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DISCRETE STOCHASTIC PROGRAMMING: CONCEPTS, EXAMPLES AND A REVIEW OF EMPIRICAL APPLICATIONS

Jeffrey Apland and Grant Hauer

DEPARTMENT OF AGRICULTURAL AND APPLIED ECONOMICS

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Jeffrey Apland and Grant Hauer*

* Associate Professor and Research Assistant, respectively, Department of Agricultural and Applied Economics, University of Minnesota. The comments by Harry Kaiser on an earlier draft are gratefully acknowledged, as are the reviews by Yongsung Cho, Paul McNamara, Kent Olson and Terry Roe. Funding to support this research came from the University of Minnesota Agricultural Experiment Station under Project No. 14-029.

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Jeffrey Apland and Grant Hauer[1]

Introduction

Mathematical programming techniques have been used extensively in analyses of decision-making and economic behavior under risk. Because of the pervasive nature of risk in farming, risk programming applications have been especially prominent in the agricultural economics literature. Risk programming models have been used to study a wide array of problems involving production, marketing, investment, technology and policy choices. The many approaches to risk analysis using mathematical programming techniques are discussed in detail by Boisvert and McCarl and summarized by Hardaker, Pandey and Patten. Mathematical programs which account for risk in components of the objective function are sometimes classified as risk programming techniques -- programming models which capture risk in the constraint functions and righthand sides can then be classified as stochastic programming techniques [Hardaker, Pandey and Patten].

This paper provides an overview of the discrete stochastic programming model. The conceptual basis of the model is presented and illustrated with numerical examples, and a summary of empirical applications of the technique is presented. The purpose is to introduce discrete stochastic programming to those familiar with risk analysis and to provide a concise review of empirical applications. The general discussion of discrete stochastic programming and its application is intended to be useful to both experienced and unexperienced practitioners of the technique.

The most widely used mathematical programming techniques for risk analysis are the EV and MOTAD models, which in their common forms provide means of addressing random variations in objective function coefficients. Other, less widely used, risk programming models include Target MOTAD [Tauer], Direct Expected Utility Maximizing Nonlinear Programming or DEMP [Lambert and McCarl], Utility Efficient Programming [Patten, Hardaker and Pannell], Mean-Gini [Yitzhaki; Okunev and Dillon], and Focus Loss [Boussard and Petit]. Although many more examples of risk programming applications appear in the literature, stochastic programming techniques are often desirable because they allow a broader range of sources of risk to be analyzed. Models which address risk in elements of the constraint set include Chance-Constrained Programming (righthand side risk) [Charnes and Cooper], a generalized quadratic programming model presented by Paris

[1] Associate Professor and Research Assistant, respectively, Department of Agricultural and Applied Economics, University of Minnesota, St. Paul, Minnesota 55108.

(righthand side risk), and an extension of the MOTAD technique developed by Wicks and Guise (constraint coefficient risk). Among the alternative stochastic programming techniques, discrete stochastic programming [Cocks; Rae, 1971a] is the most general, allowing for the analysis of risks which influence constraint function coefficients, righthand sides, and the objective function, in a dynamic framework.

In the next section of the paper, the general structure of the discrete stochastic programming model is presented. Particular attention is given to the sequence of decisions and the flow of information about random variables which influence those decisions. Issues pertaining to the objective function of stochastic programming models and issues of model size are discussed. A numerical example of a production problem will presented under various levels of information. Finally, a summary and review of empirical applications of discrete stochastic programming is presented.

Structure of the Discrete Stochastic Programming Model

Rae [1971a] is often credited with the introduction of discrete stochastic programming (DSP) in the Agricultural Economics literature and with recognizing it applicability to problems of the farm firm. Beyond its flexibility in capturing sources of risk which influence the objective function and constraint set, DSP also allows for a multi-stage decision process in which the decision maker's knowledge about random events changes through time as economic choices are made. Despite the considerable appeal of this framework for modeling real world problems, adoption of discrete stochastic programming since its formal introduction two decades ago has been relatively slow. The limitations to adoption most often cited are model size (stochastic programming matrices tend to be quite large) and related limitations of data availability, data handling and modeling time. Applications of discrete stochastic programming are, however, appearing in the agricultural economics literature with increasing frequency.

To understand the general structure of the DSP model, consider the following deterministic linear programming problem:

$$\text{Maximize:} \quad CX \qquad [1]$$

$$\text{Subject to:} \quad AX \leq b \qquad [2]$$

$$X \geq 0 \qquad [3]$$

Where: X is an n×1 vector of decision variables, C is a 1×n vector of objective function coefficients, A is an m×n matrix of constraint coefficients, and b is an m×1 vector of righthand side coefficients. Discrete stochastic programming provides a formal framework for modeling such optimization problems when elements of C, A and b are random. Discrete parameter values or states of nature are used to represent the range of possible coefficient values. The DSP framework also captures the flow of information to the decision maker about the values of objective function and constraint set parameters and matches that flow of informa-

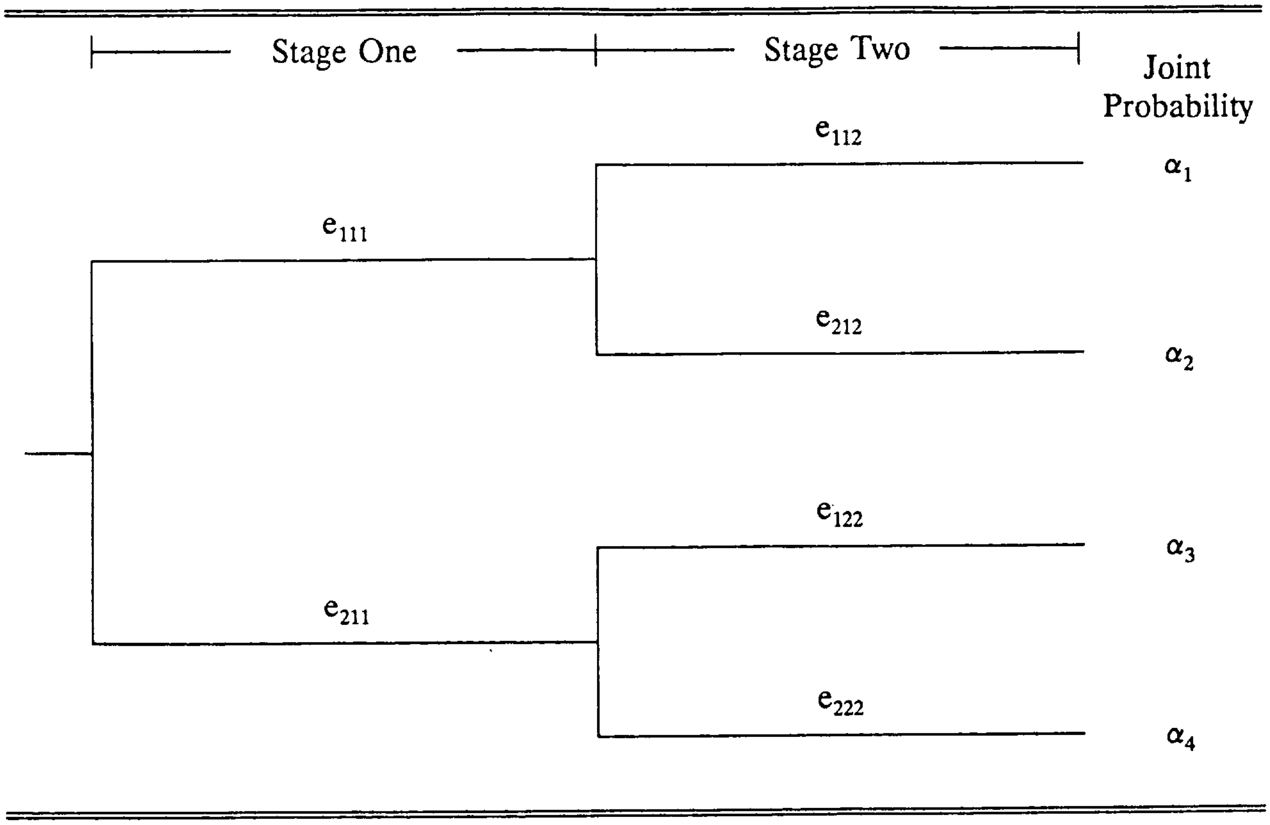

Figure 1: A Decision Tree Depicting a Two-Stage, Two-State Problem.

tion to the sequences of decisions to be made. This is done through the specification of decision stages -- time periods in which decisions are made. The sequential and stochastic framework of DSP can be represented by a decision tree. A decision tree is shown in Figure 1 for a two-stage problem with two states of nature in each stage.

Let $e_{kn_t t}$ represent the occurrence of state of nature k in stage t, where subscript n_t indicates the decision vector which will be selected with the occurrence of the kth state in stage t. In discussing DSP, the concept of an event history or state history is often useful. Here, event history at a particular stage in the decision process refers to the cumulative sequence outcomes of random events in prior stages. Thus, referring to the decision tree in Figure 1, at the end of stage two (or the beginning of stage three if there was one), one of four possible event histories will have occurred -- $\{e_{111},e_{112}\}$, $\{e_{111},e_{212}\}$, $\{e_{211},e_{122}\}$, or $\{e_{211},e_{222}\}$. Construction of the DSP matrix depends upon what Rae refers to as the information structure of the problem. The information structure is the "pattern of information receipt in relation to the decision dates" [Rae, 1971a, p. 449]. Stage t activities are assumed to be selected at the beginning of the stage. Suppose that at the beginning of stage t, the decision maker knows the outcomes of random events in stages t-i, t-i-1, ... , 1. The decision maker knows only the probabilities, conditional on known outcomes in prior stages, of outcomes in t-i+1, t-i+2, ... If i=0, the information structure may be described as complete knowledge of the past and present. Complete knowledge of the past is implied if i=1 and the decision maker has incomplete knowledge of the past if $i > 1$. It may be useful to note that for many problems, a mixed information structure is appropriate. This is the case, for example, when complete knowledge of the past characterizes the knowledge about some random variables while complete knowledge of the past and present better represents the information flow with respect to other random variables. The information structure of a problem has important implications for the construction of the DSP matrix.

Matrix Construction with Complete Knowledge of the Past. The events ($e_{kn_t t}$) shown in Figure 1 imply an information structure of complete knowledge of the past. That is, when stage one decisions are made, only the probabilities of stage one and stage two states of nature are known. When the stage two decision vector is selected, the values of random variables in stage one are known, but only the probabilities of stage two states, conditional on stage one events, are known. It is the decision vector of subscript n_t on $e_{kn_t t}$ in Figure 1 which implies an information structure of complete knowledge of the past. Note that decision vector 1 is chosen in stage one regardless of which state of nature occurs. Since the outcome of stage one random events is not known when stage one decisions are made, those decisions must be "permanently feasible" -- that is, feasible whether state 1 or state 2 occurs. The linear program for the two-stage, two-state problem under complete knowledge of the past is constructed as follows:

$$\text{Maximize:} \quad \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3 + \alpha_4 Y_4 \qquad [4]$$

$$
\text{Subject to:}\quad
\begin{array}{ccccccccc}
 & & & & A_{111}X_{11} & & & \le b_{111} & [5] \\
 & & & & A_{211}X_{11} & & & \le b_{211} & [6] \\
 & & & & & A_{112}X_{12} & & \le b_{112} & [7] \\
 & & & & & A_{212}X_{12} & & \le b_{212} & [8] \\
 & & & & & & A_{122}X_{22} & \le b_{122} & [9] \\
 & & & & & & A_{222}X_{22} & \le b_{222} & [10] \\
 & & & & -D_{111}X_{11} & +E_{12}X_{12} & & \le 0 & [11] \\
 & & & & -D_{211}X_{11} & & +E_{22}X_{22} & \le 0 & [12] \\
Y_1 & & & & -C_{111}X_{11} & -C_{112}X_{12} & & \le 0 & [13] \\
 & Y_2 & & & -C_{111}X_{11} & -C_{212}X_{12} & & \le 0 & [14] \\
 & & Y_3 & & -C_{211}X_{11} & & -C_{122}X_{22} & \le 0 & [15] \\
 & & & Y_4 & -C_{211}X_{11} & & -C_{222}X_{22} & \le 0 & [16]
\end{array}
$$

$$Y_1,\ Y_2,\ Y_3,\ Y_4,\ X_{11},\ X_{12},\ X_{22} \ \ge\ 0 \qquad [17]$$

Decision vectors $X_{n_t t}$ for the problem include X_{11} for stage one, and X_{12} and X_{22} for stage two. In stage two, activities X_{12} follow the occurrence of state 1 in stage one -- X_{22} follows state of nature 2 in stage one. Activities Y_1, Y_2, Y_3 and Y_4 are the net revenue levels for each of the four joint stage one/stage two states of nature, and α_1, α_2, α_3 and α_4 are the corresponding joint probabilities. The objective function [4] of this maximization problem is expected net revenue. Stochastic elements of the problem are captured in constraint coefficients $A_{kn_t t}$, righthand sides $b_{kn_t t}$, and net revenue coefficients $C_{kn_t t}$. Constraints [5] and [6] require stage one activities X_{11} to be feasible under both states 1 and 2. Similarly, stage two constraints [7] - [8], and [9] - [10] insure the feasibility of stage two decision vectors 1 and 2, respectively, under both of the stage two states of nature.

Constraints [11] and [12] link stage one and stage two decisions, accounting for the multi-period attributes of the decision problem. For example in a typical production problem, these constraints could provide for the transfer of resources between stages, the continuation of production processes which occur in more than one stage, and other inter-temporal linkages. Constraint [11] makes the inter-stage links given the occurrence of state 1 in stage one -- constraint [12] make the link given state 2 in stage one.

Matrix Construction with Complete Knowledge of the Past and Present. Under an information structure of complete knowledge of the past and present, the decision maker knows the outcome of stage t random events when stage t decisions are made. Thus the DSP model will have a decision vector for each discrete state of nature. Because the sequence of decisions begins with the stage one state of nature known, the complete optimal strategy (X^*_{11}, X^*_{21}, X^*_{12}, X^*_{22}, X^*_{32}, X^*_{42}) is found in the solutions to two separate linear programming problems -- one for each stage one state of nature.[2] Matrix construction proceeds as follows, with [18]-[26] corresponding to the occurrence of state 1 in stage one, and [27]-[35] corresponding to the occurrence of state 2:

Maximize: $\alpha_1 Y_1 + \alpha_2 Y_2$ [18]

Subject to:

$$A_{111}X_{11} \le b_{111} \quad [19]$$

$$A_{112}X_{12} \le b_{112} \quad [20]$$

$$A_{222}X_{22} \le b_{222} \quad [21]$$

$$-D_{111}X_{11} + E_{12}X_{12} \le 0 \quad [22]$$

$$-D_{211}X_{11} + E_{22}X_{22} \le 0 \quad [23]$$

$$Y_1 - C_{111}X_{11} - C_{112}X_{12} \le 0 \quad [24]$$

$$Y_2 - C_{111}X_{11} - C_{222}X_{22} \le 0 \quad [25]$$

$$Y_1,\ Y_2,\ X_{11},\ X_{12},\ X_{22} \ge 0 \quad [26]$$

---

[2] Rae points out the separability of the programming problems for the complete knowledge of the past case when a forecast of random events is available at the beginning of each stage. The overall model could, in that case, be separated into a subproblem for each discrete outcome of the stage one forecast [Rae, 1971a, p. 451].

$$\text{Maximize: } \alpha_3 Y_3 + \alpha_4 Y_4 \qquad [27]$$

$$
\begin{array}{lrrrrrll}
\text{Subject to:} & & A_{221}X_{21} & & & \leq & b_{221} & [28] \\
& & & A_{132}X_{32} & & \leq & b_{132} & [29] \\
& & & & A_{242}X_{42} & \leq & b_{242} & [30] \\
& & -D_{321}X_{21} & +E_{32}X_{32} & & \leq & 0 & [31] \\
& & -D_{421}X_{21} & & +E_{42}X_{42} & \leq & 0 & [32] \\
& Y_3 & -C_{221}X_{21} & -C_{132}X_{32} & & \leq & 0 & [33] \\
& Y_4 & -C_{221}X_{21} & & -C_{242}X_{42} & \leq & 0 & [34] \\
\end{array}
$$

$$Y_3,\ Y_4,\ X_{21},\ X_{32},\ X_{42} \geq 0 \qquad [35]$$

In some empirical applications of discrete stochastic programming where complete knowledge of the past and present is assumed, a single stage one state of nature is assumed and therefore only a single LP model is required [Lambert; Lambert and McCarl; Schroeder and Featherstone].

Forecasts and the Value of Information. Rae discusses how the information underlying the basic DSP problem may be augmented with forecast information. Modified in this way, the DSP model yields a strategy which is optimal given the forecast, and may, by various means, be used to estimate the value of the forecast. The level of information, as implied by the state definitions and their probabilities for example, is a fundamental characteristic of a DSP model. Procedurally, the inclusion of additional information such as that from a forecast may be accomplished through changes in the discrete probability distributions of random variables or changes in the information structure associated with the random variables. Figure 2 shows in a decision tree how the two-stage, two-state problem is altered by the availability of a forecast.

In discrete stochastic programming, forecast events, like states of nature, are characterized by discrete probability distributions. Figure 2 illustrates a case in which forecasts are received at the beginning of each stage. Each forecast has two possible outcomes. The symmetry in this example between the number of forecast events and states of nature is a matter of convenience -- the number of forecast events could be greater than or less than the number of states. The underlying information structure is complete knowledge of the past. Each stage one forecast outcome is followed by one of two stage one states of nature, so a less than perfect forecast is implied. Two stage one decision vectors are indicated -- one for each forecast event. Only one stage one decision vector is used for the no forecast problem. The forecast at the beginning of stage two will be received following one of four possible forecast/state of nature histories. Following the receipt of the stage two forecast, a decision vector is selected for each of eight possible joint events -- the no forecast problem has only two stage two vectors. For the complete knowledge of the past case, each stage two decision vector must be feasible under both stage two states of nature. In all, there are

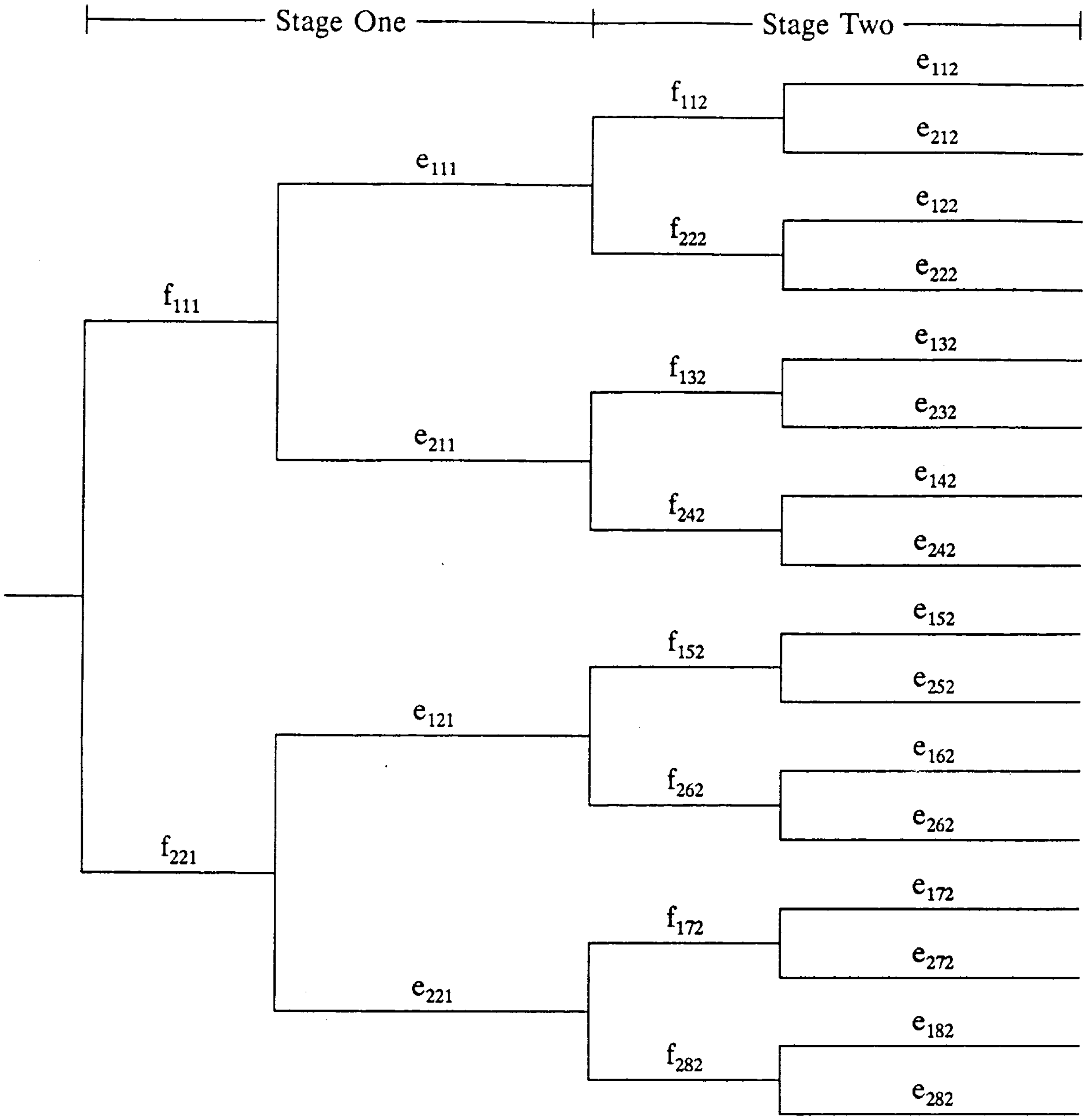

Figure 2: A Decision Tree Depicting a Two-Stage, Two-State Problem With a Forecast.

16 terminal branches in the decision tree, representing all possible forecast/state histories for the problem.

Rae points out that for the expected net revenue case discussed here, the value of the forecast information may be estimated by subtracting the objective function value for the no forecast problem from that for the model which incorporates the forecast "provided that the model includes no restraints on cash supplies" [Rae 1971a, p. 458]. If cash flow constraints are imposed, the incidence of the payment for information will influence the opportunity set and possibly the optimal solution. In this case, the value of information may be estimated by solving the model with successively increasing information charges until the objective function value is the same as for the no forecast problem. The information cost which produces an equal objective function value is the value of the forecast information. This procedure for finding the value of information could be extended to any situation in which the acquisition of information uses the firm's resources, whether operating capital or labor for activities such as "scouting" for pest infestations. When the objective is a non-linear or multi-dimensional expected utility function, the iterative procedure described above is also required. Objective function considerations are addressed in the next section.

The Objective Function in DSP

The general DSP models presented up to this point have had expected net revenue as their objective functions. In this section, various general approaches to incorporating expected utility as an objective are discussed. First, the extensions into the DSP format of the widely used EV and MOTAD objective functions are discussed. Then, more general expected utility maximization formulations are discussed, including the direct nonlinear programming approach and the separable programming approximation. Finally, other objective function issues are discussed including the implications of time and multi-dimensional utility.

EV and MOTAD Objective Functions. The EV risk programming model requires that expected utility be expressed as a function of expected income and the variance of income. Normality of the probability distribution of income and/or quadratic utility are sufficient conditions for the appropriate use of the EV objective function [Anderson, Dillon and Hardaker], but Meyer [1987] has shown that normality or quadratic utility are not necessary. The use of an EV-type objective function in DSP proceeds as follows. The occurrence of joint events is characterized by a multinomial distribution in which one of m joint events will occur for each cycle of the decision process. Let the probability of joint event j be α_j, $j=1...m$, where $\alpha_j \geq 0$ and $\alpha_1+\alpha_2+\ldots\alpha_m = 1$. Then the expected value of the jth joint event is α_j and the variance is $\sigma_j^2 = \alpha_j(1-\alpha_j)$. The covariance of joint events i and j is $\sigma_{ij} = -\alpha_i\alpha_j$ $(i \neq j)$ [Cocks]. So the EV objective function for the DSP model is:

$$\text{Maximize:} \quad \sum_{j=1}^{m} \alpha_j Y_j - \Phi \sum_{j=1}^{m} V_{ij} Y_i Y_j \qquad [36]$$

where Φ is the risk coefficient and V_{ij} is the variance $(i=j)$ or covariance $(i \neq j)$. Alternatively, the variance may be minimized subject to a minimum constraint on expected net revenue. Since the variance-covariance matrix is positive semi-definite, the DSP-EV objective function [36] is concave and a global solution to the problem is ensured. The DSP-EV model is a quadratic programming problem and may be solve using a quadratic or nonlinear programming solver, or a matrix diagonalization procedure may be used and the solution may be approximated using separable programming [McCarl and Tice].

In the MOTAD model, the standard deviation, and thus the variance, is approximated using the mean absolute deviation. A MOTAD version of the DSP model can be constructed as follows. A constraint can be added to the model which defines expected net revenue as follows:

$$\sum_{j=1}^{m} \alpha_j Y_j - \bar{Y} = 0 \qquad [37]$$

The following constraints will define elements of absolute deviations vectors d^- and d^+ for negative and positive deviations from the mean:

$$Y_j - \bar{Y} + d_j^- - d_j^+ = 0 \qquad j = 1,\ldots,m \qquad [38]$$

The mean absolute deviation of net returns is the probability weighted sum of absolute deviations. So the following MOTAD-type objective function can be used:

$$\text{Maximize:} \quad \bar{Y} - \Theta\sum_{j=1}^{m} \alpha_j\left(d_j^- + d_j^+\right) \qquad [39]$$

Where Θ is the mean absolute deviation risk coefficient.

Expected Utility Maximization and the Separable Programming Approximation. Since the DSP model includes a net revenue activity Y_j for each joint event, modification of the problem to one of expected utility maximization is straight forward. The concepts and procedures are discussed at length in Rae [1971a] and Lambert and McCarl [1985], and are summarized here. Suppose the producer's utility, expressed as a function of net revenue, is U(Y). Then expected utility is:

$$\sum_{j=1}^{m} \alpha_j U(Y_j) \qquad [40]$$

and [40] may be used as the objective function of a nonlinear programming, expected utility maximizing, DSP model. The functional form of the objective is that of the utility function. Note that since the expected utility function is separable, the problem may be readily approximated using separable programming and solved with a linear programming solver. In the separable programming formulation, a set of variables Q_{jk}, $k=1...q$ is defined for each of the nonlinear net revenue variables in the expected utility function. Each set of separable programming variables is used to represent consecutively-increasing values of a net revenue activity in a utility function approximation. Let the discrete net revenue levels used in the approximation be $\hat{Y}_{j1} < \hat{Y}_{j2} < ...\hat{Y}_{jq}$. Then the separable programming approximation of the expected utility maximizing, two-stage, two-state DSP model is constructed as follows:

$$\text{Maximize:} \quad \alpha_1\sum_{i=1}^{m} U(\hat{Y}_{1i})Q_{1i} + \alpha_2\sum_{i=1}^{m} U(\hat{Y}_{2i})Q_{2i} + \alpha_3\sum_{i=1}^{m} U(\hat{Y}_{3i})Q_{3i} + \alpha_4\sum_{i=1}^{m} U(\hat{Y}_{4i})Q_{4i} \qquad [41]$$

Subject to:

$$
\begin{array}{lllll}
A_{111}X_{11} & & & \leq b_{111} & [42] \\
A_{211}X_{11} & & & \leq b_{211} & [43] \\
 & A_{112}X_{12} & & \leq b_{112} & [44] \\
 & A_{212}X_{12} & & \leq b_{212} & [45] \\
 & & A_{122}X_{22} & \leq b_{122} & [46] \\
 & & A_{222}X_{22} & \leq b_{222} & [47] \\
-D_{111}X_{11} & +E_{12}X_{12} & & \leq 0 & [48] \\
-D_{211}X_{11} & & +E_{22}X_{22} & \leq 0 & [49]
\end{array}
$$

$$\sum_{i=1}^{m} \hat{Y}_{1i}Q_{1i} - C_{111}X_{11} - C_{112}X_{12} \leq 0 \qquad [50]$$

$$\sum_{i=1}^{m} \hat{Y}_{2i}Q_{2i} - C_{111}X_{11} - C_{212}X_{12} \leq 0 \qquad [51]$$

$$\sum_{i=1}^{m} \hat{Y}_{3i}Q_{3i} - C_{211}X_{11} - C_{122}X_{22} \leq 0 \qquad [52]$$

$$\sum_{i=1}^{m} \hat{Y}_{4i}Q_{4i} - C_{211}X_{11} - C_{222}X_{22} \leq 0 \qquad [53]$$

$$\sum_{i=1}^{m} Q_{1i} = 1 \qquad [54]$$

$$\sum_{i=1}^{m} Q_{2i} = 1 \qquad [55]$$

$$\sum_{i=1}^{m} Q_{3i} = 1 \qquad [56]$$

$$\sum_{i=1}^{m} Q_{4i} = 1 \qquad [57]$$

$$Q_{1i},\ Q_{2i},\ Q_{3i},\ Q_{4i},\ X_{11},\ X_{12},\ X_{22} \geq 0 \quad i = 1,\ldots,m \qquad [58]$$

[42]-[49] are, as before, resource and transfer constraints, and other relevant restrictions on the decision vectors. The separable programming variables Q may be interpreted as weights, which must sum to 1 by convexity constraints [54]-[57], for each of the nonlinear approximations. For a given set of decisions X, constraints [50]-[53] insure the proper accounting of net revenue for each joint event in the definition of the separable programming variables. The corresponding convex combination of net revenue levels is used in the piecewise linear approximation of the expected utility function [41]. Figure 3 illustrates a utility function and its separable programming approximation using four points-- that is, q=4. Note that the approximation becomes increasingly accurate as the number of points q is increased and or as the range of the approximation, $\hat{Y}_{j1}$ to $\hat{Y}_{jq}$, is decreased. No additional constraints are needed to accomplish this increased accuracy, however.

The separable programming technique allows the use of relatively efficient and robust linear programming solvers. For DSP models, which tend to be large and complex, the relative efficiency and reliability of LP solvers is especially attractive. However, the availability of large scale nonlinear programming solvers such as MINOS [Murtagh and Saunders] extends the viability of the nonlinear programming approach considerably.[3]

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[3] See McCarl and Önal for a discussion of nonlinear programming versus separable programming approximations.

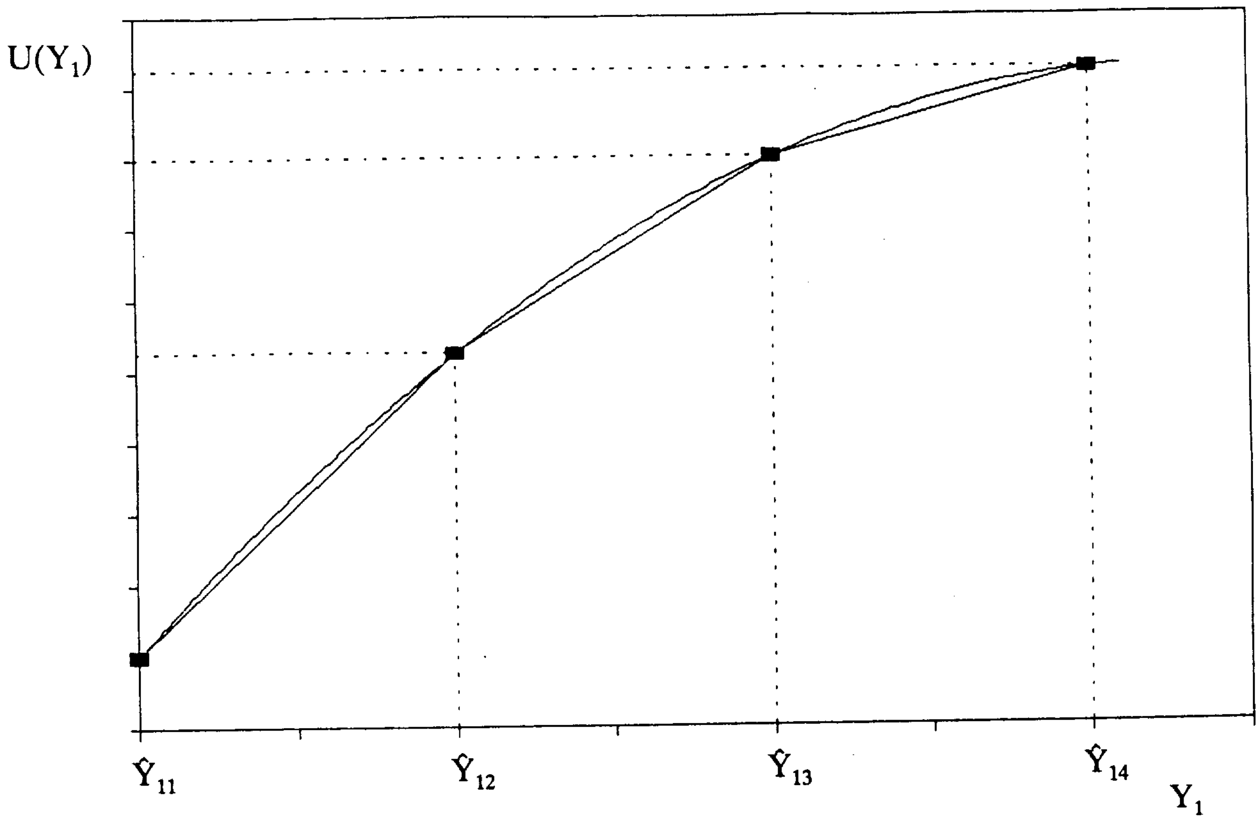

Figure 3: Illustration of the Separable Programming Approximation of the Utility Function

Patten, Hardaker and Pannell [1988] have presented an extension of the expected utility maximization approach to risk programming which is applicable to DSP. Their "utility-efficient programming" technique can be used to derive risk efficient solutions to programming problems for a relatively broad range of utility function types. For utility functions which can be expressed as the parametric sum of parts of the utility function and for which the degree of risk aversion varies with the parameter, a parametric programming approach can be applied to the problem of deriving a set of risk efficient solutions.

Other Objective Function Considerations. Since the decisions in the two-stage problem take place over two time periods, equation 40 is an objective function for expected utility of net revenue at a particular point in time. This point in time could be the beginning or the end of the planning horizon depending on whether revenues and costs are discounted or compounded. In many applications, this may be an acceptable approach. However, there may be occasions where a discounted utility approach is desirable. In this approach, incomes and costs at each decision stage are converted into utilities and then discounted. This approach might be more desirable when the planning horizon, made up of the stages of the DSP model, covers a long period of time. To implement this for the general two stage problem with complete knowledge of the past, equations [13]-[16] must be modified as follows:

$$Y_{111} \qquad - C_{111}X_{11} \leq 0 \qquad [59]$$

$$Y_{211} \qquad - C_{211}X_{11} \leq 0 \qquad [60]$$

$$Y_{112} \qquad - C_{112}X_{12} \leq 0 \qquad [61]$$

$$Y_{212} \qquad - C_{212}X_{12} \leq 0 \qquad [62]$$

$$Y_{122} \qquad - C_{122}X_{22} \leq 0 \qquad [63]$$

$$Y_{222} \qquad - C_{222}X_{22} \leq 0 \qquad [64]$$

Here, a net income variable $Y_{kn_t t}$ is specified for each state of nature k, decision vector n_t and stage t. The objective function then becomes:

$$\text{Maximize:} \quad \alpha_{111}U(Y_{111}) + \alpha_{211}\,\beta\,U(Y_{211}) + \alpha_{112}\,\beta\,U(Y_{112}) + \alpha_{212}\,\beta\,U(Y_{212}) + \alpha_{122}\,\beta\,U(Y_{122}) + \alpha_{222}\,\beta\,U(Y_{222}) \qquad [65]$$

Here $\alpha_{111} + \alpha_{211} = 1$, $\alpha_{112} + \alpha_{212} + \alpha_{122} + \alpha_{222} = 1$ and β is a discount rate. Rae [1971a] argues that this approach is impractical because of the need for future utility functions. However, if it is assumed that utility of net returns is fairly stable over time, then a discount rate could be used to account for the differences in the timing of net returns. Another method suggested by Rae is to discount returns to the beginning of the planning horizon and then convert this into a utility. If $Y_{kn_t t}$ is a vector of two or more payoff factors, rather than just net income, then equation [65] is a multi-dimensional utility function. In this case, the $C_{kn_t t}$ elements are matrices of payoff coefficients with each row representing a different payoff factor and each U function has a multi-dimensional domain.

Discrete stochastic programming allows a lot of flexibility in choosing the utility functions. For example, the U functions in equation [65] could be additively separable:

$$U(Y_{kn_t t}) = \sum_{i=1}^{n} w_i U_i(Y_{kn_t ti}) \qquad [66]$$

where the w_i's are weights and i indexes n payoff factors. More general forms of multi-attribute utility functions may also be used in DSP models depending on what independence conditions the modeler chooses to assume. Independence assumptions and their relation to various utility functional forms are discussed by Keeney and Raiffa [1976].

The Dimensionality Problem and Its Management

It is easy to see that the discrete stochastic programming matrix is large relative to its deterministic counterpart. Consider, for example, a deterministic LP model with 200 variables and 100 constraints. In the extension to a two-stage, two state DSP model with complete knowledge of the past, as stated in equations [4] through [17], the problem grows to 600 decision variables plus 4 net revenue variables, with 600 resource constraints plus 4 net revenue accounting constraints and the necessary transfer constraints. Should it be necessary to use more states of nature to adequately characterize random elements of the problem, the matrix grows substantially. With more states, it is not only necessary to further constrain decision vectors in a particular stage, but it is necessary to include more vectors in later stages to allow decisions to be made following each of the expanded number of possible event histories. Inclusion of more stages in a DSP model leads to a similar explosion in matrix size. To some extent, the general DSP model exaggerates dimensionality. In many cases, the decision vectors in each stage of a multi-stage DSP model include only a proper subset of the variables in the single decision vector of the deterministic model. Similarly, many constraints may not be needed in some stages. However, in most empirical applications, management of the size of a DSP model is an essential task.

Four areas of concern might be identified with the dimensionality problem -- data availability, model construction, model solution and interpretation of results. While the matrices of DSP models are often sparse, the models tend to contain an enormous number of constraint and objective function coefficients. In many cases, data sets which seem wholly adequate for a deterministic modeling exercise, will fall short of the requirements of a stochastic model. For example, average crop yields from time series data may work well for a deterministic farm model. In a static risk programming model (a typical EV or MOTAD model for example), the same data may be suitable for developing probability distributions of yields. When dynamic aspects of crop production are acknowledged, however, many readily available data sets may fall short. What is the probability distribution of yields when planting decisions are made? Later, after many important random events have occurred, such as spring rain, temperature, pest infestations and the like, what probability distribution of yields is appropriate to support post-planting, pre-harvest decisions, such as fertilization, pest management or forward contracting of crop sales? The construction of an adequate model of a risky decision problem depends critically on the definition of decision stages and states of nature, and places high demands on data collection. In many cases, stochastic simulation models will be necessary to produce estimates of stochastic variables. For example, one might use a biological model of plant growth to estimate yield distributions from random weather variables. Data limitations are, arguably, the most restrictive problem to the use of DSP, as well as other stochastic modeling techniques.

Model construction costs increase with model size as do the costs of interpreting optimal solutions to the model. In constructing large DSP models, the use of specialized "matrix-generators" is often warranted. Matrix generators are computer programs which read a condensed form of input data and produce the necessary input for the programming solver [see McCarl and Nuthall for a more complete discussion]. The repetitive patterns which often appear in LP matrices frequently make them well suited to computerized matrix construction -- such patterns are readily apparent in DSP models. In discrete stochastic programming, variables and constraints are repeated under various states of nature, often with identical coefficient placement but varying coefficient values. Matrix generators may be designed to construct a DSP matrix for any number of decision stages and states of nature, making changes in this significant attribute of the model less costly. Matrix generating software is generally developed in programming languages such as FORTRAN and generally focuses on a specific problem. However, technological developments in software have greatly improved the accessibility of the computer for automating the construction of mathematical programming models. Many solvers have the ability to read data from spreadsheets, thus facilitating the use of a flexible and widely-used class of software for data manipulation. Also, the development of software such as GAMS, a powerful mathematical programming language, lowers the costs of many programming applications, including DSP. The large number of variables in a DSP model can make the interpretation of optimal solutions burdensome, also. Here, software to aid in constructing solution reports and summaries is often important.

The fourth area of concern related to model size, the problem of solving a large DSP problem, is perhaps the least significant. Commercial LP codes of very large capacity are widely available, even for microcomputers. As will be seen in the review of empirical applications later in the paper, even large nonlinear DSP models have become practical with the advent of general nonlinear programming codes such as MINOS [Murtaugh and Saunders]. Linear approximations via separable programming make nonlinear models more accessible by allowing their solution with LP solvers. As discussed earlier, the expected utility maximizing DSP model may be adapted quite readily to solution by separable programming.

Because DSP models grow quickly with increases in the number of decision variables, constraints, decision stages, and the number of random variables and states of nature, the simple answer to a size management problem is to reduce the problem size in each of these dimensions. How this is done depends on the system to be modeled and the problem to be analyzed. Standard validation exercises should be used to determine acceptable levels of aggregation for each size parameter. The data problem associated with developing coefficients for a DSP model which was discussed earlier is paralleled by the problem of finding real world data with which the model can be validated. The large number of joint events which may characterize a problem implies a large number of decision variables to validate.[4] Because of the significance of the "curse of dimensionality" in DSP, the topic of size management has received some special consideration in the literature.

Anderson, Dillon and Hardaker point out that some decision variables may be judged to be suboptimal, allowing them to be eliminated from the decision vector before the model is solved [p. 229]. It may also be possible to shorten the planning horizon (that is eliminate

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[4] Helmers, Spilker and Friesen discuss a validation exercise which involves using stochastic simulation to evaluate solutions to DSP models with various number of states and decision stages.

later decision stages) if the choices in those stages do not influence decisions in the early stages. In a similar way, it may be possible to include less detail in later stages in the model without influencing optimal decision in early stages of the problem. Less detail, in this context, means fewer states of nature and more aggregated decision variables and constraints. To derive a complete optimal strategy, detail in later stages must eventually be restored, but the model may be solved with the early decisions and states taken as given.

Another promising technique for managing model size in DSP has been suggested by Yaron and Horowitz [1972b]. They present a planning model involving a series of short run (single stage) decisions which are inter-related and linked within an overall, long run planning problem. They solve a short run, single stage LP model using parametric programming techniques to derive a set of alternatives to be evaluated in the long run context. The parametric analysis concentrates on attributes of the plan for which long run considerations are crucial, thereby insuring that one of the alternative solutions is optimal in the long run plan. In approaching the problem in this way, the choice variables and constraints in each stage are dramatically simplified, thus reducing the size pressure on the DSP model. Solutions derived under such an approach could be refined once an initial solution is derived by concentrating the parametric analysis of the single stage problems in the neighborhood of the first solution.

Numerical Examples of DSP

In this section, numerical examples of discrete stochastic programming will be presented in order to illustrate elements of model structure and information concepts. All of the problems will be variants of a two stage, two state production problem. The hypothetical firm produces two products -- product A in stage one and product B in stage two. Both products require three inputs. Input 1 is available in infinitely elastic supply in each decision stage. Inputs 2 and 3 are available in fixed supply and the endowments of the fixed inputs in each stage are stochastic -- all other coefficients in the problem are deterministic. The interdependence of stage one and stage two decisions results from the fact that quantities of input 2 which are not used in stage one may be stored for use in the production of product B in stage two. The following Cobb-Douglas production functions characterize the prevailing technologies for products A and B.

$$Y_A = 3.5\, X_{1A}^{.25} X_{2A}^{.35} X_{3A}^{.40} \qquad [67]$$

$$Y_B = 4.0\, X_{1B}^{.20} X_{2B}^{.40} X_{3B}^{.40} \qquad [68]$$

Y_A is the output of product A and Y_B is the output of product B. X_{ij} is the use of input i in the production of product j. The price of input 1 is 0.5 in both stages. The price of product A is 2.5 and the price of product B is 3.5. The endowments of inputs 2 and 3 by stage and state of nature are given in Table 1. Note that in both stages, input 2 is relatively scarce in state 1 and input 3 is relatively abundant. It is assumed that the marginal probability of each state of nature in each decision stage is 0.5, and that stage one and stage two states are independent. Therefore, the probability of each joint event is 0.25.

Matrix Construction and Solutions by Information Structure. The construction of DSP models for the production problem just described will now be explained for two information structures -- complete knowledge of the past and complete knowledge of the past and present. Then optimal solutions under each information structure will be presented and compared to the case of perfect foresight. The model is set up to maximize net revenue, which can be expressed as a function of input use as follows:

$$P_A\left[3.5\, X_{1A}^{.25} X_{2A}^{.35} X_{3A}^{.40}\right] - P_1 X_{1A} + P_B\left[4.0\, X_{1B}^{.20} X_{2B}^{.40} X_{3B}^{.40}\right] - P_1 X_{1B} \qquad [69]$$

Table 1: Fixed Resource Endowments by Decision Stage and State of Nature.

	Stage One		Stage Two	
	Input 2	Input 3	Input 2	Input 3
State of Nature 1	50	125	70	175
State of Nature 2	90	75	120	125

Where P_A and P_B are the product prices and P_1 is the variable input price. The structure of the DSP model under complete knowledge of the past (CKP) is shown in Table 2.

Activities in the model are net revenue levels for each joint event, levels of input use, and levels of input transfer. Because complete knowledge of the past is assumed, decisions in each stage must be made without knowledge of the current stage state of nature. Thus, the decision vectors must be permanently feasible. The result is a single decision vector, X_{11}, in stage one and two decision vectors, X_{12} and X_{22}, in stage two -- one for each stage one state of nature. The decision vectors include levels of use for inputs 1, 2 and 3 and transfer activities for input 2. Two transfer activities are used in stage one, representing the transfer to stage two of unused amounts of input 2 under each of the possible stage one states of nature. In stage two, a single activity in each of the decision vectors transfers input 2. In vector X_{12}, the transfer occurs subsequently to the occurrence of state 1 in stage two. The transfer activity in vector X_{22} brings input 2 into stage two following state 2 in stage one.[5]

The objective, expected net revenue, is a linear function of the net revenue activities with joint probabilities as coefficients. Resource constraints limit the use plus transfers out (stage one) of the fixed inputs to no more than the endowment plus transfers in (stage two). As dictated by the prevailing information structure, two sets of resource constraints are imposed on each decision vector -- one for each state of nature in the decision stage. A transfer constraint for input 2 is needed for each stage two decision vector. A net revenue constraint for each joint event defines the corresponding net revenue activity. Note that the net revenue function [61] makes the net revenue constraints nonlinear.[6] The complete knowledge of the past model has 18 constraints, four of which are nonlinear, and 17 variables, 9 of which are nonlinear.

The structure of the DSP model under complete knowledge of the past and present (CKPP) is illustrated in Tables 3 and 4. Recall that under complete knowledge of the past and present, the model may be separated into subproblems by stage one state of nature. Table 3 shows the tableau for the first subproblem, which yields a strategy dependent on the occurrence of state 1 in stage one. Table 4 shows the second, state 2, subproblem. With complete knowledge of the past and present, there are a total of six decision vectors. In stage one there is one vector for each state of nature. In stage two there are four decision vectors -- one for each joint stage one/stage two state of nature. Elements of the decision vectors are the same as in the complete knowledge of the past model, except each stage one

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[5] To some, the definition of the transfer activities in this example may contradict the notion of permanent feasibility. However, the transfer activities are designed to account for inter temporal links in the problem and are more a consequence of resource use decisions than a management decision per se. This case demonstrates the modeling flexibility which may be achieved through the definition of specific activities in a DSP model.

[6] There is no logical reason for limiting DSP to linear programming. Practical considerations involve the availability and performance of nonlinear programming software, and the viability of linear approximation techniques.

Table 2: Tableau for the Two-Stage, Two-State Production Problem With Complete Knowledge of the Past.[†]

	Net Revenue				Stage I Vector X_{11}					Stage II Vector X_{12}				Stage II Vector X_{22}				
	Y_1	Y_2	Y_3	Y_4	$Input_1$	$Input_2$	$Input_3$	$Tran_{21}$	$Tran_{22}$	$Input_1$	$Input_2$	$Input_3$	$Tran_2$	$Input_1$	$Input_2$	$Input_3$	$Tran_2$	RHS
[4] Expected Net Revenue	.25	.25	.25	.25														
Resource Constraints:																		
[5] State 1, Stage I, Input 2						1		1										≤ 50
Input 3							1											≤ 125
[6] State 2, Stage I, Input 2						1			1									≤ 90
Input 3							1											≤ 75
[7] State 1, Stage II, Input 2											1		-1					≤ 70
Input 3												1						≤ 175
[8] State 2, Stage II, Input 2											1		-1					≤ 120
Input 3												1						≤ 125
[9] State 1, Stage II, Input 2															1		-1	≤ 70
Input 3																1		≤ 175
[10] State 2, Stage II, Input 2															1		-1	≤ 120
Input 3																1		≤ 125
Transfer of Input 2:																		
[11] State 1 in Stage I								-1					1					≤ 0
[12] State 2 in Stage I									-1								1	≤ 0
Net Revenue Constraints:																		
[13] Joint Event 1	1					—— -R(X)[‡] ——					—— -R(X)[‡] ——							≤ 0
[14] Joint Event 2		1				—— -R(X)[‡] ——					—— -R(X)[‡] ——							≤ 0
[15] Joint Event 3			1			—— -R(X)[‡] ——									—— -R(X)[‡] ——			≤ 0
[16] Joint Event 4				1		—— -R(X)[‡] ——									—— -R(X)[‡] ——			≤ 0
Non-Negativity																		

[†] Bracketed numbers on the rows of the tableau are the corresponding equation numbers from the general models presented earlier and are stated here for reference purposes.

[‡] The net revenue constraints in this model are nonlinear. Net revenue is a function of input use variables [R(X)].

Table 3: Tableau for the Two-Stage, Two-State Production Problem With Complete Knowledge of the Past and Present, Given State 1 in Stage One.†

	Net Revenue		— Stage One Vector X_{11} —				— Stage Two Vector X_{12} —				— Stage Two Vector X_{22} —				
	Y_1	Y_2	$Input_1$	$Input_2$	$Input_3$	$Tran_2$	$Input_1$	$Input_2$	$Input_3$	$Tran_2$	$Input_1$	$Input_2$	$Input_3$	$Tran_2$	RHS
[19] Expected Net Revenue	.25	.25													
Resource Constraints:															
[20] State 1, Stage One, Input 2				1		1									≤ 50
Input 3					1										≤ 125
[21] State 1, Stage Two, Input 2								1		-1					≤ 70
Input 3									1						≤ 175
[22] State 2, Stage Two, Input 2												1		-1	≤ 120
Input 3													1		≤ 125
Transfer of Input 2:															
[23] State 1 in Stage One						-1				1					≤ 0
[24] State 2 in Stage One						-1								1	≤ 0
Net Revenue Constraints:															
[25] Joint Event 1	1		——	-R(X)‡	——		——	-R(X)‡	——						≤ 0
[26] Joint Event 2		1	——	-R(X)‡	——						——	-R(X)‡	——		≤ 0
Non-Negativity															

† Bracketed numbers on the rows of the tableau are the corresponding equation numbers from the general models presented earlier and are stated here for reference purposes.

‡ The net revenue constraints in this model are nonlinear. Net revenue is a function of input use variables [R(X)].

Table 4: Tableau for the Two-Stage, Two-State Production Problem With Complete Knowledge of the Past and Present, Given State 2 in Stage One.†

	Net Revenue		— Stage One Vector X_{21} —				— Stage Two Vector X_{22} —				— Stage Two Vector X_{42} —				
	Y_3	Y_4	$Input_1$	$Input_2$	$Input_3$	$Tran_2$	$Input_1$	$Input_2$	$Input_3$	$Tran_2$	$Input_1$	$Input_2$	$Input_3$	$Tran_2$	RHS
[28] Expected Net Revenue	.25	.25													
Resource Constraints:															
[29] State 1, Stage One, Input 2				1		1									≤ 90
Input 3					1										≤ 75
[30] State 1, Stage Two, Input 2								1		-1					≤ 70
Input 3									1						≤ 175
[31] State 2, Stage Two, Input 2												1		-1	≤ 120
Input 3													1		≤ 125
Transfer of Input 2:															
[32] State 1 in Stage One						-1				1					≤ 0
[33] State 2 in Stage One						-1								1	≤ 0
Net Revenue Constraints:															
[34] Joint Event 3	1			—— -R(X)‡ ——				—— -R(X)‡ ——							≤ 0
[35] Joint Event 4		1		—— -R(X)‡ ——								—— -R(X)‡ ——			≤ 0
Non-Negativity															

† Bracketed numbers on the rows of the tableau are the corresponding equation numbers from the general models presented earlier and are stated here for reference purposes.

‡ The net revenue constraints in this model are nonlinear. Net revenue is a function of input use variables [R(X)].

vector has only one transfer activity since each vector corresponds to a unique endowment.

The two subproblems of the complete knowledge of the past and present problem each have ten constraints, including two nonlinear, net revenue constraints. There are four input transfer constraints -- one for each stage two decision vector. Each subproblem has 14 variables. There are a total of 28 variables under CKPP compared to 17 for the CKP problem. The 18 input use variables are nonlinear. An optimal strategy for the CKPP problem is formed by combining the solutions to the two subproblems, and expected net revenue is the sum of the optimal objective function values for the two subproblems.

Across states of nature, the constraints in these numerical examples differ only in the RHS values.[7] Notice, however, that if random variations in resource constraint or net revenue coefficients occur, the constraints under each state could reflect the discrete states. For this problem, potential stochastic elements include product prices, variable input prices and output elasticities.

Solutions to the two DSP problems just described appear in Table 5, along with a third solution for the perfect foresight case. For the perfect foresight (PF) case, it is assumed that while random variation occurs in the fixed resource endowments, the decision maker knows the outcome of random events in both stages when a decision cycle begins. The perfect foresight strategy was constructed from the solutions to a deterministic model of the problem solved for each of the possible joint events. Thus, the perfect foresight strategy includes eight decision vectors -- one for each stage and joint event. In moving from the complete knowledge of the past case to the complete knowledge of the past and present case, and finally, to perfect foresight case, the decision maker has increasing levels of information. This fact is reflected in the increasing optimal net revenue values of $2,540 (CKP), $3,014 (CKPP), and $3,021 (PF).

Under complete knowledge of the past, 255.3 units of product A are produced. If state 1 occurs in stage one, 25.4 units of input 2 are transferred to stage two. If state 2 occurs, 65.4 units are transferred. As a result, 671.8 units of product B are produced in stage two following stage one state 1, but 800.4 units are produced following state 2. With complete knowledge of the past and present, output levels of product A are greater under both states, 383.5 and 269.7, respectively, than in the CKP case. This result reflects the added flexibility in resource allocation when permanent feasibility is relaxed. The improved information is reflected in the use and transfer activities for input 2, also. More of input 2 is used under both states (32.8 and 27.7, respectively) than in the CKP case (24.6). When input 2 is known to be scarce (state 1), 17.2 units are transferred -- 8.2 units fewer than in the CKP case. When input 2 is relatively abundant, 62.3 units (3.1 less than CKP) are transferred.

Incorporation of a Forecast. Two new solutions are reported in Table 6 along with the complete knowledge of the past solution. For all three of the problems, complete

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[7] The constraint coefficients for the stage I input transfer activities in the CKP problem are an exception, as explained earlier.

Table 5: Optimal Solutions for the Two-Stage, Two-State Production Problem With Various Information Structures.

	Stage One						Stage Two					
			Input Use						Input Use			
Expected Net Revenue	Decision Vector	Output of Product A	Input 1	Input 2	Input 3	Input 2 Transfer	Decision Vector	Output of Product B	Input 1	Input 2	Input 3	Input 2 Transfer
Complete Knowledge of the Past												
$2,028	1	141.7	50.6	17.1	75.0	32.9, 72.9*	1	660.3	396.2	102.9	125.0	32.9
							2	778.1	466.8	142.9	125.0	72.9
Complete Knowledge of the Past and Present												
$2,399	1	215.1	76.8	23.3	125.0	26.7	1	757.4	454.4	96.7	175.0	26.7
							2	788.4	473.0	146.7	125.0	26.7
	2	149.1	53.2	19.0	75.0	71.0	3	914.3	548.6	141.0	175.0	71.0
							4	899.4	539.6	191.0	125.0	71.0
Complete Foresight												
$2,403	1	189.7	67.8	17.8	125.0	32.2	1	778.5	467.1	102.2	175.0	32.2
	2	250.5	89.5	32.3	125.0	17.7	2	763.8	458.3	137.7	125.0	17.7
	3	132.7	47.4	14.8	75.0	75.2	3	927.8	556.7	145.2	175.0	75.2
	4	170.8	61.0	25.5	75.0	64.5	4	884.1	530.5	184.5	125.0	64.5

* Transfer of input 2 given state of nature 1 and 2, respectively, in stage one.

Table 6: Optimal Solutions to the Two-Stage, Two-State Production Problem With Complete Knowledge of the Past for the Myopic Case, the No-Forecast Case, and With a Forecast.

	Stage One						Stage Two					
			Input Use						Input Use			
Expected Net Revenue	Decision Vector	Output of Product A	Input 1	Input 2	Input 3	Input 2 Transfer	Decision Vector	Output of Product B	Input 1	Input 2	Input 3	Input 2 Transfer
Complete Knowledge of the Past, Myopic Case												
$1,942	1	234.0	83.6	50.0	75.0	0.0, 40.0	1	544.5	326.7	70.0	125.0	0.0
							2	682.6	409.6	110.0	125.0	40.0
Complete Knowledge of the Past												
$2,028	1	141.7	50.6	17.1	75.0	32.9, 72.9	1	660.3	396.2	102.9	125.0	32.9
							2	778.1	466.8	142.9	125.0	72.9
Complete Knowledge of the Past, With Forecast												
$2,233	1	205.0	73.2	21.0	125.0	29.0, 69.0	1	766.2	459.7	99.0	175.0	29.0
							2	647.6	388.5	99.0	125.0	29.0
							3	907.9	544.7	139.0	175.0	69.0
							4	767.3	460.4	139.0	125.0	69.0
	2	138.6	49.5	16.3	75.0	33.7, 73.7	5	784.3	470.6	103.7	175.0	33.7
							6	662.8	397.7	103.7	125.0	33.7
							7	923.2	553.9	143.7	175.0	73.7
							8	780.2	468.1	143.7	125.0	73.7

knowledge of the past is the underlying information structure. Results are reported for a "myopic" case in which the problem is solved using two independent, single stage, mathematical programming models. With the inter-stage links ignored, excess amounts of the transferable input are considered valueless in stage one. Therefore, use of input 2 in stage one is 50 -- the entire endowment associated with the state of nature most limiting for that input. The occurrence of state 2 in stage one results in a 40 unit surplus of input 2 which is added to the stage two endowments of the input for joint events 3 and 4 -- stage two decision vector 2. A comparison to the two stage CKP case reveals that in the myopic solution, production in stage one is too high, with 234.0 units of product A versus 141.7 units. In addition both inputs 1 and 2 are over-utilized in stage one, 83.6 and 50.0 units, respectively, versus 50.6 and 17.1. Correspondingly, production of product B in stage two is too low -- in the myopic case 544.5 units are produced following state 1 in stage one and 682.6 following state 2, compared to 660.3 and 778.1 for the dynamic, CKP problem. The $86 difference in the objective function value between the two-stage CKP solution and the myopic solution may be interpreted as the value of the probabilistic information about stage two resource endowments and production alternatives. The third solution reported in Table 6 is for a complete knowledge of the past problem with information augmented by forecasts.

It is assumed that a forecast is made at the beginning of each stage. Each forecast has one of two discrete outcomes. When forecast event j occurs at the beginning of stage t, the endowment of input 3 is known to be the state of nature j endowment. Thus, with the forecast information, the decision maker has complete knowledge of the past and present with respect to the endowment of input 3. It is assumed that the marginal probability of each forecast event is 0.5 regardless of the event history, and the probabilities of the states of nature are as before. Thus each of the sixteen possible combinations of forecast events and states of nature are equally likely -- the probability of each joint event is 0.0625. The problem here fits the general case illustrated in the decision tree of Figure 2. Two stage one decision vectors are selected -- one for each forecast outcome. Subsequent to each of the four possible combinations of stage one forecasts and states, a stage two forecast is received. With the two possible stage two forecast events, the stage two decision vector is selected with one of eight possible forecast and state histories known. Therefore, eight stage two decision vectors are selected to construct a complete strategy.

The optimal value of the objective function with forecast information is $2,233 -- up from $2,028 in the no forecast case. The $205 increase in net revenue may be interpreted as the value of information, assuming that resources expended to acquire the information do not limit the feasibility of the optimal strategy. The average production of products A and B is 171.8 and 780.0, respectively, for the forecast case, compared to 141.7 and 719.2 without the forecast. Most of the increase in productivity is attributable to the increased flexibility in the allocation of input 3 which results from the improved information about its availability. However, the forecast results in a small adjustment in the transfer of input 2 from stage one to stage two. With forecast event 1 in stage one, indicating the relative abundance of input 3 in that stage, more of input 2 is used and thus less is transferred than in the no forecast case. Similarly, when forecast event 2 establishes the relative scarcity of input three, relatively more of input 2 is saved for later use.

Review of DSP Applications

The purpose of this section is to give a brief survey of the types of applications for which DSP has proved a useful framework in the past, focusing mostly on the characteristics of the models. Applications selected for review were taken from various agricultural economics journals. The review is based on information in the journal articles as well as information from a survey of the authors and discussions with authors. DSP was first introduced in agricultural economics by Rae in 1971, but was used very infrequently as a tool until recently. Tables 7 and 8 provide summaries of the most notable applications appearing in the literature.

Given the relatively small number of examples of DSP found in the literature, it is difficult to classify the applications. However, for discussion purposes, the applications in Tables 7 and 8 might be thought to fall into two categories: production unit problems and regional problems. All the production unit problems are based on activities at the farm level and focus on the decisions and objectives of farm owners. The only regional-level applications are the plant location analysis by Brown and Dynan, and the irrigation development paper by McCarl and Parandvash. Both of these models have two stages with long run capital investment decisions taking place in the first stage. Short run operating decisions are modeled in both the first and second stage with decisions in the second stage made dependent on random events in the first stage.

The farm-level problems explore decisions of three types: production, marketing and finance. Some of the applications are purely production problems. For example, Rae's paper contains only harvesting, cropping and labor hiring decisions [Rae, 1971b]. The papers written by Apland; Apland, McCarl and Baker; Kaiser et al.; Olson and Mikesell; and Garoian, Conner and Scifres are also examples concerned exclusively with production. All of these papers include sources of risk that affect production decisions. Examples from these papers include: field days, yield and price risk [Apland]; effectiveness of prescribed burns in controlling undesired plant competition in rangeland [Garoian, Conner and Scifres]; and rangeland forage yield [Olson and Mikesell]. These papers also include price risk, however they cannot be considered marketing problems because each of these applications considers only one marketing option.

Two of the paper's are exclusively marketing problems. Schroeder and Featherstone include cash sale, hedging and put option activities in their paper "Dynamic Marketing and Retention Decisions for Cow-Calf Producers". Stochastic variables are product prices at four stages of production. Lambert and McCarl also present a pure marketing problem but their paper is concerned with the marketing of grain.

The rest of the papers contain mixes of finance and production decisions, marketing and production decisions, and finance and marketing decisions. The finance and production papers include Featherstone, Preckel and Baker [1991], Leatham and Baker [1988], and Yaron and Horowitz [1972a]. Featherstone, Preckel and Baker's model includes production decisions concerning crops and hogs and an array of capital structure and finance decisions including land purchase, share rent production, machinery sale and purchase, building purchases and sales, off farm investment and labor hiring. The model also contains variables tracking machinery assets, hog facility assets, hog assets, land ownership, debt, and owner's equity. Sources of risk include crop and hog prices and interest rates. Through these stochastic variables the author's are able to model liquidity risk, collateral risk, and credit reserve risk. Leatham and Baker's model also includes production and investment activities but in addition there are activities that model alternatives for farm loans including fixed rate, adjustable rate, and fixed rates hedged with interest rate options. Yaron and Horowitz's paper considers production decisions for irrigated crops, borrowing and lending, and alternative capital investments [1972a].

Papers that include marketing and production decisions include Kaiser and Apland [1989] and Lambert. In Kaiser and Apland's paper marketing decisions include cash grain sales at harvest and after storage, as well as grain sales by hedging. Lambert's paper emphasizes production decisions. Marketing decisions in this paper are concerned more with timing of calf sales through retention decisions than with marketing alternatives such as futures or put options. Stochastic variables in these papers include field days, field rates and crop yields in Kaiser and Apland, and prices in both papers.

Turvey and Baker present a marketing and finance model, although they also consider some production decisions. Finance and capital structure aspects of the model include decisions for land purchase and sale, cash renting, acquiring debt, investing in liquid assets, debt repayment, and asset liquidation. Marketing decisions include cash crop sales, futures options and put options. These decisions are made subject to stochastic crop yields and prices.

Information structures used in these applications are either complete knowledge of the past or complete knowledge of the past and present. We are unaware of any DSP model in the agricultural economics literature which uses incomplete knowledge of the past as an information structure. The choice of information structure tends to be problem specific. However, there is one pattern that emerges. Production activities in these applications are modeled with either complete knowledge of the past or complete knowledge of past and present, depending on the problem. However, marketing and finance decisions tend to be modeled with complete knowledge of the past and present as an information structure. The reason is that the current and past values of stochastic variables upon which these decisions depend -- usually interest rates and product prices -- are usually known at the time of the decision. Hence applications such as those by Turvey and Baker, and Schroeder and Featherstone, which involve marketing and finance decisions, tend to use complete knowledge of the past and present as an information structure.

The objective functions in these applications vary from linear expected net revenue or cost functions to nonlinear EV or direct utility functions -- none of the studies reviewed used a multi-dimensional utility function. Kaiser and Apland, Kaiser et al. , and Leatham and Baker use MOTAD formulations. Olson and Mikesell use an EV approach in one of their formulations while Lambert uses target MOTAD. Rae [1971b] was the first to use an expected utility approach in the objective function, implemented using separable programming. With the recent advent of reliable and powerful non-linear algorithms, direct inclusion of expected utility with a variety of functional forms is becoming standard practice [Featherstone, Preckel and Baker; Lambert and McCarl; Schroeder and Featherstone; Turvey and Baker].

Recent applications have also become very large and have begun to incorporate non-linearities in both objective functions and in production activities. For example, Turvey and Baker had over 15,000 variables and 9,000 constraints, while Featherstone, Preckel and Baker had over 6,000 variables (900 being nonlinear) with over 4,000 constraints. Lambert's paper on calf production and retention decisions contains non-linearities in the production model for calves. Integer variables have also been introduced into DSP applications with Brown and Dynan's paper. Given the dimensions of these applications, it appears that model size is becoming less of an obstacle for applications in agricultural economics. In fact, few of the author's mention efforts to manage matrix size. Garoian, Conner and Scifres do describe a way of limiting the number of variables in their model by establishing rules that restrict the number of feasible burning schedules. The rules were established based on results from field experiments for their particular management problem.

Although the number of applications is small, discrete stochastic programming has been used effectively by agricultural economists for a wide variety of problems. Equally diverse are the various dimensions of empirical models reviewed. Of the sixteen models discussed in the review, fourteen had two, three or four decision stages. However, one model had seven [Lambert] and another had ten stages [Garoian, Conner and Scifres]. The number of terminal branches or complete event histories used in the problems ranged from three to over five thousand -- not surprisingly, the largest of the programming matrices were the models which had the most terminal branches. The range of model sizes suggests that for many applications of DSP, analysts have found a desired level of model performance well within the capabilities of current mathematical programming software.

Table 7: Summary of Empirical Applications of DSP.

Author(s). Year. Title	Objective	Information Structure	Description of the Study and the Model
Apland. 1993. The Use of Field Days in Economic Models of Crop Farms.	Expected Net Revenue	Complete Knowledge of Past and Present	The impacts of field days risk on a representative corn and soybean farm are analyzed. The model has three decision stages -- the first two are for production decisions, the third is for marketing. Multiple production periods are used within each of the stages. Activities include field operations in various periods and grain sales. Land, labor, tractor time, planter time and harvester time are constrained.
Apland, McCarl and Baker. 1981. Crop Residue for Energy Generation: A Prototype Application to Midwestern USA Grain Farms.	Expected Net Revenue	Complete Knowledge of Past and Present	The model is used to derive a farm-level crop residue supply function with available field days as a random variable. The first stage in the two-stage model is deterministic -- field days during harvest periods, in the second stage, are stochastic. Land, labor and machine constraints are imposed on grain and residue production activities. Crop residue supply, stochastic due to field days variation, is derived by parametrically altering the residue price.
Brown and Drynan. 1986. Plant Location Analysis Using Discrete Stochastic Programming.	Expected Net Cost	Complete Knowledge of the Past	This model examines long run plant location and size decisions for the Queensland cattle slaughtering industry of Australia. In this two-stage model, plant location and size decisions are made in the first stage and short run operating decisions are made in both the first and second stages. In the second stage a range of possible random supply and demand scenarios are realized. Thus long run investment decisions account for long term uncertainty with respect to supply and demand.
Featherstone, Preckel and Baker. 1991. Modeling Farm Financial Decisions in a Dynamic and Stochastic Environment.	Expected Utility of Wealth with Various Functional Forms	Complete Knowledge of Past and Present	This paper describes a DSP model used to analyze farm capital structure decisions. The model finds the capital structure and production plan that maximizes expected utility given various states of liquidity risk, collateral risk and credit reserve risk. Possible decisions include production of crops and hogs, land purchases and sales, share rent crop production, machinery sales and purchases, building purchases and sales, off farm investment, and labor hiring.

Continued . . .

Table 7: Summary of Empirical Applications of DSP, Continued.

Author(s). Year. Title	Objective	Information Structure	Description of the Study and the Model
Garoian, Conner, and Scifres. 1987. A Discrete Stochastic Programming Model to Estimate Optimal Burning Schedules on Rangeland.	Expected Net Revenue	Complete Knowledge of the Past	McCartney Rose -- a plant infesting cattle rangeland in Texas -- can be controlled using prescribed fire. This application addresses the problem of determining an optimal burn scheduling. Decisions in each stage of the model are "burn" and "don't burn". These decisions are made over ten one year planning periods with "good" or "bad" weather events each period. The weather events determine the probability of effective burns. With ten decision stages the model could easily have become un-manageably large. Therefore, the author's developed a set of scheduling rules to eliminate unlikely courses of action based on probable decisions after successful and unsuccessful burns.
Kaiser and Apland. 1989. DSSP: A Model of Production and Marketing Decisions on a Midwestern Crop Farm.	MOTAD	Complete Knowledge of Past and Present	The paper discusses a DSP model of a midwestern US corn-soybean farm. The model was constructed to analyze the implications of risk aversion for participation in commodity programs [see Kaiser and Apland, 1987]. The three stage model is deterministic in stage one. Field days, yields and harvest prices are stochastic in stage two -- post-harvest prices are stochastic in stage three. Activities include field operations for crop production, cash grain sales and grain sales by hedging.
Kaiser et al. 1993. A Farm-Level Analysis of Economic and Agronomic Impacts of Gradual Climate Warming.	MOTAD	Complete Knowledge of Past and Present	The effects of climate change on optimal cropping decisions for a southern Minnesota grain farm are explored in this paper. Activities include spring plowing, planting, fall plowing and harvesting. Sources of risk include field days, crop yield, grain drying costs and crop prices. The model was solved for four climate change scenarios which affected the means and variabilities of the sources of risk.
Lambert. 1989. Calf Retention and Production Decisions Over Time.	Maximize Expected Returns and Target MOTAD	Complete Knowledge of Past and Present	This paper presents four alternative formulations for a calf retention, production and marketing model. The models have six stages corresponding to winter months and one stage corresponding to the summer months of the production cycle. Activities include feed use during the winter, sales and retention decisions during the winter, and sales at the end of the summer. The stochastic variables are final product prices.

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Table 7: Summary of Empirical Applications of DSP, Continued.

Author(s). Year. Title	Objective	Information Structure	Description of the Study and the Model
Lambert, and McCarl. 1989. Sequential Marketing of White Wheat Marketing Strategies.	Expected Income and Expected Utility with Various Functional Forms	Complete Knowledge of Past and Present	The model represents a wheat marketing problem with stochastic prices. Marketing activities include immediate cash sales, future cash sales, and sales for future delivery.
Leatham and Baker. 1988. Farmers' Choice of Fixed and Adjustable Interest Rate Loans.	MOTAD	Complete Knowledge of Past and Present	This application looks at fixed versus adjustable rate borrowing decisions for a farm business. In this model, the farmer can choose between fixed-rate and adjustable-rate loans and loans with fixed-rates hedged with interest rate options. Borrowing, production and investment activities are chosen in an environment of uncertain product prices, interest rates, yields and weather.
McCarl and Parandvash. 1988. Irrigation Development Versus Hydroelectric Generation: Can Interruptible Irrigation Play a Role?	Expected Net Benefits (Integral of the Demand Function)	Complete Knowledge of the Past	This application explores irrigation/hydropower tradeoffs in the Pacific-Northwest region of the United States. The model is a very simple two stage DSP model. In the first stage, activities include irrigation and thermal power investments which must be made without knowledge of the level of water flow in the second stage. In the second stage, decisions are made about the operation of existing and newly constructed facilities.
Olson and Mikesell. 1988. The Range and Stocking Decision and Stochastic Forage Production.	Expected Net Return and EV	Mixed: Complete Knowledge of Past, and Complete Knowledge of Past and Present	This model is concerned with finding optimal herd size for a ranch in northern California. States of nature are defined for forage yield which is contingent on random weather events. Activities include number of animals in the breeding herd, calves in each season, stockers in each season, amount of forage resources available, amount of forage resources transferred between periods, and purchases of hay.

Continued . . .

Table 7: Summary of Empirical Applications of DSP, Continued.

Author(s). Year. Title	Objective	Information Structure	Description of the Study and the Model
Rae. 1971. An Empirical Application and Evaluation of Discrete Stochastic Programming in Farm Management.	Expected Utility Using Separable Programming	Complete Knowledge of the Past	This paper presents a DSP application that represents a decision problem for a fresh vegetable farmer. Activities include crop production and labor hiring decisions. The outcome of these decisions, measured in terms of revenue, depends on random weather and price variables. Separable programming is used to approximate the expected utility function.
Schroeder and Featherstone. 1990. Dynamic Marketing and Retention Decisions for Cow-Calf Producers.	Expected Utility with Negative Exponential Utility Function	Complete Knowledge of Past and Present	This application examines calf retention and marketing activities for cow-calf producers. Marketing activities include cash sales, hedging and put options. Stochastic variables are prices on calves, yearlings, and slaughter steers and heifers. Corn feed prices are also stochastic. Optimal calf retention and marketing decisions depend on expected prices, available marketing alternatives and risk aversion.
Turvey and Baker. 1990. A Farm-Level Financial Analysis of Farmers' Use of Futures and Options Under Alternative Farm Programs.	Expected Utility with Power and Log Functional Forms	Complete Knowledge of Past and Present	This model is used to analyze the effect of government farm policies such as target prices and loan rates for grain on optimal hedging strategies. The model also considers the impact of farm capital structure and risk aversion on hedging. Stochastic variables include corn and soybean yields and harvest prices in stage one and post harvest prices in stage two.
Yaron and Horowitz. 1972. A Sequential Model of Growth and Capital Accumulation of a Farm Under Uncertainty.	Expected Income	Complete Knowledge of the Past	This model distinguishes between activities concerned with long run and short run planning on a small family farm with irrigated crops in southern Israel. Long run activities involve fixed asset and capital accumulation while short run plans are concerned with current production as constrained by long run decision about capital and fixed assets. States of nature are aggregated into a single variable -- economic success/failure -- in both stages.

Table 8: Dimensions of Discrete Stochastic Programming Models.

Author(s). Year. Title	Decision Stages	States of Nature by Stage	Terminal Branches or Joint Events	Activities	Constraints
Apland. 1993. The Use of Field Days in Economic Models of Crop Farms.	3	1, 4, 8	128	1,501	1,305
Apland, McCarl and Baker. 1981. Crop Residue for Energy Generation: A Prototype Application to Midwestern USA Grain Farms.	2	1, 5	5	1,213	416
Brown and Dynan. 1986. Plant Location Analysis Using Discrete Stochastic Programming.	2	1, 3	3	900	200
Featherstone, Preckel and Baker. 1991. Modeling Farm Financial Decisions in a Dynamic and Stochastic Environment.	4	9, 45, 225, 900	900	6,225	4,262
Garoian, Conner, and Scifres. 1987. A Discrete Stochastic Programming Model to Estimate Optimal Burning Schedules on Rangeland.	10	3, 5, ... 381	381	?	?
Kaiser and Apland. 1989. DSSP: A Model of Production and Marketing Decisions on a Midwestern Crop Farm.	3	1, 10, 100	100	1,770	1,028
Kaiser et al. 1993. A Farm-Level Analysis of Economic and Agronomic Impacts of Gradual Climate Warming.	2	3, 10	30	166	449
Lambert. 1989. Calf Retention and Production Decisions Over Time.	7	1, 2, 4, 8, ... 64	64	1,546	762
Lambert and McCarl. 1989. Sequential Modeling of White Wheat Marketing strategies.	4	1, 3, 18, 108	108	1,320	812

Continued . . .

Table 8: Dimensions of Discrete Stochastic Programming Models, Continued.

Author(s). Year. Title	Decision Stages	States of Nature by Stage	Terminal Branches or Joint Events	Activities	Constraints
Leatham and Baker. 1988. Farmers' Choice of Fixed and Adjustable Interest Rate Loans.	3	2, 2, 2	8	700	1,000
McCarl and Parandvash. 1988. Irrigation Development Versus Hydroelectric Generation: Can Interruptible Irrigation Play a Role?	2	40, 40	40	500	200
Olson and Mikesell. 1988. The Range and Stocking Decision and Stochastic Forage Production.	3	4, 12, 36	36	203	161
Rae. 1971. An Empirical Application and Evaluation of Discrete Stochastic Programming in Farm Management.	3	2, 5, 5	50	730	838
Schroeder and Featherstone. 1990. Dynamic Marketing and Retention Decisions for Cow-Calf Producers.	4	1, 5, 75, 900	900	4,063	3,100
Turvey and Baker. 1990. A Farm-Level Financial Analysis of Farmers' Use of Futures and Options Under Alternative Farm Programs.	2	225, 5,625	5,625	15,704	9,888
Yaron and Horowitz. 1972. A Sequential Model of Growth and Capital Accumulation of a Farm Under Uncertainty.	2	2, 2	4	330	275

Summary

Stochastic programming is a flexible technique for analyzing decision problems under risk. Compared to more widely used risk programming models, the DSP model allows for the analysis of a broader range of risk sources, by allowing random variations in coefficients of the constraint set, often resource requirements and supplies, as well as objective function coefficients. Further, DSP allows decisions to be made in a sequential fashion with information concerning sources of risk entering the decision process at various times. This sequential decision framework, and the ability to capture information availability in a variety of ways, make DSP well suited to a variety of firm-level problems. The technique may be effectively applied to public resource planning problems also.

A realistic representation of decision variables and constraints in a DSP model often leads to large programming models. In many cases, the models may have nonlinear objective functions or technical constraints, also. A review of empirical applications of discrete stochastic programming reveals that many analysts have constructed acceptable models well within the technical limits of available linear and nonlinear programming solvers. The critical issues, then, in determining the viability of DSP in particular applications appear to be the cost of model construction and the availability of data. Automation of the model building process is critical, whether through the use of specialized matrix generating computer programs, mathematical programming software which links to spreadsheet and database management applications, or flexible mathematical programming languages such as GAMS [Brooke, Kendrick and Meeraus]. Well maintained technical and economic databases are critical to the effective use of DSP as well as other risk modeling techniques. Where data limitations are especially critical, effective use may be made of simulation techniques to synthetically generate random states of nature for model coefficients.

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