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Public debt and time preferences: Insolvency, excessive saving and in between*

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Public debt and time preferences: Insolvency, excessive saving and in between*

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Abstract

Democracy tends to cultivate short-sighted politicians, for whom the horizon extends more or less till next election. This feature gives rise to a discrepancy between the time rate of discount of a country's polity and the interest rates at which the country borrows. I show how this discrepancy induces public debt swelling. Moreover, if the discrepancy exceeds a certain threshold, public debt will accumulate to the point of insolvency and, to make matter worse, this (unfortunate) state of affairs will be approached at a *finite time*. Conversely, if budget decision makers are so far-sighted that their time rate of discount is smaller than the relevant interest rate, the country becomes an excessive saver. If the polity's time rate of discount falls neither below the market interest rate nor exceeds it too much, equilibrium will be reached at a debt-to-GDP ratio between insolvency and excessive saving. Economic growth exacerbates the debt accumulation problem, making insolvency more likely compared to a ceteris-paribus stationary economy.

"Democracy is the worst form of government, except for all the others that have been tried from time to time." - Winston Churchill.

Keywords: Public debt; democracy; time preferences; insolvency

JEL classification: C61, H61, H63

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1 Introduction

Excessive buildup of public debt has become increasingly worrisome in recent decades and advanced democracies seem particularly vulnerable to the disease. The gross government debt of Japan will reach 236 percent of GDP this year and the comparable figures for the United States and the Euro area are 107 percent and 90 percent, respectively, with Greece, Italy, Ireland, Portugal and Belgium at or above the 100 percent mark (IMF 2012, Table 1.1).[1] Dealing with the problem requires understanding the causes of the malaise and the literature offers a variety of explanations, including intergenerational redistribution motives, intra-generational distribution conflicts, and debt buildup as a strategic commitment device (see the survey of Alesina and Perotti 1995, and references therein). In this work I offer an explanation based on the discrepancy between the time rate of discount of a country's polity and the interest rate at which the government borrows. I find that if this discrepancy exceeds a certain margin, government debt will accumulate to the point of insolvency and, to make matter worse, this unfortunate state of affairs will be approached at a *finite time*.

Far from being coincidental, a positive discrepancy between polity's impatience and market interest rates stems from the short-sightedness of politicians inherent in democratic systems, as crisply put in:

> Politicians themselves have, for the most part, short time horizons. For most of them, each election presents a critical point, and the primary problem they face is getting past this hurdle. ..... This is not to say that politicians never look beyond the next election in choosing courses of action, but only that such short-term consider-

[1] These figures pertain to official debt (held by the public) and are but fractions of the broader, fiscal gap indebtedness (see Kotlikoff and Burns 2012, for a discussion of the U.S. fiscal gap).

> ations dominate the actions of most of them. Such features are, of course, an inherent and necessary attribute of a democracy. But when this necessary attribute is mixed with a fiscal constitution that does not restrain the ordinary spending and deficit creating proclivities, the result portends disaster. (Buchanan and Wagner 1977, p. 166.)

Empirical evidence on the link leading from political horizons to public debt swelling can be found in Roubini and Sachs (1989) and Grilli et al. (1991). The former constructed an indicator of political fragmentation in a group of OECD countries, based on the number of parties, and found that government tenure significantly affects public debt. The latter found that longer-lived governments have smaller deficits. The model developed here explains the underlying mechanism.

Time preferences in general and short-termism in particular are succinctly represented by discount rates and a polity's time rate of discount is inversely related to the length of the period it expects to hold power. In advanced democracies this period ranges between a few months and 8-10 years, giving rise to polity's time rate of discounts that often exceed market rates. I show how this discrepancy gives rise to public debt buildups. Particularly, I find that if the polity's time rate of discount exceeds the market interest rate at which the government borrows by a certain margin, government debt will accumulate to the point of insolvency and, to make matter worse, this unfortunate state of affairs will be approached at a *finite time*.

It seems natural to expect that economic growth would be conducive for debt handling, as a richer economy can more easily service any given debt. I find the opposite: economic growth often exacerbates insolvency prospects. The reason is that growth alters borrowing incentives in a way that motivates transfers from (wealthier) future generations to the present. As a result, the

bound on the discrepancy between the polity's time rate of discount and the market interest rate under which the country is doomed to becoming insolvent changes in such a way that, ceteris paribus, a growing economy is more likely to be driven to insolvency than its stationary counterpart.

If the polity's discount rate is smaller than the market interest rate, the country becomes a net saver, eventually reaching an excessive saving limit. If the polity's discount rate neither falls below the market interest rate nor exceeds it too much, equilibrium will be reached at a debt-to-GDP ratio between insolvency and excessive saving.

The idea that differences in impatience across economic agents underlie borrowing-lending and debt patterns has recently been used by Eggertsson and Krugman (2012) to explain several episodes observed in the current economic slump and to rationalize certain cures. Here, the impatience discrepancy is between that of politicians (or budget decision makers) and of the public at large (as represented by the market interest rate) and is shown to drive the dynamics of public debt.

A related strand of literature is concerned with governments' defaults (see Reinhart and Rogoff 2009, for a comprhensive account and historical perspectives). On the one hand, defaults are a consequence of public debt buildups. On the other hand, defaults may well be a legitimate cause (particularly of sovereign debt buildup) if taken by (yet perfectly solvent) governments as a viable course of action in dealing with future excessive debts. The often (surprisingly) low cost by which countries get away with defaults suggests that the latter (moral hazard) role of the option to default may also contribute to public debt buildup.[2]

[2]See Acharya and Rajan (2012) for further evidence and recent literature. These authors

My results regarding the conditions under which an economy is driven to the insolvency brink are obtained under the assumption that defaults are not feasible. Relaxing this assumption, allowing defaults, can only exacerbate the insolvency prospects (make it more likely to happen or increase the pace at which it is reached) but will not otherwise change the results. This is so because the consequences of insolvency can only be harsher without the option to default than with it. Thus, a government that drives its country to the brink when the option to default is not feasible (e.g., too costly) will certainly do so when the option to default is taken as a viable course of action in the future.

Since the aim here is to examine long run trends of public debt processes, stochastic fluctuations are ignored. This, no doubt, is an abstraction but one that allows grasping the essence of the underlying currents and focusing sharply on the role of short-sighted politicians. The next section presents the model's basic ingredients, explains why, in democratic societies, individuals tend to behave more impatiently when acting as politicians than when engaging in ordinary market transactions, and shows how this phenomenon leads to public debt swelling and insolvency. Section 3 introduces default risks and shows that the ensuing risk premium function has a self-correcting role but is unlikely to be solvency-proof. Section 4 shows that growth tends to exacerbate the insolvency prospects by motivating transfers from the (would be wealthier) future to the present. Some policy implications are discussed in the concluding section and the appendix contains technical derivations.

observe that penalties are painful mostly immediately following a default; they use this observation to explain why myopic governments, wishing to avoid the short-term consequences, tend not to default.

2 The basic setup

I begin with a simple model of government spending, where the government faces an exogenous stream of income[3] and uses it to finance its expenses, of which some are mandatory and some discretionary.[4] Let $y(t)$, $t \geq 0$, denote the discretionary part of the income stream, i.e., total income minus mandatory expenses (henceforth income and discretionary income are used interchangeably). The deterministic income flow $y(t)$ is assumed positive and fluctuates around a constant value (growth will be considered in Section 4). The value at time t of the income stream from time t onward is

$$Y(t) = \int_t^\infty y(\tau) e^{-r(\tau - t)} d\tau, \tag{2.1}$$

where r is the market interest rate facing the government, assumed constant.

The government's discretionary budget at time t is

$$b(t) = y(t) - x(t) \geq 0, \tag{2.2}$$

where $x(t) \leq y(t)$ is the surplus (if positive) or deficit (if negative). The budget $b \in \mathbb{R}_+$ generates the instantaneous utility $u : \mathbb{R}_+ \mapsto \mathbb{R}$, satisfying

$$u'(\cdot) > 0, \quad u''(\cdot) < 0, \quad \text{and} \quad \lim_{b \to \infty} u'(b) = 0. \tag{2.3}$$

The utility $u(\cdot)$ reflects the preferences of the budget decision-makers, namely

[3]The government's income can be taken as a constant share of GDP. This is consistent with Barro's (1979) constant tax prescription, resulting from the desire to smooth consumption over time when the government decides on tax rates and takes spending parametrically.

[4]See, for example, the mandatory-discretionary breakdown of USA's proposed 2013 budget in http://www.nytimes.com/interactive/2012/02/13/us/politics/2013-budget-proposal-graphic.html.

the country's polity, and accounts for households' (voters') preferences via the latter effect on elections outcomes. In a democratic society, thus, the polity's instantaneous utility will resemble that of a representative household (voter).

A budget policy $\{x(t) \leq y(t),\, t \geq 0\}$, or equivalently $\{b(t) \geq 0,\, t \geq 0\}$, generates the payoff

$$\int_0^\infty u(b(t))e^{-\rho t}dt, \tag{2.4}$$

where ρ is the polity's time rate of discount (impatience). Unlike $u(\cdot)$, which (for reasons mentioned above) is likely to represent households' preferences, there are good reasons for ρ to be substantially higher than the discount rate of (most) market participants. This is so because ρ is inversely related to the polity's planning horizon (i.e., the time length expected to remain in office) and this horizon extends more or less till next election, which in advance democracies is substantially shorter than the horizon of ordinary households.

The relation between ρ and the polity's planning horizon can be demonstrated in the context of the following simple setting (Yaari 1965). Let T be the polity's random time horizon (the time remaining in office) and suppose that the probability of not surviving beyond $[t, t+\Delta]$ given survival up to time t is $\tilde{\rho}\Delta = Pr\{T \leq t+\Delta | T > t\} = \frac{f_T(t)\Delta}{1-F_T(t)} + o(\Delta)$, where $F_T(t)$ and $f_T(t) = F_T'(t)$ are the distribution and density functions of T, respectively, and $o(\Delta)$ is a term that approaches zero faster than Δ (i.e., $\lim_{\Delta\to 0} \frac{o(\Delta)}{\Delta} = 0$). Thus, $\tilde{\rho} = -d\ln(1-F_T(t))/dt$, giving $F_T(t) = 1 - e^{-\tilde{\rho}t}$. Assuming that the polity's utility is $u(b(t))$ while in office and zero otherwise, its objective is

$$E\left\{\int_0^T u(b(t))e^{-\rho_0 t}dt\right\},$$

where E is expectation with respect to T and $\rho_0 \geq 0$ represents innate (natural) impatience common to politicians and ordinary citizens (households) alike. Taking the expectation, using $F_T(t) = 1 - e^{-\tilde{\rho}t}$, gives

$$E\left\{\int_0^T u(b(t))e^{-\rho_0 t}dt\right\} = \int_0^\infty u(b(t))e^{-\rho t}dt,$$

where $\rho = \tilde{\rho} + \rho_0$. Since $\tilde{\rho} = 1/E\{T\}$, the polity's impatience ρ is inversely related to its expected time in office: the shorter it expects to stay in office, the more impatient (i.e., higher ρ) it becomes. Where elections occur more frequently, the span of the polity's planning horizon becomes accordingly shorter and the ensuing discount rate higher.

The budget surpluses/deficits accumulate to form the outstanding debt $D(t)$ and the latter evolves in time according to

$$\dot{D}(t) = rD(t) - x(t), \tag{2.5}$$

I consider the case were the debt servicing policy pertains to total debt (domestic and external).[5]

A budget deficit (when $x(t) < 0$) requires borrowing and the government

[5]Although governments have a salient motive to prefer domestic over foreign creditors (e.g., because domestic creditors vote whereas foreigners do not), separating the two is not always possible. In developed economies, with a sophisticated and (internationally) integrated financial sector, domestic and foreign owners of government debt cannot be easily identified (see, e.g., Guembel and Sussman 2009, Broner et al. 2010) and this feature hampers the ability of governments to treat external and domestic debts separately. For emerging economies, with limited international exposure, letting $D(t)$ stand for sovereign debt, might be more appropriate. In the latter case, the cost of servicing the domestic debt becomes part of the mandatory expense (that have been subtracted from government's income at the outset) and is not included in the discretionary income.

can borrow freely at the going rate r as long as

$$D(t) \leq Y(t). \tag{2.6}$$

If (2.6) is violated, the country's net worth

$$W(t) = Y(t) - D(t) \tag{2.7}$$

is negative. Now, $W(t)$ satisfies

$$W(t+s) \leq e^{rs} W(t) \quad \text{for all} \quad s \geq 0, \tag{2.8}$$

equality holding if the entire income is used to service the debt from time t onward (i.e., $x(t+\tau) = y(t+\tau)$ for all $\tau \geq 0$).[6] Thus, a negative net worth today implies that future net worths will become ever more negative and the government will not be able to pay the interest on its debt, let alone the principal, even when its entire income is allocated to service the debt now and forever. Under such circumstances, borrowing becomes impossible, as no (private) lender will be willing to lend at any rate.

The country, thus, becomes insolvent when $W(t) = Y(t) - D(t) = 0$ and I refer to (2.6) as the insolvency constraint. Without defaulting on some of the debt (unilaterally or by consent), (2.8) implies that the insolvent state $W(t) = 0$ is trapping, in that debt cannot be reduced below $Y(t)$ and the entire income is doomed to service the debt forever (i.e., $b(t+\tau) = 0$ for

[6] To verify (2.8), integrate (2.5) from t to $t+s$ to obtain $D(t+s)e^{-r(t+s)} - D(t)e^{-rt} = -\int_t^{t+s} x(\tau)e^{-r\tau}d\tau \geq -\int_t^{t+s} y(\tau)e^{-r\tau}d\tau$, where the inequality follows from $x(\tau) \leq y(\tau)$. Noting (2.1), $\int_t^{t+s} y(\tau)e^{-r\tau}d\tau = e^{-rt}\left[Y(t) - e^{-rs}Y(t+s)\right]$ and the inequality can be expressed as $e^{-rs}D(t+s) - D(t) \geq e^{-rs}Y(t+s) - Y(t)$, which noting (2.7) gives (2.8).

all $\tau \geq 0$). The loss of the option to borrow and the trapping property should deter governments from reaching insolvency. Suppose that defaults are not feasible, so the insolvency state is trapping. Will a (currently perfectly solvent) government reach insolvency as a planned outcome? The answer, it is shown below, is in the affirmative. The reason is the short-sightedness of governments which, far from being coincidental, is an inherent attribute of democracy.

A negative debt occurs when the accumulated surpluses exceed (in current value) the accumulated deficits, in which case the country is a net lender. However, lending cannot be extended without limit, since the demand for credit (domestic and global) is finite. The lending constraint is imposed in terms of the country's net worth as

$$W(t) = Y(t) - D(t) \leq \bar{W}, \tag{2.9}$$

where $\bar{W}$ is possibly very large but finite. The bound $\bar{W}$ is the excessive saving limit.

A budget policy is feasible if $x(t) \leq y(t)$, or equivalently $b(t) \geq 0$, and $D(t) \in [Y(t) - \bar{W}, Y(t)]$ for all $t \geq 0$. The optimal policy is the feasible policy that maximizes (2.4) subject to (2.5), given $D(0) < Y(0)$. Let $D^*(t)$, $t \geq 0$, represent the optimal debt process and $W^*(t) = Y(t) - D^*(t)$ the corresponding net worth trajectory. Define

$$\hat{W} = \begin{cases} \bar{W} & \text{if } \rho - r < 0 \\ W(0) = Y(0) - D(0) & \text{if } \rho - r = 0 \\ 0 & \text{if } \rho - r > 0 \end{cases} . \tag{2.10}$$

Then:

Proposition 1. *Suppose (2.3) holds. Then: (i) $W^*(t)$ converges monotonically to a steady state at $\hat{W}$ from any initial state $W(0) \in (0, \bar{W}]$. (ii) If $\rho < r$, the steady state $\bar{W}$ will be reached at a finite time. (iii) If $\rho = r$, the steady state is entered instantly (at the initial time) and the optimal policy is to maintain a constant budget $b = rW(0)$. (iv) If $\rho > r$, the country is doomed to become insolvent ($\hat{W} = 0$) and this (unfortunate) state of affairs will occur at a finite time or asymptotically (as $t \to \infty$), depending one whether $u'(0)$ is finite or infinite, respectively.*

The proof is presented Appendix B.

The interest rate r reflects the time preferences of market participants (Ramsey 1928), i.e., $r = \rho_0$, where ρ_0 was defined above as an innate impatience or the utility discount rate of a representative household. A case can be made for a benevolent polity that freely chooses the discount rate to set $\rho = r$. The optimal policy in this case is to maintain a constant budget $b(t) = b = rW(0) = r[Y(0) - D(0)]$. Thus, recalling that $x(t) = y(t) - b(t)$, when income is low (during recession periods), $x(t)$ is appropriately reduced by borrowing and running a budget deficit, and during boom periods a surplus $x(t) > 0$ occurs. This property stems from the diminishing marginal utility of budget, which operates to smooth out the budget trajectory over time.

The case $\rho > r$ occurs when the polity's time horizon is shorter than the planning horizon of (most) market participants. In this case the country is doomed to become insolvent. Moreover, when $u'(0)$ is finite (a likely property given that the income is net of the mandatory spending), insolvency will occur at a finite time. Avoiding this outcome requires some form of external con-

straints on government spending. Examples include the Stability and Growth Pact, which specifies limits on deficits and public debts for the 27 member states of the European Union, and the United States' "ceiling" on public debt.

3 Default risk

In actual practice, a country will encounter difficulty borrowing at the riskless rate r long before it becomes insolvent. As soon as potential lenders begin to doubt the country's ability and determination to service its debt, they will demand a risk premium to balance out their concerns. Consequently, the interest rate at which the government borrows includes a risk premium that depends on the debt-income ratio. To allow a sharp focus on the effects of default risk, a constant income stream $y(t) = y$ is assumed.[7] Normalizing y to unity implies that the budget $b(t)$, the surplus/deficit $x(t)$ and the ensuing debt $D(t)$ are all measured as income shares (in this section, "debt" and "debt-income ratio" are used interchangeably).

Governments borrow by issuing bonds at prices that vary with the bonds' characteristics. A detailed account of such term structure is beside the present scope. It is expedient in the present context to consider the case where the government constantly recycles debt by issuing short-term bonds whose price includes a risk premium, denoted h, that depends on the debt-income ratio. As long as the debt does not exceed some critical income share, no default risk is perceived (by potential lenders) and $h = 0$. As debt increases above this threshold, the risk of default becomes real and h increases at an increasing rate. Without loss of generality, the threshold debt above which the risk

[7] This is equivalent to assuming that a-priori (independent of the budget policy) income is stabilized at the annual-equivalent flow $y = rY(0)$.

premium is positive is assumed zero.[8] Thus, $h(\cdot)$ satisfies:

$$h(D) = 0 \text{ for } D \leq 0; \quad h'(D) > 0 \text{ and } h''(D) > 0 \text{ for } D > 0. \tag{3.1}$$

The function $h(\cdot)$ varies from country to country and reflects (potential lenders) beliefs regarding the government ability and determination to service its debt.

The interest cost of a debt D is $[r + h(D)]D$ and debt evolves in time according to

$$\dot{D}(t) = [r + h(D(t)]D(t) - x(t). \tag{3.2}$$

Let $\bar{D}$ be the debt level satisfying

$$[r + h(\bar{D})]\bar{D} = y = 1 \tag{3.3}$$

(recall the normalization $y = 1$). Since $x(t) \leq 1$, at debt level $\bar{D}$, the interest cost $[r + h(\bar{D})]\bar{D}$ consumes the entire income and any increase in debt above $\bar{D}$ implies that the debt will increase indefinitely (the right-hand side of (3.2) remains positive even when $x(t) = 1$), in which case the government becomes insolvent. In actual practice, as soon as debt reaches $\bar{D}$, borrowing becomes impossible (no potential lender will be found), implying that, barring a default,

$$D(t) \leq \bar{D}. \tag{3.4}$$

The insolvency bound $\bar{D}$ is trapping, in that once the debt-income ratio reaches $\bar{D}$, without defaults, the country is doomed to allocate its entire income to cover the interest cost now and forever. Facing this situation, the gov-

[8] This assumption simplifies notation and can be relaxed, by allowing a positive threshold, without any effect on the nature of the results.

ernment may default, provided the consequences (penalties, trade sanctions, reputation loss, use of military force) are not too harsh. Suppose defaults are not feasible. Can a perfectly solvent government reach insolvency while acting "optimally" when it is fully aware of the consequences (that there is no way out of insolvency and instant gratifications are obtained at the expense of posterity's well-being)? As in the previous case, the answer is in the affirmative. All it takes is that the polity's impatience rate ρ exceeds the risk-adjusted interest rate $r + \psi(\bar{D})$, where

$$\psi(D) = h(D) + h'(D)D \tag{3.5}$$

is the marginal cost of the default risk. If this is the case when defaults are not feasible (e.g., because their cost is prohibitive), if anything, governments will be less reluctant to reach insolvency when defaults are not ruled out at the outset.

A negative debt means that the country is a net lender and the lower bound $\underline{D} = (Y - \bar{W})/y \leq 0$ applies (see discussion of $\bar{W}$ below (2.9)), i.e.,

$$D(t) \geq \underline{D}\,. \tag{3.6}$$

The budget management problem can be formulated as

$$\max_{x(t)\leq 1} \int_0^\infty u(1 - x(t))e^{-\rho t}dt \tag{3.7}$$

subject to (3.2) and $D(t) \in [\underline{D}, \bar{D}]$, given $D(0) \in [\underline{D}, \bar{D})$.

Define

$$\hat{D} = \begin{cases} \underline{D} & \text{if } \rho < r \\ \min(D(0), 0) & \text{if } \rho = r \\ \psi^{-1}(\rho - r) & \text{if } r < \rho \leq r + \psi(\bar{D}) \\ \bar{D} & \text{if } \rho > r + \psi(\bar{D}) \end{cases} \tag{3.8}$$

where $D(0)$ is the initial debt (notice that the equation $\psi(D) = \rho - r$ admits a unique solution $\psi^{-1}(\rho - r) \in (0, \bar{D}]$ when $0 < \rho - r \leq \psi(\bar{D})$). The optimal debt process $D^*(t)$ is characterized in:

Proposition 2. *Suppose (2.3) and (3.1) hold. Then: (i) $D^*(t)$ converges monotonically to a steady state at $\hat{D}$ from any initial debt $D(0) \in [\underline{D}, \bar{D})$. (ii) If $\rho < r$, the steady state $\hat{D} = \underline{D}$ will be reached at a finite time. (iii) If $\rho = r$ then: if $D(0) \leq 0$, the steady state $\hat{D} = D(0)$ is entered instantly; if $D(0) > 0$, the steady state $\hat{D} = 0$ will be reached asymptotically (as $t \to \infty$). (iv) If $r < \rho \leq r + \psi(\bar{D})$, the steady state $\hat{D} = \psi^{-1}(\rho - r) \in (0, \bar{D}]$ will be reached asymptotically. (v) If $\rho > r + \psi(\bar{D})$, the country is doomed to become insolvent ($\hat{D} = \bar{D}$) and this (unfortunate) state of affairs will occur at a finite time or asymptotically as $u'(0)$ is finite or infinite, respectively.*

The proof is presented in Appendix C.

As expected, default risk plays no role when $\rho < r$, in which case debt is negative (or will eventually turn negative) and the risk premium vanishes. When $\rho \geq r$, the risk effects are pronounced. Consider first the case $\rho = r$, where the polity's impatience ρ coincides with the riskless market rate r. In this case, no debt with a positive risk (i.e., $h(D) > 0$) will prevail in the long-run. If the initial debt is nonnegative, the optimal policy is to retain debt at its initial level (by running a balance budget). If the initial debt $D(0)$ is

positive, debt will be gradually reduced, approaching zero asymptotically (as $t \to \infty$).

When $\rho > r$, the equilibrium debt is positive, but as long as $\rho < r + \psi(\bar{D})$, debt will not reach the insolvency level $\bar{D}$. If $\rho \geq r + \psi(\bar{D})$, insolvency is inevitable (the equilibrium debt equals $\bar{D}$) and will occur at a finite time if the inequality is strong ($\rho > r + \psi(\bar{D})$) and $u'(0)$ is finite (since the discretionary income is net of the mandatory expenses, a finite $u'(0)$ is plausible).

Comparing with the results of Section 2, it is seen that the risk premium function $h(\cdot)$ mitigates the tendency of politicians to drive their country to the brink, in that the critical polity's impatience rate ρ above which the country will sooner or later become insolvent is higher with risk premium than without. As a market phenomenon, the risk premium is thus a self-correcting mechanism, though far from being solvency-proof, as frequent episodes of bailout-seeking countries reveal.

4 Growth

I turn now to examine effects of exogenous economic growth on public debt buildup and insolvency prospects.[9] Suppose the economy grows at a constant rate $g > 0$:

$$y(t) = e^{gt}, \tag{4.1}$$

where the normalization $y(0) = 1$ is used. Total debt is $D(t)$ and

$$d(t) = D(t)e^{-gt} \tag{4.2}$$

[9]There may also be a causality link running from debt to growth (see Reinhart and Rogoff 2010), which is not considered here.

is the debt-income ratio. Likewise, $b(t)$ and $x(t)$ represent, respectively, the budget and surplus/deficit at time t, expressed as income shares, so the total budget is $b(t)e^{gt}$, the total suprlus/deficit is $x(t)e^{gt}$ and $b(t) = 1 - x(t)$.

The risk premium function $h(\cdot)$ is the same as in the stationary case (specified in (3.1)), with the debt-income ratio d as its argument:

$$h(d) = 0 \text{ for } d \leq 0; \; h'(d) > 0 \text{ and } h''(d) > 0 \text{ for } d > 0. \tag{4.3}$$

The interest cost associated with a (total) debt $D(t)$ is $[r_g + h(d(t))]D(t)$ and $D(t)$ evolves in time according to

$$\dot{D}(t) = [r_g + h(d(t))]D(t) - x(t)e^{gt}, \tag{4.4}$$

where r_g is the (riskless) interest rate (which may differ from its stationary economy counterpart r, hence the subscript g). Differentiating (4.2) with respect to time, using (4.4), gives

$$\dot{d}(t) = [r_g - g + h(d(t))]d(t) - x(t). \tag{4.5}$$

Noting (4.5), the insolvent debt-income ratio $\bar{d}$ is defined by

$$[r_g - g + h(\bar{d})]\bar{d} = 1. \tag{4.6}$$

Since $x(t) \leq 1$, when $d(t) = \bar{d}$, the interest payments consume the entire income and any debt above $\bar{d}$ will increase without bound. The debt level $\bar{d}$ is trapping, in that it cannot be reduced (without the government defaulting

on some of its debt). Thus, barring defaults,

$$d(t) \leq \bar{d}. \tag{4.7}$$

A negative d occurs when the country becomes a net lender. In a growing global economy, potential lending increases at the rate g and equals $\underline{D}e^{gt}$, where $\underline{D} = (Y - \bar{W})/y$ (see equation (3.6)). Thus,

$$d(t) \geq \underline{d} = \underline{D}. \tag{4.8}$$

The utility flow generated by the budget $(1 - x(t))e^{gt}$ takes the form

$$\frac{((\zeta + 1 - x(t))e^{gt})^{1-\eta} - 1}{1 - \eta} = \frac{(\zeta + 1 - x(t))^{1-\eta} e^{-(\eta-1)gt} - 1}{1 - \eta}$$

where $\eta > 0$ is the elasticity of marginal utility (or the inverse of the intertemporal elasticity of substitution) and $\zeta \geq 0$ is a nonnegative parameter (the logarithmic form is used when $\eta = 1$). The present value of the utility flow is

$$\int_0^\infty \frac{(\zeta + 1 - x(t))^{1-\eta}}{1 - \eta} e^{-(\rho+(\eta-1)g)t} - \frac{1}{(1 - \eta)\rho} ,$$

which, using

$$u(b) = \frac{(\zeta + b)^{1-\eta}}{1 - \eta} \tag{4.9}$$

and ignoring the constant term, can be expressed as

$$\int_0^\infty u(1 - x(t))e^{-(\rho+(\eta-1)g)t}. \tag{4.10}$$

A feasible budget policy satisfies $x(t) \leq 1$ at all times. The optimal policy

is the feasible policy that maximizes (4.10) subject to (4.5) and $d(t) \in [\underline{d}, \bar{d}]$, given $d(0) \in [\underline{d}, \bar{d})$.

The marginal cost of risk corresponding to $d(t)$ is

$$\psi(d) = h(d) + h'(d)d, \tag{4.11}$$

and $\hat{d}$ is defined by:

$$\hat{d} = \begin{cases} \underline{d} & \text{if } \rho < r_g - \eta g \\ \min(d(0), 0) & \text{if } \rho = r_g - \eta g \\ \psi^{-1}(\rho + \eta g - r_g) & \text{if } r_g - \eta g < \rho \leq r_g - \eta g + \psi(\bar{d}) \\ \bar{d} & \text{if } \rho > r_g - \eta g + \psi(\bar{d}) \end{cases} \tag{4.12}$$

(note that $\psi^{-1}(\rho + \eta g - r_g) \in (0, \bar{d}]$ when $0 < \rho + \eta g - r_g \leq \psi_g(\bar{d})$).

The optimal debt process $d^*(t)$ is characterized in:

Proposition 3. *Suppose (4.3) and (4.9) hold. Then: (i) $d^*(t)$ converges monotonically to a steady state at $\hat{d}$ from any initial debt $d(0) \in [\underline{d}, \bar{d})$. (ii) If $\rho < r_g - \eta g$, the steady state $\hat{d} = \underline{d}$ will be reached at a finite time. (iii) If $\rho = r_g - \eta g$ then: if $d(0) \leq 0$, the steady state $\hat{d} = d(0)$ is entered instantly; if $d(0) > 0$, the steady state $\hat{d} = 0$ will be reached asymptotically (as $t \to \infty$). (iv) If $r_g - \eta g < \rho \leq r_g - \eta g + \psi(\bar{d})$, the steady state $\hat{d} = \psi^{-1}(\rho + \eta g - r_g) \in (0, \bar{d}]$ will be reached asymptotically. (v) If $\rho > r_g - \eta g + \psi(\bar{d})$, the country is doomed to become insolvent ($\hat{d} = \bar{d}$) and this (unfortunate) state of affairs will occur at a finite time when $\zeta > 0$ (i.e., $u'(0)$ is finite).*

The proof is presented in Appendix D.

Comparing with the the results of the stationary economy (Section 3), economic growth affects insolvency prospects in two ways. First it changes the (riskless) interest rate from r to r_g. Second, it changes the condition characterizing the equilibrium debt-income ratio $\hat{d}$ (note in particular the conditions leading to insolvency, where $\hat{d} = \bar{d}$). The equilibrium interest rate satisfies (Ramsey 1928) $r_g = \rho_0 + \eta g$, thus $r_g = r + \eta g$, where r is the equilibrium interest rate in a stationary economy ($g = 0$) and ρ_0 is the representative agent's utility discount rate (discussed in Section 2). Substituting $r_g = r + \eta g$ in (4.6) and comparing with (3.3), one verifies that $\bar{d} < \bar{D}$ if $\eta > 1$, $\bar{d} = \bar{D}$ if $\eta = 1$ and $\bar{d} > \bar{D}$ if $\eta < 1$.

A stationary economy is doomed for insolvency when $\rho > r + \psi(\bar{D})$ (Proposition 2(v)). The corresponding condition for a growing economy, according to Proposition 3(v), is $\rho > r_g - \eta g + \psi(\bar{d}) = r + \psi(\bar{d})$. Thus, the effect of growth on a country's insolvency prospects boils down to the relation between $\psi(\bar{D})$ and $\psi(\bar{d})$. In particular, $\eta > 1$ implies $\bar{d} < \bar{D}$ and $\psi(\bar{d}) < \psi(\bar{D})$, in which case growth exacerbates the insolvency prospects by lowering the upper bound on ρ above which insolvency is inevitable. When $r + \psi(\bar{d}) < \rho < r + \psi(\bar{D})$, the same polity will drive a growing economy ($g > 0$) to insolvency but will retain a stationary economy ($g = 0$) perfectly solvent. This situation is depicted in Figure 1.

The explanation for this result rests on the role of η as a measure of aversion to intergenerational inequality: The introduction of growth means that future generations will be richer, and high aversion to intergenerational inequality ($\eta > 1$) induces redistribution from (wealthier) future generations to the present; such a redistribution, which takes the form of borrowing, pushes an economy further towards insolvency. The magnitude of η is a subtle (and

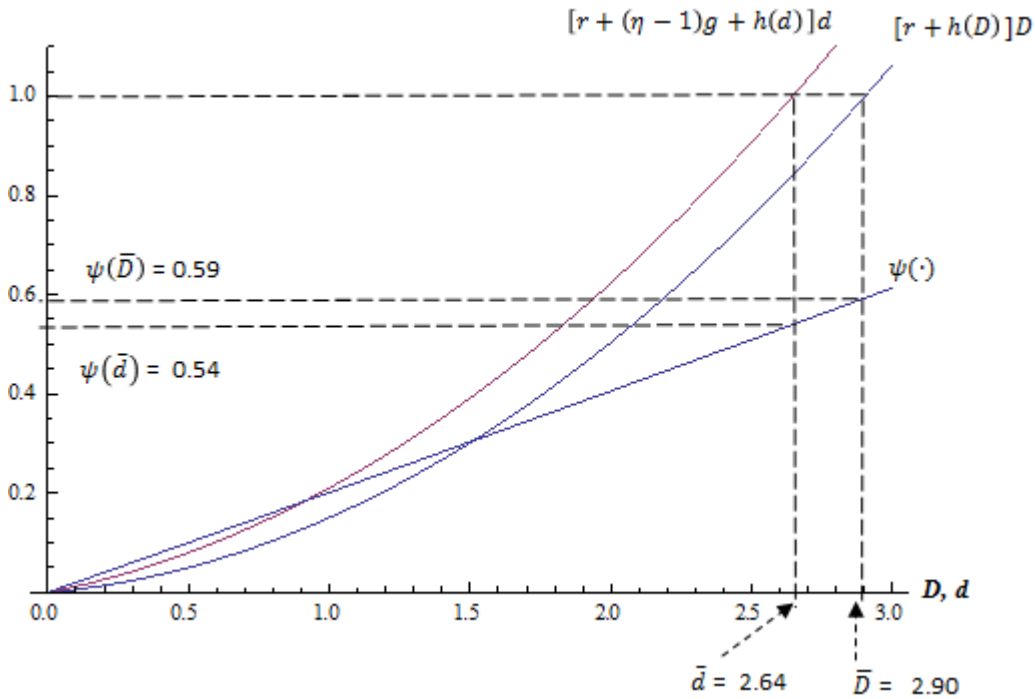

Figure 1: The curves $[r+h(D)]D$ and $[r+(\eta-1)g+h(d)]d$ correspond to stationary and growing economies, respectively, with an exponential $h(\cdot)$ and parameters $r = 0.05$, $\eta = 4$ and $g = 0.02$. $\bar{D}$ and $\bar{d}$ solve, respectively, $[r + h(\bar{D})]\bar{D} = 1$ and $[r + (\eta - 1)g + h(\bar{d})]\bar{d} = 1$, giving $\bar{D} = 2.90 > \bar{d} = 2.64$ and $\psi(\bar{D}) = 0.59 > \psi(\bar{d}) = 0.54$. A polity whose impatience ρ falls between $r + \psi(\bar{d})$ and $r + \psi(\bar{D})$, i.e., $0.59 < \rho < 0.64$, would drive the growing economy to insolvency at a finite time but retain the stationary economy perfectly solvent.

contested) issue (see Stern 2008, and references cited therein); empirical evidence suggests $\eta > 1$ (Hall 1988).

5 Discussion and conclusion

The tendency of advanced democracies to accumulate excessive debt is attributed (at least partly) to the high impatient rate of politicians. This feature implies that the politicians' (budget decision makers') time rate of discount

exceeds the interest rate at which the government borrows and this discrepancy, it is shown, induces public debt swelling and gives rise to insolvency when it exceeds a certain threshold. Moreover, often economic growth exacerbates the debt swelling problem and makes insolvency more pervasive. Far from being coincidental, a discrepancy between politicians' discount rates and market interest rates is an inherent attribute of democracy, where the former is inversely related to the length of the period a government expects to remain in office and these periods could be quite short.

A popular remedy entails rules (in the form of norms, laws, constitutional amendments) to restrain deficits and debt buildup as discussed, e.g., in Buchanan and Wagner (1977, Chapters 10-12). Examples include the Stability and Growth Pact, specifying limits on deficits and public debts for the 27 member states of the European Union, or the ceiling on public debt in the United States. The former has originally set deficit and debt limits at 3 percent and 60 percent of GDP, respectively, and was latter updated; the latter has recently been updated last Summer. This approach seems to work well while the restraining rules are not binding. As soon as the limits begin to bite, the current polity has a tendency to relax them (particularly if it is not the one that imposed the rules in the first place). This is an example of the "rules-rather-than-discretion" dilemma (Kydland and Prescott 1977), where optimal policies are inconsistent and consistent policies are suboptimal. In the present context, external ex-ante rules are constantly updated ex-post by short-sighted politicians. An upper bound on deficits is particularly tempting to update when the constraint is binding, since this occurs at the wrong time – when the economy suffers (in a recession) and badly needs oxygen (budget infusion). Perhaps a policy that sets a lower bound on the surplus when the

economy thrives might be less challenging, as it is easier to restrict spending during booms than during busts.

Given that the source of the problem is the short sightedness of politicians induced by democracy's "rules-of-the-game," possible remedies could exploit the advantages of democracy in transparency and information disclosure to mitigate this shortcoming. Legislations with detrimental impacts on future generations could be evaluated and disclosed to the general public by an independent, unbiased agency, inducing lawmakers to reconsider before they cast their vote.[10]

Finally, where the short-horizon feature is potentially lethal, more authority should be delegated to independent, professional civil servants with longer time perspectives. A prime example is the delegation of the authority to set monetary policy to an a-political, professional agency – the central bank. Of course, fiscal policy is arguably the most pronounced manifestation of political priorities and should be determined by elected politicians, but professional civil servants should have the authority to impose external limits in certain, pre-specified circumstances.

[10]In 2001, Israel's Knesset (parliament) formed such an agency, called the Posterity's Commission ('*Netzivut Ha'dorot Ha'baim*'), whose role was to monitor impacts on future generations of legislative processes. Regretfully, the commission was abolished in 2010.

Appendix

A A useful property

The proofs of the propositions will make use of the following (useful) property. Consider the infinite-horizon, autonomous problem

$$\max_{q(t)\in\mathcal{A}\subset\mathbb{R}} \int_0^\infty f(Q(t), q(t))e^{-\rho t}dt$$

subject to

$$\dot{Q}(t) = g(Q(t), q(t))$$

and $\underline{Q} \leq Q(t) \leq \bar{Q}$, given $Q(0) \in (\underline{Q}, \bar{Q})$. It is assumed that an optimal solution (not necessarily unique) exists. Suppose there exists a function $M : [\underline{Q}, \bar{Q}] \mapsto \mathcal{A}$, satisfying

$$g(Q, M(Q)) = 0 \; \forall \, Q \in [\underline{Q}, \bar{Q}].$$

Let $V(Q) \equiv f(Q, M(Q))/\rho$ and define $L : [\underline{Q}, \bar{Q}] \mapsto \mathbb{R}$ by

$$L(Q) = \frac{f_q(Q, M(Q))}{g_q(Q, M(Q))} + V'(Q), \tag{A.1}$$

where subscript q indicates partial derivative with respect to q. Then:

Property 1. (i) *The optimal* $Q(t)$ *process converges monotonically to a steady state* $\hat{Q} \in [\underline{Q}, \bar{Q}]$ *from any initial* $Q(0) \in (\underline{Q}, \bar{Q})$. (ii) *When* $L(\cdot)$ *is non-*

increasing:

$$\hat{Q} = \begin{cases} \bar{Q} & \textit{if } L(\bar{Q}) > 0 \\ L^{-1}(0) & \textit{if } \underline{Q} \leq L^{-1}(0) \leq \bar{Q} \textit{ and } L(\cdot) \textit{ is decreasing} \\ \underline{Q} & \textit{if } L(\underline{Q}) < 0 \end{cases} \cdot \tag{A.2}$$

Proof. The proof is based on the observation that optimal state trajectories of (a single state) infinite-horizon, autonomous problems are monotonic. Thus, if the state is bounded, the optimal path must converge to a steady state (see details in Tsur and Zemel 2001, pp. 484-485). □

B Proof of Proposition 1

Proof. (i): Equations (2.1), (2.2) and (2.5) give

$$\dot{W}(t) = rW(t) - b(t) \tag{B.1}$$

and the budget problem can be reformulated as

$$\max_{b(t)\geq 0} \int_0^\infty u(b(t))e^{-\rho t}dt \tag{B.2}$$

subject to (B.1) and $W(t) \in [0, \bar{W}]$, given $W(0) > 0$. This is an infinite-horizon, autonomous problem with a bounded state, thus, according to Property 1, $W^*(t)$ converges to a steady state. To show that $W^*(t)$ converges to $\hat{W}$, defined in (2.10), note that (A.1) specializes in this case to

$$L(W) = u'(rW)\frac{r-\rho}{\rho}. \tag{B.3}$$

Thus, noting (A.2), the optimal steady state is $\bar{W}$ or 0 as $\rho - r < 0$ or $\rho - r > 0$, respectively. The case $\rho = r$ requires some care, since $L(W) = 0$ and equation (A.2) does not identify the steady state. It is verified in (iii) below that $\hat{W} = W(0)$.

The proofs of $(ii) - (iv)$ will benefit from the following reformulation of the budget problem. Given that $W^*(t)$ converges to a steady state, the budget problem can be reformulated as

$$\max_{\{T, b(t) \geq 0\}} \int_0^T u(b(t))e^{-\rho t} + e^{-\rho T} u(\hat{b})/\rho$$

subject to (B.1) and $W(t) \in [0, \bar{W}]$, given $W(0) \in (0, \bar{W}]$, where T is the steady state entrance time and

$$\hat{b} = rW(T) \tag{B.4}$$

is the steady state budget. The current-value Hamiltonian for this problem is

$$H(t) = u(b(t)) + \mu(t)(rW(t) - b(t)),$$

where $\mu(t)$ is the current-value costate variable, and necessary conditions for (interior) optimum include

$$u'(b(t)) = \mu(t), \tag{B.5}$$

$\dot{\mu}(t) - \rho\mu(t) = -r\mu(t)$, giving

$$\mu(t) = \mu_0 e^{(\rho - r)t}, \tag{B.6}$$

and the transversality condition

$$e^{-\rho T}[H(T) - u(\hat{b})] = 0. \tag{B.7}$$

(ii) Suppose $\rho < r$. Then, according to part (i), the optimal $W(t)$ process converges monotonically to $\bar{W}$. To show that $\bar{W}$ must be reached at a finite time, suppose otherwise (that it will be approached asymptotically as $t \to \infty$). Then, (2.3) and (B.5)-(B.6) imply that $b(t) \to \infty$ and there exists some finite time t such that $b(t+\tau) > r\bar{W} \geq rW(t+\tau)$ for all $\tau \geq 0$. Thus, noting (B.1), from time t onward, $W(t)$ decreases, violating part (i). It is concluded that the steady state $\bar{W}$ must be reached at a finite time.

With a finite T, the transversality condition (B.7) requires $b(T) = \hat{b} = r\bar{W}$, implying, noting (B.5)-(B.6), $u'(r\bar{W}) = \mu_0 e^{(\rho-r)T}$ or

$$r\bar{W} = u'^{-1}\left(\mu_0 e^{(\rho-r)T}\right). \tag{B.8}$$

Integrating (B.1) from 0 to T, using (B.5)-(B.6) and $W(T) = \bar{W}$, gives

$$\bar{W}e^{-rT} = W(0) - \int_0^T u'^{-1}\left(\mu_0 e^{(\rho-r)t}\right) e^{-rt} dt. \tag{B.9}$$

The parameters T and μ_0 are obtained from conditions (B.8)-(B.9).

(iii) When $\rho = r$, conditions (B.5)-(B.6) imply that $\mu(t) = \mu_0$ and $b^*(t) = b^*,\ t \in [0, T]$, where b^* is constant. Suppose $b^* \neq rW(0)$. Setting $b^* < rW(0)$ implies, noting (B.1), that $W^*(t) = e^{rt}(rW(0) - b^*)/r + b^*/r$ increases and will reach $\bar{W}$ at a finite time. Likewise, when $b^* > rW(0)$, $W^*(t)$ decreases and will reach zero at a finite time. In either case, condition (B.7) implies

$H(T) - u(\hat{b}) = 0$ or

$$u(b^*) + u'(b^*)[rW(T) - b^*] - u(\hat{b}) = 0, \tag{B.10}$$

where

$$\hat{b} = \begin{cases} 0 & \text{if } W(T) = 0 \\ r\bar{W} & \text{if } W(T) = \bar{W} \end{cases}. \tag{B.11}$$

Now, when $b^* > rW(0)$, $W(T) = 0$ and $b^* > \hat{b} = 0$; when $b^* < rW(0)$, $W(T) = \bar{W}$ and $b^* < \hat{b} = r\bar{W}$. In both cases (B.10) is violated due to the strict concavity of $u(\cdot)$. Thus, $b^* \neq rW(0)$ cannot be optimal, implying that $b^* = rW(0)$ is the optimal policy, i.e., the steady state is entered instantly as claimed.

(iv) Suppose $\rho > r$. Noting (B.3), $L(W) < 0$ for all $W \in [0, \bar{W}]$ and Property 1 implies that the optimal $W(t)$ process converges monotonically to the insolvent state $W = 0$. Conditions (B.5)-(B.6) imply then that the insolvent steady state will be reached at a finite time or asymptotically (as $t \to \infty$) depending on whether $u'(0)$ is finite or infinite, respectively. If $u'(0)$ is finite, T and μ_0 are determined by the conditions

$$\int_0^T u'^{-1}\left(\mu_0 e^{(\rho-r)t}\right) e^{-rt} dt = W(0) \tag{B.12}$$

and

$$u'^{-1}\left(\mu_0 e^{(\rho-r)T}\right) = 0. \tag{B.13}$$

If $u'(0)$ is infinite, $T = \infty$ and μ_0 is determined by condition (B.12). $\square$

C Proof of Proposition 2

Proof. (i) The budget Problem (3.7) is an infinite-horizon, autonomous problem with a bounded state, thus the optimal state trajectory $D^*(t)$, according to Property 1(i), converges to a steady state in $[\underline{D}, \bar{D}]$. I show that this steady state equals $\hat{D}$, defined in (3.8). With D and x corresponding, respectively, to Q and q of Property 1, and the functions $u(1-x)$ and $[r+h(D)]D - x$ corresponding, respectively, to $f(Q,q)$ and $g(Q,q)$ of Property 1, the $L(\cdot)$ function corresponding to problem (3.7) is

$$u'(1-[r+h(D)]D)[\rho - r - \psi(D)].$$

Since $u'(1-[r+h(D)]D) > 0$ for all $D \in [\underline{D}, \bar{D}]$, Property 1 can be applied with

$$L(D) = \rho - r - \psi(D).$$

Suppose $\rho < r$. Then, $L(D) < 0$ for all $D \in [\underline{D}, \bar{D}]$ and equation (A.2) implies that the optimal steady state is $\hat{D} = \underline{D}$, in agreement with (3.8).

Suppose $\rho = r$. Then, since $L(\cdot)$ is decreasing and negative over $(0, \bar{D}]$, the steady state cannot fall in this interval (otherwise (A.2) is violated), implying that $[\underline{D}, 0]$ is the interval of feasible steady states. Thus, if $D(0) \leq 0$, it is known from the outset that $D^*(t)$ will remain in the interval $[\underline{D}, 0]$, over which $h(D)$ vanishes. The problem, then, reduces to a riskless problem (with a zero risk premium) and the proof of Proposition 1(iii) can be repeated (with the obvious modifications) to show that $\hat{D} = D(0)$ and the steady state is entered instantly. If $D(0) > 0$, then $D^*(t)$ must decrease monotonically in order to exit the interval $(0, \bar{D}]$ (which does contain the steady state), eventually

reaching zero. As soon as as debt equals zero, the above argument implies that the steady state has been entered.

Suppose $r < \rho \leq r + \psi(\bar{D})$. Since $L(D) = \rho - r - \psi(D) > 0$ for $D \in [\underline{D}, 0]$, Property 1(ii) rules out the possibility that the optimal steady state falls in $(\underline{D}, 0]$. Thus, the monotonicity property ensures that $D^*(t)$ enters $(0, \bar{D}]$ (which happens instantly if $D(0) > 0$) and remains in this interval. Applying Property 1 to Problem (3.7) with the state D restricted to lie in $[0, \bar{D}]$ gives $\hat{D} = L^{-1}(0) = \psi^{-1}(\rho - r)$. Notice that $\psi^{-1}(\rho - r) \in (0, \bar{D}]$ when $0 < \rho - r \leq \psi(\bar{D})$.

When $\rho > r + \psi(\bar{D})$, equation (A.2) implies that the optimal steady state falls at the upper bound: $\hat{D} = \bar{D}$. This completes the proof of part (i).

The proof of parts $(ii) - (v)$ will make use of the following reformulation of the budget problem. Given (i), the budget problem (3.7) can be rendered as

$$\max_{\{T,\, x(t) \leq 1\}} \int_0^T u(1 - x(t))e^{-\rho t} dt + e^{-\rho T} u(1 - \hat{x})/\rho \tag{C.1}$$

subject to (3.2), given $D(0) = D \in [\underline{D}, \bar{D})$, where

$$\hat{x} \equiv (r + h(\hat{D}))\hat{D} \leq 1, \tag{C.2}$$

the strict inequality holding when $\hat{D} < \bar{D}$ or, noting part (i), when $\rho - r < \psi(\bar{D})$. The current-value Hamiltonian (dropping the time argument for convenience) is

$$H = u(1 - x) + \lambda[(r + h(D))D - x]$$

and necessary conditions for optimum include

$$u'(1-x) = -\lambda \tag{C.3}$$

$$\dot{\lambda} - \rho\lambda = -\lambda[r + h(D) + h'(D)D] \tag{C.4}$$

and the tranversality condition (associated with the choice of T)

$$e^{-\rho T}[H(T) - u(1-\hat{x})] = 0. \tag{C.5}$$

When $\lambda < 0$, Condition (C.4) can be expressed as

$$\dot{\lambda}/\lambda = \rho - r - \psi(D). \tag{C.6}$$

($\lambda = 0$ implies, noting (C.3) and (2.3), $x = -\infty$ and is ruled out in all cases of interest.)

(ii) Suppose $\rho < r$. I show that T is finite. Recall that in this case $\hat{D} = \underline{D}$ and $\rho - r - \psi(D) \leq \rho - r < 0$ for all $D \geq \underline{D}$. Suppose that T is infinite. Then, from part (i), $D(t)$ decreases monotonically toward $\hat{D} = \underline{D}$ while (C.6) implies

$$\dot{\lambda}(t)/\lambda(t) = \rho - r - \psi(D(t)) \leq \rho - r < 0 \; \forall \, t \geq 0.$$

Thus, $\lambda(t)$ approaches zero at a rate faster or equal to that of $e^{(\rho-r)t}$. Condition (C.3) then implies, noting (2.3), that $x(t) \to -\infty$. Thus, there exists some finite τ such that $x(t) < -[r + h(\underline{D})]\underline{D}$ for all $t > \tau$, implying (noting (3.2)) that $D(t)$ increases from time τ onward, contradicting part (i). It is concluded that T is finite.

(iii) Suppose $\rho = r$. The claim trivially holds when $D(0) \leq 0$, in which

case, according to part (i), the steady state is entered instantly. When $D(0) > 0$, the proof that the approach to the steady state $\hat{D} = 0$ is asymptotic is the same as the proof of (iv), since the latter proof holds also for $r = \rho$ when $D(0) > 0$.

(iv) Suppose $r \leq \rho \leq r + \psi(\bar{D})$. I show that T cannot be finite when $D(0) \neq \hat{D}$. Suppose otherwise, that T is finite. The transversality condition (C.5) then requires

$$u(1 - x(T)) + u'(1 - x(T))[x(T) - \hat{x}] - u(1 - \hat{x}) = 0,$$

implying, in light of the strict concavity of $u(\cdot)$, that (see (C.2))

$$x(T) = \hat{x} \equiv (r + h(\hat{D}))\hat{D} \leq 1,$$

equality holding only if $\hat{D} = \bar{D}$, which according to part (i) occurs when $\rho - r = \psi(\bar{D})$.

For the present (infinite-horizon, autonomous) problem, one can express the optimal x policy as a function of the state, say $\tilde{x}(D)$, and the time trajectory of $D(t)$ is given by the solution of (3.2):

$$T - t = \int_{D(t)}^{\hat{D}} \frac{dk}{(r + h(k))k - \tilde{x}(k)} \tag{C.7}$$

for any $0 \leq t < T$ (note that $D(T) = \hat{D}$). Expanding $(r + h(k))k - \tilde{x}(k)$ around $k = \hat{D}$, using $(r + h(\hat{D}))\hat{D} = \hat{x} = \tilde{x}(\hat{D})$, gives

$$\begin{aligned} (r + h(k))k - \tilde{x}(k) &= [r + h(\hat{D}) + h'(\hat{D})\hat{D} - \tilde{x}'(\hat{D})](k - \hat{D}) + o(k - \hat{D}) \\ &= (\rho - \tilde{x}'(\hat{D}))(k - \hat{D}) + o(k - \hat{D}), \end{aligned} \tag{C.8}$$

where use has been made of $r+h(\hat{D})+h'(\hat{D})\hat{D} = r+\psi(\hat{D}) = \rho$, and $o(\cdot)$ represents a term that goes to zero faster than its argument (i.e., $o(k-\hat{D})/(k-\hat{D}) \to 0$ as $k-\hat{D} \to 0$). It is verified below that:

Lemma 1. *Suppose $\rho - r \in [0, \psi(\bar{D})]$. Then, $\tilde{x}'(\hat{D})$ is finite and satisfies*

$$\tilde{x}'(\hat{D}) \geq \rho,$$

equality holding only if $\hat{D} = \bar{D}$.

Suppose $\hat{D} < \bar{D}$, so $\tilde{x}'(\hat{D}) - \rho > 0$, and consider the case $D(0) < \hat{D}$ (so $D^*(t)$ increases toward $\hat{D}$). If T is finite, then for every (small) $\varepsilon > 0$ there exists some time $t_\varepsilon < T$ such that $D(t_\varepsilon) = \hat{D} - \varepsilon$ and $\hat{D} - D^*(t) < \varepsilon$ for all $t > t_\varepsilon$. From (C.7) and (C.8) I obtain

$$T - t_\varepsilon = \frac{1}{\tilde{x}'(\hat{D}) - \rho} \int_{\hat{D}-\varepsilon}^{\hat{D}} \frac{dk}{\left(1 + \frac{o(\hat{D}-k)}{\hat{D}-k}\right)(\hat{D}-k)} \tag{C.9}$$

Now, there exists small enough $\varepsilon > 0$ satisfying $|o(\varepsilon)/\varepsilon| \leq 1$, such that $|\frac{o(\hat{D}-k)}{\hat{D}-k}| \leq 1$ for all $k \in [\hat{D}-\varepsilon, \hat{D}]$. Thus, from (C.9),

$$T - t_\varepsilon \geq \frac{1}{2(\tilde{x}'(\hat{D}) - \rho)} \int_{\hat{D}-\varepsilon}^{\hat{D}} \frac{dk}{\hat{D}-k}$$

and the integral on the right diverges for every positive ε, contradicting the assumption that T is finite. The case $D(0) > \hat{D}$ (where $D^*(t)$ decreases toward $\hat{D}$) is similarly verified.

If $\rho - r = \psi(\bar{D})$, then $\hat{D} = \bar{D}$, $\tilde{x}'(\hat{D}) = \rho$ (Lemma 1) and (C.9) specializes

to

$$T - t_\varepsilon = \int_{\hat{D}-\varepsilon}^{\hat{D}} \frac{dk}{\frac{o(\hat{D}-k)}{\hat{D}-k}(\hat{D}-k)}$$

As above, for small enough $\varepsilon > 0$, $|o(\hat{D}-k)/(\hat{D}-k)| \leq 1$ for all $k \in [\hat{D}-\varepsilon, \hat{D}]$ and the above integral diverges, contradicting the assumption that T is finite.

(v) When $\rho > r+\psi(\bar{D})$, part (i) implies that $D(t)$ approaches a steady state at $\hat{D} = \bar{D}$, during which $\rho - r - \psi(D) \geq \rho - r - \psi(\bar{D}) > 0$. Condition (C.6) then implies that $-\lambda$ increases at a rate equal to or larger than $\rho - r - \psi(\bar{D}) > 0$ and (C.3) implies that $u'(1-x)$ increases at the same rate and, noting (2.3), $x(t) \to 1$. Thus, the insolvent state $\bar{D}$ will be reached at a finite time or asymptotically as $u'(0) < \infty$ or $u'(0) = \infty$, respectively. □

Proof of Lemma 1. Differentiating (C.3) with respect to time, using (C.6), gives

$$\dot{x} = \sigma(x)(1-x)[\rho - r - \psi(D)], \tag{C.10}$$

where

$$\sigma(x) \equiv \frac{u'(1-x)}{-u''(1-x)(1-x)} \tag{C.11}$$

is the inverse of the elasticity of marginal utility (or the intertemporal elasticity of substitution). Combining $\dot{x} = \tilde{x}'(D)\dot{D} = \tilde{x}'(D)[(r+h(D))D - \tilde{x}(D)]$ and (C.10) gives

$$\tilde{x}'(D) = \sigma(\tilde{x}(D))(1-\tilde{x}(D))\frac{\rho - r - \psi(D)}{(r+h(D))D - \tilde{x}(D)}\,. \tag{C.12}$$

Note that $\dot{x} = \tilde{x}'(D)\dot{D}$ and (C.10) imply that the sign of $\tilde{x}'(D)$ equals the sign of $[\rho - r - \psi(D)]/\dot{D}$. When $D(0) < \hat{D}$, both $\rho - r - \psi(D)$ and $\dot{D}$ are positive,

and when $D(0) > \hat{D}$ they are both negative. Thus, $\tilde{x}'(D) > 0$ at D values near $\hat{D}$ and (using the continuity of $\tilde{x}'(D)$) $x'(\hat{D}) \geq 0$.

To obtain $\tilde{x}'(\hat{D})$, l'Hôpital's rule is applied to express (C.12) as

$$\tilde{x}'(\hat{D}) = \sigma(\hat{x})(1-\hat{x})\frac{\psi'(\hat{D})}{\tilde{x}'(\hat{D}) - \rho}$$

or

$$\tilde{x}'(\hat{D})[\tilde{x}'(\hat{D}) - \rho] = \sigma(\hat{x})(1-\hat{x})\psi'(\hat{D})\,, \tag{C.13}$$

where use has been made of $r + h(\hat{D}) + h'(\hat{D})\hat{D} = r + \psi(\hat{D}) = \rho$ and $\tilde{x}(\hat{D}) = \hat{x} = (r + h(\hat{D}))\hat{D}$. When $\hat{D} < \bar{D}$, the right-hand side of (C.13) is positive (since $1 - \hat{x} > 0$), implying that $\tilde{x}'(\hat{D}) - \rho > 0$. When $\hat{D} = \bar{D}$, the right-hand side of (C.13) vanishes (since $1 - \hat{x} = 0$), implying that either $\tilde{x}'(\hat{D}) = 0$ or $\tilde{x}'(\hat{D}) - \rho = 0$. The former case is ruled out since it implies (by continuity of $\tilde{x}'(D)$) that $\tilde{X}'(D) - \rho$ is negative at D values near $\hat{D}$, which in turn (noting (C.13)) implies that $\tilde{X}'(D)$ is negative at D values near $\hat{D}$, contradicting $x'(\hat{D}) \geq 0$ (found above). I conclude that $\tilde{x}'(\hat{D}) - \rho = 0$ when $\hat{D} = \bar{D}$. □

D Proof of Proposition 3

Proof. (i) The budget problem entails finding $\{x(t) \leq 1,\, t \geq 0\}$ that maximizes (4.10) subject to (4.5) and $d(t) \in [\underline{d}, \bar{d})$, given $d(0) \in [\underline{d}, \bar{d})$. This is an infinite-horizon, autonomous problem with a bounded state, thus, according to Property 1(i), the optimal state trajectory $d^*(t)$ converges monotonically to a steady state in $[\underline{d}, \bar{d}]$. I show that this steady state equals $\hat{d}$, defined in (4.12). With d and x corresponding, respectively, to Q and q of Property 1, the functions $u(1-x)$ and $[r_g - g + h(d)]d - x$ corresponding, respectively, to

$f(Q,q)$ and $g(Q,q)$ of Property 1, and the discount rate $\rho+(\eta-1)g$, the $L(\cdot)$ function corresponding to the budget problem is

$$u'(1-(r_g-g+h(d))d)(\rho+\eta g-r_g-\psi(d)).$$

Since $u'(1-(r_g-g+h(d))d)>0$ for all $d\in[\underline{d},\bar{d}]$, Property 1 can be applied with

$$L(d)=\rho+\eta g-r_g-\psi(d).$$

Suppose $\rho<r_g-\eta g$. Then, $L(d)<0$ for all $d\in[\underline{d},\bar{d}]$ and equation (A.2) implies that the optimal steady state is $\hat{d}=\underline{d}$, in agreement with (4.12).

Suppose $\rho=r_g-\eta g$. Then, since $L(\cdot)$ is decreasing and negative over $(0,\bar{d}]$, the steady state cannot fall in this interval (otherwise (A.2) is violated), leaving $[\underline{d},0]$ as the interval of feasible steady states. Thus, if $d(0)\leq 0$, it is known from the outset that $d^*(t)$ will remain in the interval $[\underline{d},0]$, over which $h(d)$ vanishes. The problem, then, reduces to a riskless problem (with a zero risk premium) and the proof of Proposition 1(iii) can be repeated (with the obvious modifications) to show that $\hat{d}=d(0)$ and the steady state is entered instantly. If $d(0)>0$, then $d^*(t)$ must decrease monotonically in order to exit the interval $(0,\bar{d}]$ (which does not contain the steady state), eventually reaching zero. As soon as a zero debt is reached, the above argument implies that the steady state has been entered.

Suppose $r_g-\eta g<\rho\leq r_g-\eta g+\psi(\bar{d})$. Since $L(d)=\rho-(r_g-\eta g)>0$ for $d\in[\underline{d},0]$, Property 1(ii) rules out the possibility that the optimal steady state falls in $[\underline{d},0]$. Thus, the monotonicity property ensures that $d^*(t)$ enters $[0,\bar{d}]$ (which happens instantly if $d(0)>0$) and remains in this interval. Applying

Property 1 to the budget Problem with the state d restricted to lie in $[0, \bar{d}]$ gives $\hat{d} = L^{-1}(0) = \psi^{-1}(\rho - r_g + \eta g)$. (Notice that $\psi^{-1}(\rho - r_g + \eta g) \in (0, \bar{d}]$ when $0 < \rho - r_g + \eta g \leq \psi(\bar{d})$.)

When $\rho > r - \eta g + \psi(\bar{d})$, $L(d) > 0$ for all feasible d values and equation (A.2) implies that the optimal steady state falls at the upper bound, i.e., $\hat{d} = \bar{d}$. This completes the proof of (i).

With obvious modifications, the proofs of $(ii) - (v)$ proceed along the same steps as the proofs of Proposition $2(ii) - (v)$ and are therefore omitted. $\square$

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