follows Let Ω be a set (interpreted as a sample space) and F a collection (that is, a set) of subsets of Ω such that (1) $\Omega \in F$, (2) $\Omega \backslash A \in F$ whenever $A \in F$, and (3) $\bigcup_{i=1}^{\infty} A_i \in F$ whenever $\{A_i\}_{i=1}^{\infty}$ is a sequence of elements of F Then, F is called a *Borel field* F plays the role of a collection of events to which probabilities can be assigned By ensuring that F is closed under various set-theoretic operations on the events in it, conditions 1-3 guarantee that certain natural logical combinations of events in F will also be in F For example, application of 1-3 to the set-theoretic identity $A \cap B = \Omega \backslash [(\Omega \backslash A) \cup (\Omega \backslash B)]$ implies that $A \cap B$, the event whose occurrence amounts to the joint occurrence of A and B, is in F whenever A and B are

Borel fields have an interpretation as "information structures" in the following sense For simplicity, let the sample space Ω be the interval [0,1], let F be the smallest Borel field over Ω that includes among its elements the intervals [0,1/2) and [1/2,1] (so that $F = \{\varnothing, [0,1/2), [1/2,1], [0,1]\}$), and let F' be the smallest Borel field over Ω that includes among its elements the intervals [0,1/4), [1/4,1/2), and [1/2,1] (so that $F' = \{\varnothing, [0,1/4), [1/4,1/2), [1/2,1], [0,1/2), [1/4,1], [0,1/4) \cup [1/2,1], [0,1]\}$) Suppose an outcome ω_0 in Ω is realized, but all that is to be revealed to us is the identity of an event in F that has thereby occurred (that is, the identity of an event $E \in F$ for which $\omega_0 \in E$) Then, the most that we could potentially learn about the location of ω_0 in Ω would be either that ω_0 lies in [0,1/2) or that ω_0 lies in [1/2,1] However, if we were instead to be told the identity of an event in F' that has occurred, we would have the possibility of learning certain additional facts about ω_0 not available through F For example, we might learn that the event [0,1/4) in F' has occurred, so that $\omega_0 \in [0,1/4)$

Observe that, in this example, F' contains every event in F and additional events not in F That is, F is a strict subset of F' Thus, F' offers a richer supply of events to help us home in on the realized state of the world, ω_0 In this sense, whenever any Borel field is a subset of another, the second may be interpreted to be at least as informative as (and, in the case of strict inclusion, more informative than) the first

A particularly important Borel field over the real line $\mathbb{R}$ is denoted B and defined as follows First, note that the set of all subsets of $\mathbb{R}$ is a Borel field that contains all intervals as elements Second, observe that the intersection of any number of Borel fields over the same set is itself a Borel field over that set Define B as the intersection of *all* Borel fields over $\mathbb{R}$ that contain all intervals as elements Then, B is itself a Borel field over $\mathbb{R}$ containing all intervals as elements Moreover, it is the "smallest" such Borel field, since it is a subset of each such Borel field The elements of B are known as *Borel sets*

Probability Measures and Probability Spaces

Let P be a nonnegative real-valued function whose domain is a Borel field F over a set Ω Then, P is called a *probability measure* on Ω and Ω (or, alternatively, the triple (Ω,F,P) is called a *probability space* if (1′) $P(\Omega) = 1$ and (2′) $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ whenever $\{A_i\}_{i=1}^{\infty}$ is a sequence of elements of F that are pairwise disjoint (that is, for which $i \neq j$ implies $A_i \cap A_j = \varnothing$) Condition 2′ asserts that the probability of the occurrence of exactly one event out of a sequence of pairwise incompatible events is the sum of the individual probabilities Probability measures on $\mathbb{R}$ having domain B are called *Borel* probability measures

Random Variables

Finally, suppose (Ω,F,P) is a probability space and r a real-valued function with domain Ω Then, r is called a *random variable* if, for every Borel set B in $\mathbb{R}$, $\{\omega \mid \omega \in \Omega \text{ and } r(\omega) \in B\} \in F$ (A function r satisfying this condition is said to be *measurable* with respect to F) The effect of the measurability condition is to ensure that a situation like "random crop yield will lie in the interval I" corresponds to an element of F and can thus be assigned a probability by P

A random variable measurable with respect to a Borel field F can be interpreted as depending only on the information inherent in F For example, in finance theory, the flow of information over a time interval [0,T] is represented by a family of Borel fields F_t $(0 \leq t \leq T)$ satisfying (among other conditions) the requirement that F_s be a subset of F_t whenever $s \leq t$ (information is nondecreasing over time) Correspondingly, the moment-to-moment price of a commodity is represented by a family of random variables p_t $(0 \leq t \leq T)$ such that, for each t, p_t is measurable with respect to F_t In this manner, the price at time t is portrayed as depending only on the information available in the market at that time (For additional details, see Dothan, 1990) Similarly, in stochastic dynamic policy models, a decisionmaker's contingent decisions over time are represented by a family of random variables r_t related to an increasing family of Borel fields F_t by the requirement that each r_t be measurable with respect to F_t

Notwithstanding its name, a random variable is not random, and it is not a variable It is a function, a set of ordered pairs of a certain type Randomness or variability are aspects not of random variables themselves but, rather, of the *interpretations* we imagine when we use random variables to model real phenomena For example, when we model a farmer's crop yield, we use a random variable (hence, a function), r, to represent *ex ante* yield, but we use a function *value*, $r(\omega)$, to represent *ex post* yield What determines ω? We interpret nature as having "randomly" selected ω from the probability space on which r is defined

Agricultural economists often represent stochastic production through forms such as $f(x) + \varepsilon$, where $x \in \mathbb{R}^n$ is interpreted as a vector of inputs and ε is interpreted as a random disturbance Despite superficial appearances, such a construct is not a sum of a production function and a random variable Rather, it is a *random field* (Ivanov and Leonenko, p 5) To characterize it in precise terms, suppose f is a (production) function and ε a random variable Define a function Φ having domain $D_\varepsilon \times D_f$ by $\Phi((\omega,x)) = f(x) + \varepsilon(\omega)$ for each $\omega \in D_\varepsilon$ and each $x \in D_f$ Then, Φ is a formal representation of stochastic production with additive error, and various functions defined in terms of Φ represent specific aspects of stochastic production For example, for each $x \in D_f$, the random variable $\Phi(\ ,x)$ defined on D_ε by $[\Phi(\ ,x)](\omega) = \Phi((\omega,x))$ represents *ex ante* production under the input x Similarly, for each $\omega \in D_\varepsilon$, the function $\Phi(\omega,\ )$ defined on D_f by $[\Phi(\omega,\ )](x) = \Phi((\omega,x))$ represents the effect of input choice on *ex post* production (that is, on the particular *ex post* production associated with nature's random "selection" of ω)

Another example of a random field is provided by the idea of signaling in principal-agent theory (Spremann, 1987, p 26) Suppose the effort expended by an economic agent (for instance, the effort expended by a producer to ensure the safety of food) is unobservable by the principal (here, the consumer), but some "noisy function of" the effort *can* be observed Such a signal of hidden effort may be defined formally as follows Let h be the (real-valued) observer function (its domain is the set of allowable effort levels) and let ε be a random variable Then, the function $z\ D_h \times D_\varepsilon \to \mathbb{R}$ defined for each $e \in D_h$, $\omega \in D_\varepsilon$ by $z(e,\omega) = h(e) + \varepsilon(\omega)$ is a random field that serves as a monitoring signal of effort

When a risk situation can be represented by a random variable, it can equally well be represented by infinitely many distinct random variables (For example, there exist infinitely many distinct normal random variables having mean 0 and variance 1, each defined on a different probability space) For this reason, random variables cannot model situations of risk uniquely However, to every random variable r defined on a probability space $(\Omega,\mathcal{F},P)$, there corresponds a unique Borel measure, P_r, on $\mathbb{R}$ satisfying $P_r(B) = P(\{\omega \mid \omega \in \Omega \text{ and } r(\omega) \in B\})$ for each Borel set B (P_r is called the *probability distribution* of r) In addition, there corresponds a unique function F_r $\mathbb{R} \rightarrow [0,1]$, the *cumulative distribution function* (*c d f*) of r, such that $F_r(t) = P(\{\omega \mid \omega \in \Omega \text{ and } r(\omega) \leq t\})$ for each $t \in \mathbb{R}$ P_r and F_r contain the same probabilistic information as r, but in a form more convenient for certain computational purposes

The c d f of a random variable r is always (1) nondecreasing and (2) continuous on the right at each point of $\mathbb{R}$ In addition, (3) $\lim_{t\to-\infty} F_r(t) = 0$ and $\lim_{t\to\infty} F_r(t) = 1$ Conversely, any function F $\mathbb{R} \rightarrow [0,1]$ enjoying properties 1-3 can be shown to be the c d f of some random variable Thus, we are free to view the set of all c d f's as simply the set of all functions F $\mathbb{R} \rightarrow [0,1]$ satisfying 1-3

Random Functions

Random variables, Borel measures, and c d f's are tools for representing the chance occurrence of scalars They can be generalized to n-dimensional random vectors, probability measures on $\mathbb{R}^n$, and n-dimensional c d f's to represent the chance occurrence of vectors in $\mathbb{R}^n$ However, even more general tools are sometimes needed for the conceptualization of risk in agriculture For example, the yield of a corn plant depends on, among other things, the surrounding temperature over the period of growth It is reasonable to express this temperature as a real-valued function, τ, defined on some time interval, [0,t] Yet τ, as a constituent of weather, must be regarded as determined by chance Thus, the probability distribution of the plant's yield depends on the probability distribution by which nature "selects" the temperature function Just as the probability distribution of a random variable is a probability measure defined on a set of numbers, this notion of the probability distribution of a random function finds its natural expression in the form of a probability measure defined on a probability space of functions

Similarly, consider stochastic crop production, CP_L, over a region, L, in the plane $\mathbb{R}^2$ Since yield, like weather, can vary over a region, it is appropriate to define CP_L not merely as the traditional "acreage times yield" but, rather, as the integral over L of a yield (or production density) function defined at each point of L That is, suppose Ω is a probability space representing weather outcomes, X a set of input vectors, and Y $\Omega \times X \times L \rightarrow \mathbb{R}_+$ a stochastic pointwise yield function such that, for each choice $x \in X$ of inputs and each location $\lambda \in L$, the function $Y(\ ,x,\lambda)$ $\Omega \rightarrow \mathbb{R}_+$ (interpreted as the *ex ante* yield at the location λ given input choice x) is a random variable[2] Then, for each weather outcome $\omega \in \Omega$, the corresponding *ex post* crop production over L given input vector x can be expressed as $CP_L(\omega,x) = \int_L Y(\omega,x,\lambda)d\lambda$ whenever the integral exists However, the integrand (the *ex post* pointwise yield function $Y(\omega,x,\cdot)$ $L \rightarrow \mathbb{R}_+$) is determined by chance, since it is parameterized by ω Thus, the probability distribution (if it exists) of $CP_L(\ ,x)$, that is, of *ex ante* production over L given x, depends on the probability distribution by which nature selects the integrand The latter notion is, again, expressed naturally by a probability measure defined on a probability space of functions, in this case functions mapping the region L into $\mathbb{R}$

Individual Choice Under Risk

Like the theory of consumer demand, the theory of choice under risk begins with an ordering that expresses an individual's preferences among the elements of a designated set In demand theory, that set consists of vectors representing commodity bundles In risk theory, it consists of mathematical constructs (random variables, c d f's, probability measures, or the like) capable of representing situations of risk

Preference Orderings

Suppose $\succcurlyeq$ is a relation such that $D_{\succcurlyeq} = R_{\succcurlyeq}$ (in which case $\succcurlyeq$ is a subset of $D_{\succcurlyeq} \times D_{\succcurlyeq}$ and relates elements of $D_{\succcurlyeq}$ to elements of $D_{\succcurlyeq}$) Write $a \succcurlyeq b$ to signify that $(a,b) \in \succcurlyeq$ Then, $\succcurlyeq$ is called a *preference ordering* if it is complete (that is, $a \succcurlyeq b$ or $b \succcurlyeq a$ for any elements a,b of $D_{\succcurlyeq}$) and transitive (that is, for any elements a,b,c of $D_{\succcurlyeq}$, $a \succcurlyeq c$ whenever $a \succcurlyeq b$ and $b \succcurlyeq c$) When $\succcurlyeq$ is a preference ordering, the assertion $a \succcurlyeq b$ is read "a is weakly preferred to b" and interpreted to mean that the economic agent either prefers a to b or is indifferent between a and b

[2]Y constitutes our third example of a random field

Though individuals' preferences are often considered empirically unobservable, there is nothing indefinite about the *concept* of a preference ordering. In contemporary economic theory, preference orderings are mathematical objects, and they can be examined, manipulated, and compared as such. For example, $\geqslant$, the ordinary numerical relation "greater than or equal to," is a preference ordering of $\mathbb{R}$. Formally, as a set of ordered pairs, it is simply the closed half-space lying below the line $y = x$ in the plane $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. Thus, it can be compared as a geometric object to other subsets of $\mathbb{R}^2$ that signify preference orderings of $\mathbb{R}$. This geometric perspective can be invoked in investigating whether two preference orderings are the same, whether they are near one another, and so forth. Similarly, preference orderings of other sets S, including sets of c.d.f.'s or other representations of risk, can be studied as geometric objects in $S \times S$. In this context is to be found the formal meaning (if not the econometric resolution) of such empirical questions as "have consumer preferences for red meat changed?" or "are poor farmers more risk averse than wealthy farmers?"

Lotteries and Convexity

What properties are appropriate to require of a set of risk representations before a preference ordering of it can be defined? Expected utility theory imposes only one restriction: the set of risk representations must be closed under the formation of compound lotteries.

A "lottery" may be viewed as a game of chance in which prizes are awarded according to a preassigned probability law. Suppose a lottery L offers prizes L_1 and L_2 having respective probabilities p and $1 - p$ of occurring. If L_1 and L_2 are themselves lotteries, L is called a *compound* lottery.

Consider a farmer whose crops face an insect infestation having probability p of occurring. Assume weather to be random. Then, the farmer would receive one income distribution with probability p, another with probability $1 - p$. This situation has the form of a compound lottery.

What expected utility theory requires of the domain of a preference ordering is that whenever two lotteries with monetary prizes lie in the domain, any compound lottery formed from them must lie in it as well. Now, mathematically, lotteries L_1, L_2 with numerical prizes can be represented by c.d.f.'s C_1, C_2. If the internal structure of a compound lottery is ignored and only the distribution of the lottery's final numerical prizes is considered, then the compound lottery L offering L_1 and L_2 as prizes with probabilities p and $1 - p$ is represented by the c.d.f. $pC_1 + (1-p)C_2$. Thus, the requirement that the domain of a preference ordering be closed under compounding is expressed formally by the requirement that, whenever c.d.f.'s C_1, C_2 lie in the domain, any convex combination $pC_1 + (1-p)C_2$ of them must lie in it as well. However, within the vector space over $\mathbb{R}$ of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (Hoffman and Kunze, 1961, pp. 28-30), $pC_1 + (1-p)C_2$ is nothing but a point on the line segment joining C_1 and C_2. Thus, this entire line segment is required to lie in the domain whenever its endpoints do. In short, the domain is required to be convex (Kreyszig, 1978, p. 65).

The ability of c.d.f.'s to represent compound lotteries as convex combinations is shared by Borel probability measures but not by random variables. Thus, expected utility theory and the related risk literature usually deal with c.d.f.'s or probability measures rather than random variables. Formally, the term *lottery* is commonly used to denote either a c.d.f. or a probability measure, depending on context. For this article, we define a lottery to be a c.d.f. A *lottery space* is a convex set of lotteries. By a *risk preference ordering*, we mean a preference ordering whose domain is a lottery space.

Keep in mind that not all situations of individual choice in the presence of risk are appropriately modeled by the simple optimization of a risk preference ordering. Risk preference orderings are intended to compare risks and risks only. By contrast, a consumer's decision whether to obtain protein through consumption of peanut butter (a potential source of aflatoxin) or chicken (a potential source of salmonella) involves questions of taste as well as risk. Unless these influences can be separated, standard risk theories—expected utility or otherwise—will not apply.

Choice Sets and the Modeling of Information

As a result of budget constraints or other restrictions dictated by particular circumstances, an individual's choices under risk will generally be confined to a strict subset of the lottery space $D_{\geqslant}$ termed the choice set. It is this set on which are ultimately imposed a model's assumptions concerning what is known versus unknown, certain versus uncertain to the economic agent.

Several areas of concern to agricultural economists—food safety, nutrition labeling, grades and standards, and product advertising—are intimately tied to the economics of information (and, by extension, to the economics of uncertainty). The inability of consumers to detect many food contaminants unaided, for example, limits producers' economic incentives to compete on the basis of food safety Government policy aims both to reduce risks to consumers and to provide information about what risks do exist How, though, can assumptions about, or changes in, a consumer's information or uncertainty be incorporated explicitly into a mathematical model? The agent's choice set would often appear to be the proper vehicle for representing these factors For example, when the agent is assumed to be choosing under certainty, the choice set is confined to constructs representing certainties, such as c d f's of constant random variables When the agent is assumed to be choosing under risk, representations of certainty are *excluded* from the choice set, and only those lotteries are allowed that conform to the economic and probabilistic assumptions of the model The choice set of lotteries in a model of behavior under risk plays no less important a role than the set of feasible, budget-constrained commodity bundles in a model of consumer demand In each case, the optimum achieved by the economic agent is crucially dependent on the set over which preferences are permitted to be optimized

Utility Functions on Lottery Spaces

Let $\geqslant$ be a risk preference ordering A function U $D_{\geqslant} \rightarrow \mathbb{R}$ is called a *utility function* for $\geqslant$ if, for any elements a,b of $D_{\geqslant}$, $U(a) \geqslant U(b)$ if and only if $a \geqslant b$ A function U $D_{\geqslant} \rightarrow \mathbb{R}$ is called *linear* if $U\big(tL_1 + (1-t)L_2\big) = tU(L_1) + (1-t)U(L_2)$ whenever $L_1, L_2 \in D_{\geqslant}$ and $0 \leqslant t \leqslant 1$

Linearity in the above sense must be distinguished from the notion of linearity customarily applied to mappings defined on vector spaces (Hoffman and Kunze, 1961, p 62) Indeed, a lottery space cannot be a vector space since, for example, the sum of two c d f's is not a c d f Rather, the assumption that a function U $D_{\geqslant} \rightarrow \mathbb{R}$ is linear in our sense is analogous to the assumption that a function g $\mathbb{R} \rightarrow \mathbb{R}$ has a straight-line graph, that is, that g is both concave ($g(\lambda x + (1-\lambda)y) \geqslant \lambda g(x) + (1-\lambda)g(y)$ whenever $x, y \in D_g$ and $0 \leqslant \lambda \leqslant 1$) and convex ($g(\lambda x + (1-\lambda)y) \leqslant \lambda g(x) + (1-\lambda)g(y)$ whenever $x, y \in D_g$ and $0 \leqslant \lambda \leqslant 1$), or, equivalently, that $g(\lambda x + (1-\lambda)y) = \lambda g(x) + (1-\lambda)g(y)$ whenever $x, y \in D_g$ and $0 \leqslant \lambda \leqslant 1$ [3] Restated for a function U $D_{\geqslant} \rightarrow \mathbb{R}$, the latter condition expresses precisely the concept of linearity introduced above

An assumption of linearity requires, in effect, that a compound lottery be assigned a utility equal to the expected value of the utilities of its lottery prizes Though stated for a convex combination of two lotteries, the formula in the definition of linearity is easily shown to extend to a convex combination of n lotteries For example, we can use the convexity of $D_{\geqslant}$ to express $p_1L_1 + p_2L_2 + p_3L_3$, a convex combination of three elements of $D_{\geqslant}$, as a convex combination of *two* elements of $D_{\geqslant}$, obtaining (under the conventions $p_2 + p_3 \neq 0$, $p_2' \equiv p_2/(p_2+p_3)$, $p_3' \equiv p_3/(p_2+p_3)$)

$$
\begin{aligned}
U(p_1L_1 + p_2L_2 + p_3L_3) &= U\big(p_1L_1 + (p_2+p_3)(p_2'L_2 + p_3'L_3)\big) \\
&= p_1U(L_1) + (p_2+p_3)U(p_2'L_2 + p_3'L_3) \\
&= p_1U(L_1) + (p_2+p_3)p_2'U(L_2) + (p_2+p_3)p_3'U(L_3) \\
&= p_1U(L_1) + p_2U(L_2) + p_3U(L_3)
\end{aligned}
$$

A similar argument applied recursively to a convex combination of n elements of $D_{\geqslant}$ can be used to establish the general case

Utility functions allow questions about risk preference orderings to be recast into equivalent questions about real-valued functions defined on lottery spaces The benefit of this translation is most apparent when the utility function can itself be expressed in terms of another "utility function" that maps not lotteries to numbers but *numbers* to numbers, for then the techniques of calculus can be applied It is on utility functions of the latter type that the attention of agricultural economists is usually focused

Although such wholly numerical utility functions are frequently described as "von Neumann-Morgenstern" utility functions, von Neumann and Morgenstern (1947) were concerned with assigning utilities to lotteries, not numbers Using a very general concept of lottery, they demonstrated that any risk preference ordering satisfying certain plausible behavioral assumptions can be represented by a linear utility function They did not prove, nor does it follow from their assumptions, that a linear utility function, U, must give rise to a numerical function u such that the utility of an arbitrary lottery L has an expected utility integral representation of the form $U(L) = \int_{-\infty}^{\infty} u(t)dL(t)$

Conditions guaranteeing that a linear utility function has an integral representation of this type were given by Grandmont (1972) However, one of these conditions fails to hold for the lottery space of all c d f's having finite mean, which is a natural lottery space on which to consider risk aversion

Numerical Utility Functions

What is the general relationship between numerical utility functions and the more fundamental utility

[3] Such a function g is linear as a vector space mapping if and only if $g(0) = 0$

functions defined on lottery spaces? To examine this question, we introduce the following definitions

For each $r \in \mathbb{R}$, the lottery δ_r defined by

$$\delta_r(t) = \begin{cases} 0 \text{ if } t \leq r \\ 1 \text{ if } t > r \end{cases}$$

is called *degenerate* δ_r is the c d f of a constant random variable with value r Thus, it represents "r with certainty "

Suppose U is a utility function for a risk preference ordering $\succcurlyeq$ whose domain, $D_{\succcurlyeq}$, contains all degenerate lotteries Define a function u $\mathbb{R} \to \mathbb{R}$ by

$$u(r) = U(\delta_r),$$

for each $r \in \mathbb{R}$ We call u the *utility function induced on* $\mathbb{R}$ *by* U u is a numerical function that importantly, encapsulates the action of U under certainty

A lottery L is called *simple* if it is a convex combination of a finite number of degenerate lotteries, that is, if there exist degenerate lotteries $\delta_{r_1}, \ \ , \delta_{r_n}$ and nonnegative numbers $p_1, \ \ , p_n$ such that $\sum_{i=1}^{n} p_i = 1$ and $L = \sum_{i=1}^{n} p_i\delta_{r_i}$ In this case, L is the c d f of a random variable taking the value r_i with probability p_i $(i = 1, \ \ , n)$

Now, let U be a linear utility function whose domain contains all degenerate lotteries Then u, the utility function induced on $\mathbb{R}$ by U, is defined Moreover, by the convexity property of a lottery space, the domain of U contains all simple lotteries

Consider any simple lottery $L \equiv \sum_{i=1}^{n} p_i\delta_{r_i}$, and let X be a random variable whose c d f is L Then, the composite function $u \circ X$ is a random variable taking the value $u(r_i)$ with probability p_i $(i = 1, \ \ , n)$, and it follows that

$$U(L) = \sum_{i=1}^{n} p_i U(\delta_{r_i})$$

$$= \sum_{i=1}^{n} p_i u(r_i)$$

$$= E(u \circ X),$$

that is, U(L) is the expected value of $u \circ X$[4] However, unless additional restrictions (such as those of Grandmont (1972)) are imposed, U(L) cannot, in general, be expressed as the expectation of the induced utility function when L is *not* simple In fact, a significant part of U is *independent* of its induced utility function and therefore independent of U's utility assignments under certainty We turn next to this subject—the structural distinctness within a linear utility function of its "certainty part" and a portion of its "uncertainty part" (Weiss, 1987, 1992)

Decomposition of Linear Utility Functions

A linear utility function can be decomposed into a "continuous part" and a "discrete part " The latter encodes all aspects of U relating to behavior under certainty Unless additional restrictions are imposed, the former is entirely independent of behavior under certainty

To describe this decomposition satisfactorily, we require the following definitions A lottery is called *continuous* if it is continuous as an ordinary function on $\mathbb{R}$ A lottery L is called *discrete* if it is a convex combination of a sequence of degenerate lotteries, that is, if there exist a sequence $\{\delta_{r_i}\}_{i=1}^{\infty}$ of degenerate lotteries and a sequence $\{p_i\}_{i=1}^{\infty}$ of nonnegative numbers such that $\sum_{i=1}^{\infty} p_i = 1$ and $L = \sum_{i=1}^{\infty} p_i\delta_{r_i}$ Such a lottery L is the c d f of a random variable taking the value r_i with probability p_i $(i = 1, 2, 3, \ \ )$ Every simple lottery (and thus every degenerate lottery) is discrete

Now, every lottery L has a decomposition

$$L = p_L L_c + (1-p_L)L_d,$$

such that $0 \leq p_L \leq 1$, L_c is a continuous lottery, and L_d is a discrete lottery (Chung, 1974, p 9) (Such decompositions occur naturally in the economics of risk, as when an agricultural price support or other insurance mechanism truncates a random variable whose c d f is continuous, leading to a "piling up" of probability mass at one point See Weiss, 1987, pp 69-70, or Weiss, 1988) Moreover, p_L is unique, L_c is unique if $p_L \neq 0$, and L_d is unique if $p_L \neq 1$ It follows that if U is a linear utility function whose domain contains L, L_c, and L_d, then $U(L) = p_L U(L_c) + (1-p_L)U(L_d)$ Thus, U is entirely determined by its action on continuous lotteries and its action on discrete lotteries If, moreover, the domain of U contains all degenerate lotteries and U is countably linear over such lotteries in the sense that $U(L) = \sum_{i=1}^{\infty} p_i U(\delta_{r_i})$ for any discrete lottery $L = \sum_{i=1}^{\infty} p_i\delta_{r_i}$ in its domain, then U

[4]In the applied literature, $u \circ X$ is often incorrectly identified with u $u \circ X$ is a random variable, while u is not

is entirely determined by its action on continuous lotteries and its action at certainties

The foregoing remarks show how function values of U can be decomposed, but they do not indicate how U itself, as a function, can be decomposed A full description of this functional decomposition cannot be given here In brief, however, one uses the rule $U^*(pL) \equiv pU(L)$ to extend U to a new function, U^*, defined on an enlarged domain consisting of all product functions pL for which $0 \leq p \leq 1$ and $L \in D_U$ (such product functions are called *sublotteries*) Then (assuming $L \in D_U$ implies (1) $L_c \in D_U$ if $p_L \neq 0$ and (2) $L_d \in D_U$ if $p_L \neq 1$), U^* has a decomposition

$$U^* = U_c^* + U_d^*,$$

into unique functions U_c^* and U_d^* that are defined and linear over sublotteries, map the zero sublottery to itself, and depend only on the continuous or discrete part, respectively, of a sublottery (see Weiss, 1987)

We have described how a linear utility function can be resolved into its continuous and discrete parts Conversely, one can *construct* a linear utility function out of a linear utility function defined over continuous lotteries and a linear utility function defined over discrete lotteries In fact, if V_1 is a linear function defined over all continuous lotteries and V_2 a bounded real-valued function defined over all degenerate lotteries, a function V can be defined at any lottery

$$L = p_L L_c + (1-p_L) \sum_{i=1}^{\infty} p_i \delta_{r_i},$$

by the rule

$$V(L) \equiv p_L V_1(L_c) + (1-p_L) \sum_{i=1}^{\infty} p_i V_2(\delta_{r_i})$$

V will be a linear utility function for the preference ordering $\succcurlyeq$ defined for all pairs of lotteries by $L_1 \succcurlyeq L_2$ if and only if $V(L_1) \geq V(L_2)$ In this manner, one can construct risk preference orderings for which the utilities assigned to continuous lotteries are independent of those assigned to certainties—in short, risk preference orderings for which, in appropriate choice sets, behavior under risk is independent of, and cannot be predicted from, behavior under certainty

This construction provides a useful illustration of why the traditional, graphical approach to risk is inadequate the graph of the utility function induced on $\mathbb{R}$ (by V) provides no information concerning, say, the individual's risk preferences among normal c d f's One also sees from this construction that the utility function induced on $\mathbb{R}$ by a linear utility function need not itself be linear (in the sense of having a straight-line graph)

Risk Aversion

Risk aversion is a purely ordinal notion, a property of risk preference orderings Suppose $\succcurlyeq$ is a risk preference ordering such that each lottery, L, in $D_{\succcurlyeq}$ has a finite mean, E(L), for which $\delta_{E(L)} \in D_{\succcurlyeq}$ Then, $\succcurlyeq$ is called *risk averse* if, for each $L \in D_{\succcurlyeq}$, $\delta_{E(L)} \succcurlyeq L$ That is, an individual is risk averse if a guaranteed payment equal to the expected value of a lottery is always (weakly) preferred to the lottery itself

Risk aversion is often identified with the concavity of a numerical utility function, and this characterization plays an important role in applied risk studies The techniques of the preceding paragraphs, however, demonstrate that the equivalence is not universally valid Since a linear utility function can be constructed using independent selections of its induced utility function and its continuous part, it is easy to construct a risk preference ordering that is not risk averse but is represented by a linear utility function whose induced utility function is strictly concave In addition, while risk aversion does indeed imply concavity of the induced utility function, it is nevertheless possible to construct a risk-averse preference ordering $\succcurlyeq$, a linear utility function V representing $\succcurlyeq$, and a numerical function v *strictly convex* on [0,1] such that, for any continuous lottery L on [0,1] (that is, for which $L(0) = 0$ and $L(1) = 1$), one has

$$V(L) = \int_0^1 v(t)dL(t)$$

This example seems contrary to "common knowledge" about risk aversion, but its real lesson is that there is more to risk aversion and to other risk concepts than can be captured by the traditional approaches

A correct description of the relationship between risk aversion and the concavity of numerical utility functions can be given using the concept of continuous preferences (Weiss, 1987, 1990) Let us call a utility function U for a risk preference ordering $\succcurlyeq$ *continuous* if, for any lottery L in $D_{\succcurlyeq}$ and any sequence $\{L_i\}_{i=1}^{\infty}$ of lotteries in $D_{\succcurlyeq}$ converging to L in distribution (that is, for which $\lim_{i\to\infty} L_i(x) = L(x)$ for each point x at which L is continuous), one has $\lim_{i\to\infty} U(L_i) = U(L)$ We call a preference ordering continuous if it can be represented by a continuous utility function Now, suppose ≽ is a risk preference ordering represented by a linear utility function having an induced utility function u Then, (1) if ≽ is risk averse, u is concave, while (2) if u is concave *and* ≽ *is continuous*, then ≽ is risk averse For proofs, see Weiss (1987)

Statement 2 shows that the assumption of continuous risk preferences is sufficient to ensure the equivalence between concave numerical utility functions and risk-averse preferences Note, however, that continuity of ≽ is not guaranteed by continuity of u In fact, no assumption concerning u alone can guarantee the continuity of either U or ≽ (Weiss, 1987) Rather, only through assumptions at a more abstract level, beyond the "visible" or "graphable" part of ≽ or U embodied in u, can the continuity of risk preferences be assured Here, again, we see the limitations of traditional approaches as a theoretical foundation for empirical risk analysis

Beyond Linearity: Machina's "Generalized Expected Utility Theory"

Machina (1982) provided an important generalization of expected utility theory by showing that many of the results of the classical theory extend, in an approximate sense, to nonlinear utility functions His findings, which have attracted attention among agricultural economists (note, for example, Machina, 1985), exemplify the contribution of modern mathematical concepts to risk theory

At an intuitive level, Machina's work is grounded in the idea that a function f $\mathbb{R} \rightarrow \mathbb{R}$ differentiable at a point x_0 is locally linear in the sense that the line tangent to the graph of f at $(x_0, f(x_0))$ approximates the graph near this point That is, if T_{x_0} $\mathbb{R} \rightarrow \mathbb{R}$ is the function whose graph is this tangent line, then T_{x_0} approximates f near x_0

Machina exploited a simple but powerful idea a differentiable utility function should also be locally linear Since linearity of the utility function of a preference ordering is the essence of expected utility theory, such local linearity ought to impart at least local (and possibly global) expected utility-type properties to any smooth risk preference ordering, that is, to any risk preference ordering representable by a differentiable utility function

What, though, is to be meant by the "differentiability" of a utility function of a preference ordering? After all, such functions are defined not on the real line or even on $\mathbb{R}^n$, but on a space of lotteries, of cumulative distribution functions An answer is provided by the concept of "Fréchet differentiability" (Luenberger, 1969, p 172, Nashed, 1966), the natural notion of differentiability for a real-valued function defined on a normed vector space (Kreyszig, 1978, p 59) To motivate a definition, consider that ordinary differentiability of a function f $\mathbb{R} \rightarrow \mathbb{R}$ at a point x_0 can be characterized by the following condition there exists a continuous function g $\mathbb{R} \rightarrow \mathbb{R}$, linear in the vector space sense (so that $g_{x_0}(tx+y) = tg_{x_0}(x) + g(y)$ and, in particular, $g_{x_0}(0) = 0$), such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - g_{x_0}(x-x_0)}{x - x_0} = 0 \qquad (1)$$

Indeed, when the stated condition holds, the restrictions on g imply that g must be of the form $g_{x_0}(x) = \alpha_{x_0} x$ for some $\alpha_{x_0} \in \mathbb{R}$, and equation (1) thus reduces to

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \alpha_{x_0} ,$$

implying the differentiability of f at x_0 Conversely, if f is differentiable at x_0, the above condition is satisfied by the function g_{x_0} defined by $g_{x_0}(x) = f'(x_0)x$

The limit appearing in equation (1) makes use of division by $x - x_0$, an operation having no counterpart for vectors in a general vector space However, equation (1) can be expressed in the equivalent form

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - g_{x_0}(x-x_0)}{|x - x_0|} = 0 \qquad (2)$$

The division by an absolute value introduced in this reformulation (and, more particularly, the absolute value itself) *does* have a vector space counterpart, whose description follows

A *norm*, $\| \ \|$, is a real-valued function defined on a vector space and satisfying the following conditions (stated for arbitrary vectors x, y and an arbitrary scalar $r \in \mathbb{R}$) (1) $\|x\| \geq 0$, (2) $\|x\| = 0$ only if x is the zero vector, (3) $\|rx\| = |r| \, \|x\|$, (4) $\|x+y\| \leq \|x\| + \|y\|$ A norm is a kind of generalized absolute value for a vector space Intuitively, $\|x\|$ is the distance between x and the zero vector, while $\|x-y\|$ is the distance between x and y

Now, let V be a real-valued function defined on a vector space **V** equipped with a norm $\| \ \|$ (Functions of this type are often called *functionals*) Then, we say V is *Fréchet differentiable at* $v_0 \in \mathbf{V}$ if there exists a real-valued function Λ_{v_0}, both continuous (in the sense that $\|v-v^*\| \rightarrow 0$ implies $|\Lambda_{v_0}(v) - \Lambda_{v_0}(v^*)| \rightarrow 0$) and linear (in the vector space sense) on **V**, such that

$$\lim_{v \to v_0} \frac{V(v) - V(v_0) - \Lambda_{v_0}(v-v_0)}{\|v - v_0\|} = 0 \qquad (3)$$

We say V is *Fréchet differentiable* if it is Fréchet differentiable at v for each $v \in \mathbf{V}$. Observe that equation (3) is a direct parallel to equation (2).

The preceding definition provides a straightforward approach to Fréchet differentiability. However, just as in the definition of differentiability on the real line, slight modifications to the underlying assumptions are needed when V is defined only on a subset of **V**. This limitation on V is typical within expected utility theory, because utility functions for preference orderings are defined only on lottery spaces, and the latter, while *subsets* of a vector space (for example, the vector space of all linear combinations of c.d.f.'s), are not *themselves* vector spaces. We omit the complicating details. The essential point is that Fréchet differentiability at v_0 can be defined as long as (1) V is defined at all vectors near in norm-distance to v_0, and (2) Λ_{v_0} is linear and continuous over small (that is, small-norm) difference vectors of the form $v-v_0$, $v \in D_V$.

A statement of Machina's main result can now be given. Assuming $M > 0$, let L be the lottery space consisting of all c.d.f.'s on the closed interval [0,M] (that is, all c.d.f.'s L for which $L(0) = 0$ and $L(M) = 1$). Let $\|\ \|$ be the "L^1 norm" $L^1[0,M]$ (Kreyszig, 1978, p. 62), for which

$$\|L-L^*\| = \int_0^M |L(t)-L^*(t)|\,dt,$$

whenever $L, L^* \in L$. (Note: the symbol "L^1" is standard and independent of our use of "L" to denote a lottery.) Let V be a Fréchet differentiable function defined on L. (Observe that V is automatically a utility function for the risk preference ordering $\geqslant$ defined on L by $L \geqslant L^*$ if and only if $V(L) \geqslant V(L^*)$.) Then, for any $L_0 \in L$, there exists a function $U(\ ,L_0)$: $[0,M] \rightarrow \mathbb{R}$ such that

$$\lim_{\|L-L_0\|\rightarrow 0} \frac{V(L) - V(L_0)}{\int_0^M U(t,L_0)dL(t) - \int_0^M U(t,L_0)dL_0(t)} = 1$$

Thus, when an individual moves from L_0 to a nearby lottery L, the difference in the V-utility values is nearly equal to the difference in the expected values of $U(\ ,L_0)$ with respect to L and L_0. In this sense, the individual behaves essentially like an expected utility maximizer with "local utility function" $U(\ ,L_0)$.

Machina also showed how various local properties (that is, properties of the local utility functions) can be used to derive global properties (that is, properties of the utility function V itself). In so doing, he demonstrated that many of the standard results of expected utility analysis remain valid under weaker assumptions than previously realized.

The applicability of Fréchet differentiation in economics is not limited to risk theory. For example, Lyon and Bosworth (1991) use Fréchet differentiation to investigate the generalized cost of adjustment model of the firm in an infinite dimensional setting. They call into question the acceptance within received theory of a disparity in the slopes of static and dynamic factor demand functions. Their results, if correct, would have implications for agricultural economics studies that have relied on the received theory to interpret their empirical findings (Vasavada and Chambers, 1986, Howard and Shumway, 1988).

Conclusions

The theory of individual choice under risk is a subject in ferment. Spurred on by the contributions of Machina and others, researchers are actively seeking an empirically more realistic paradigm to describe behavior under risk. Their search deserves the attention and participation of agricultural economists.

Today, the frontier of research on behavior under risk employs such mathematical tools as measure theory and functional analysis. Other techniques, including those of differential geometry (Russell, 1991), are on the horizon. What is certain is that the economic analysis of uncertainty is now drawing on technical methods of increased generality and sophistication.

Readers wishing to explore this subject further should benefit from the references already cited. In addition, a more extensive introduction to the contemporary, set-theoretic style of mathematical reasoning used in this article may be found in (Smith, Eggen, and St. Andre, 1986).

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