Dynamic Factor Demands Using Intertemporal Duality

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Abstract. Intertemporal duality can be used for empirical research to derive a system of optimal choice functions (dynamic factor demands and output supplies) consistent with an explicit dynamic optimization framework While the literature on intertemporal duality focuses on infinite-horizon autonomous problems, many applied problems cannot be analyzed within this framework This article uses intertemporal duality to specify a system of optimal choice functions for a broader and less restrictive set of intertemporal planning problems

Keywords. Intertemporal duality, dynamic factor demands, Hamilton-Jacobi equation, dynamic optimization

Agricultural production is inherently uncertain and dynamic Lags exist between variable input use and output realization, and biological and manufactured assets are managed over time To analyze such dynamic production processes, agricultural economists have continually searched for improved empirical methods to analyze shortrun and longrun decisions and explain response to price, policy, and technical changes

Optimal resource allocation and production has become a common issue analyzed at the theoretical level (1, 2, 6, 13, 15, 19, 30) [1] Given the profound influence of static duality theory on theoretical and empirical investigations of firm and consumer behavior (4, 5, 8, 9, 12, 21, 31), literature on duality relationships for dynamic optimization problems has also grown (3, 10, 14, 17, 26) For certain dynamic optimization problems, duality relationships provide a convenient method for modeling optimal choice functions (output supplies, consumption and factor demands, investment demands)

For example, in the context of an adjustment-cost model of the firm, Epstein develops the duality between a production function and the maximized present value of profits, which is then exploited to derive the firm's system of investment and factor demands via the dynamic analogue of Hotelling's Lemma (17) Cooper and McLaren provide similar results in the context of consumer theory (14) Chambers and Lopez generalize Epstein's problem and extend their results to a dynamic model of the financially constrained farm household and to a model of optimal fisheries management (10) Taylor and Monson, and Vasavada and Chambers use this approach to study investment in U S agriculture (33, 36) The dynamic duality approach could be applied to many intertemporal planning problems where assets are managed over time, including animal husbandry, mining industries, and forestry or agro forestry production

Cooper and McLaren distinguish three types of duality relationships for dynamic optimization problems (1) atemporal duality, which refers to the relationship between instantaneous functions at one point in time, such as between utility and indirect utility functions, (2) temporal duality, which refers to the relationship between the present values of sequences of functions (optimal value functions), such as between the maximized present value of utility and the minimized present value of expenditures over a given time horizon, and (3) intertemporal duality, which refers to the relationship between an instantaneous function and a corresponding optimal value function While atemporal and temporal duality are essentially equivalent to that surveyed by Diewert (14), intertemporal duality provides a convenient method for modeling optimal choice functions

Intertemporal duality is based on the Hamilton-Jacobi equation, which links the optimal value function to the instantaneous function via a static optimization problem The dynamic analogues of Hotelling's Lemma or Roy's Identity can be found by applying the envelope theorem to the Hamilton-Jacobi equation, which can be written in various forms depending upon the structure of the intertemporal problem While the literature focuses on infinite-horizon autonomous problems, there are many applied issues that cannot be analyzed within this framework [2] Therefore, intertemporal duality could be applied to a broader range of planning problems

[1] Italicized numbers in parentheses cite sources listed in the References at the end of this article

[2] Optimal control problems are said to be autonomous when the current value Hamiltonian is not an explicit function of time, in which case the solution to the problem involves solving an autonomous system of ordinary differential equations For example, consider an ordinary differential equation of the form $dx/dt = g(x, u, t)$, $x(t_0) = x_0$, where x is a state variable, t is time, and u is a control variable This system is said to be stationary if g is not an explicit function of time $(dx/dt = g(x, u))$, is free when u=0 for all $t \geq t_0$, and is autonomous when it is stationary and free (7, p 448)

The main objectives of this article are to discuss the various types of applied problems that could be analyzed with the intertemporal duality approach, and to provide convenient forms of the Hamilton-Jacobi equation for these problems The results of this article can be used as the foundation for further theoretical and empirical applications of the intertemporal duality approach [3] I show in the autonomous case how dynamic factor demands for infinite-horizon autonomous problems can be determined Using an adjustment-cost model of the firm based on the method outlined for the autonomous case, I then develop a convenient form of the Hamilton-Jacobi equation for infinite-horizon nonautonomous problems and provide the appropriate analogue of Hotelling's Lemma

Duality for Finite-Horizon Problems

Consider the following finite-horizon nonautonomous problem A

$$J(x_o, t_o t_1, b) = \underset{u \in U}{\text{Max}} \int_{t_o}^{t_1} f(x, u, t, b) dt \qquad (1)$$

$$\text{s t} \quad dx/dt = g(x, u, t, b), \; x(t_o) = x_o, \; x(t_1) \text{ free,}$$

where t is time, t_o is initial time, t_1 is terminal time, u(t) are control variables, x(t) are state variables, x_o are initial states, b are constant parameters, such as prices, taxes, or other policy variables, f is the intermediate function, dx/dt = g are the state equations, and U is the control set Given a set of conditions to ensure a solution to problem A, the optimal value function $J(x_o, t_o, t_1, b)$ is defined as the optimal value of the objective functional for the problem with initial state x_o that begins at t_o and ends at t_1, given the parameters b

Chavas and others used a variation of problem A to model the present net value of a biological asset (hogs)(12) In that case, u is variable inputs, x is the state of the asset (weight), b is a vector of prices and the discount rate, f(x, u, t, b) is the net revenue of flow products obtained from the asset, dx/dt = g is a biological growth function, t_o is time of purchase, and t_1 is time of sale Net revenues are an explicit function of t if the flow of outputs varies with the age of the asset While Chavas and others (12) considered the case of animal replacement, the asset could just as

well be a perennial crop or a tree from which products are obtained (milk, coffee, firewood, palm oil, gum arabic, oranges)

Finite-horizon investment problems also occur when a farmer leases a farm for a fixed period of time In that case, u represents variable inputs, x represents the farm capital stock, and net investment dx/dt equals gross investment less depreciation

In general, the maximum principle could be used to solve for the optimal choices of u in problem A over the period t_o to t_1, but in practice an analytical solution is usually difficult to obtain However, at any initial time $t_o \le t_1$, it is well known that the Hamilton-Jacobi equation for problem A takes the general form (20, 22)

$$-\frac{\partial J}{\partial t_o} = \underset{u}{\text{Max}} \left[f(x_o, u, t_o, b) \right.$$

$$\left. + \frac{\partial J}{\partial x_o}(x_o, t_o, t_1, b) g(x_o, u, t_o, b) \right] \qquad (2)$$

The significance of the Hamilton-Jacobi equation is that the maximizing u in equation 2, $u^* = u^*(x_o, t_o, t_1, b)$, are the optimal controls to problem A for the initial time t_o Differentiating equation 2 with respect to b and using the envelope theorem provides the dynamic analogue of Hotelling's Lemma

$$-\frac{\partial^2 J}{\partial t_o \partial b} = \frac{\partial f}{\partial b} + \frac{\partial^2 J}{\partial x_o \partial b} g + \frac{\partial J}{\partial x_o} \frac{\partial g}{\partial b} \qquad (3)$$

Given forms for J, f, and g, and an error structure, u^* can be, in principle, estimated from equation 3 by using simultaneous equation techniques for implicit functions (18), although certain functional forms allow more direct estimation procedures Thus, intertemporal duality allows the optimal choice functions, $u^* = u(t_o, t_1, x_o, b)$, to be derived from J, f, and g without the need to solve problem A Because equation 3 is a nonlinear function of the variables u and x and the parameters b, further assumptions on the functions f and g can improve the empirical tractability of the system [4]

Equation 2 is the general form of the Hamilton-Jacobi equation In the remainder of this article, I explore some simple variations of problem A that allow the Hamilton-Jacobi equation and, therefore, the

analogues of Hotelling's Lemma to be written more simply For example, consider the following infinite-horizon autonomous problem B·

$$J(x_0, t_0 t_1, b) = \underset{u}{Max} \int_{t_0}^{t_1} f(x, u, b) dt \qquad (4)$$

s t $dx/dt = g(x, u, b)$, $x(t_0) = x_0$, $x(t_1)$ free

Because time is not an explicit argument in f and g (problem B is autonomous), the Hamiltonian evaluated at the optimizing u is constant for all t, $t_0 \le t \le t_1$ (20) Therefore, the Hamilton-Jacobi equation can be written as·

$$-\frac{\partial J}{\partial t_0} = H^*, \qquad (5)$$

where H* is the Hamiltonian for problem B evaluated at the optimal u Integrating both sides with respect to initial time t_0 and evaluating over the interval t_0 to t_1 yields:

$$\frac{J(t_0, x_0, t_1, b) - J(t_1, x_0, t_1, b)}{(t_1 - t_0)} = H^* \qquad (6)$$

But, $J(t_1, x_0, t_1, b) = 0$ because it is defined as the optimal value of the objective functional for the problem beginning and ending at time t_1, and there is no scrap value The term $1/(t_1 - t_0)$ converts the sum J received every t_1-t_0 periods into a constant flow every period Depending upon the situation, it could be more useful to derive the analogue of Hotelling's Lemma from either equation 5 or 6.

Duality for Infinite-Horizon Problems: The Autonomous Case

The specific form of problem A in much of the literature on duality theory and dynamic factor demands is some variation of problem C

$$J(t_0 k_0, p) = \underset{I \ge 0}{Max} \int_{t_0}^{\infty} e^{-rt}[f(k, I) - pk] dt \qquad (7)$$

s.t $dx/dt = I - \alpha k$, $k(t_0) = k_0$,

where I(t) is investment in capital k(t), p is the rental price of capital normalized with respect to output price, α is the depreciation rate; f is the production function, $dk/dt = I - \alpha k$ is the capital stock equation of motion, and r is the firm's discount rate J is not written as an explicit function of α and r to reduce notational clutter

Problem C describes an adjustment-cost model of the firm with static-price expectations (see *17, 24, 27, 35*)[5] However, the structure of problem C is also similar to (1) models of the extractive firm, where k is the mineral stock, I is the rate of extraction, and dk/dt = -I (see *11*), (2) farm-level models of soil conservation, where k is topsoil depth, I is erosion, s is natural regeneration, and dk/dt = s - I (see *25*), and (3) forest-harvesting models, where k is tree biomass, g(k) is the tree-growth function, I is the harvest rate, and dk/dt = g(k) - I (see *13*)

There are two important distinctions between problem C and problem A First, problem C is autonomous in the sense that its current-value Hamiltonian is not an explicit function of time (time enters only through the discount term) And second, problem C has an infinite time-horizon beginning at any time t_0, but discounted to time 0 Given these two conditions, the optimal value function for problem C can be written as

$$J(t_0 k_0, p) = e^{-rt_0} V(k_0, p), \text{ where}$$

$$V(k_0, p) = \underset{I \ge 0}{max} \int_{t_0}^{\infty} e^{-r(t-t_0)} [f(k, I) - pk] dt \qquad (8)$$

s t $dk/dt = I - \alpha k$, $k(t_0) = k_0$

The optimal value function V in equation 8 is not written as an explicit function of t_0 because $\partial V/\partial t_0 = 0$ (2, 22) Since $-\partial J/\partial t_0 = rV\exp(-rt_0)$ and $\partial J/\partial k_0 = [\partial V/\partial k_0][\exp(-rt_0)]$, the Hamilton-Jacobi equation for an infinite-horizon autonomous problem at an arbitrary time t_0 (22, pp 241-2) can be written as

$$rV(k_0, p) = \underset{I \ge 0}{Max} \left[f(k_0, I) - pk_0 + \frac{\partial V}{\partial k_0}(k_0, p)[I - \alpha k_0] \right] \quad (9)$$

In the literature, initial time is usually considered to be $t_0 = 0$, in which case J = V, and the Hamilton-Jacobi equation for problem C can be written as

$$rJ(k_0, p) = \underset{I \ge 0}{Max} \left[f(k_0, I) - pk_0 + \frac{\partial J}{\partial k_0}(k_0, p)[I - \alpha k_0] \right] (10)$$

The maximizing investment decisions from 10, $I^* = I(k_0, p)$, are the optimal choices for problem C at $t_0 = 0$. By the envelope theorem, the derivative of equation 10 with respect to p yields, after rearranging, the intertemporal analogue of Hotelling's Lemma for problem C

$$I^*(k_0, p) = \left[\frac{\partial^2 J}{\partial k_0 \partial p}\right]^{-1} \left[r\frac{\partial J}{\partial p} + k_0\right] + \alpha k_0 \qquad (11)$$

Equation 11 provides a simple way to derive systems of investment equations that are consistent with an infinite-horizon autonomous control problem. In contrast to problem A, since the objective functional in problem C is linear in p and the state equation is linear in I (and independent of p), the optimal investment I^* can be written only in terms of the indirect objective function J. While the theoretical investment equation 11 is simple to derive, it is potentially nonlinear in variables and parameters and, as a result, may be difficult to estimate. Epstein (17) includes variable inputs into the analysis, derives the properties of J, and also discusses the issue of functional forms for J

Two recent studies that apply the adjustment-cost model of the firm to U S agriculture are Taylor and Monson, and Vasavada and Chambers (33, 36) Based on theoretical models similar to problem C, both studies use intertemporal duality to specify dynamic factor demands and output supply [6] Each study proceeds by specifying a functional form for J, imposing conditions on J to ensure consistent aggregation (which is discussed in 36), and then deriving the analogues of equation 11 based on the Hamilton-Jacobi equation for infinite-horizon autonomous problems

Duality for Infinite-Horizon Problems: The Nonautonomous Case

When Vasavada and Chambers (36) or Taylor and Monson (33) actually estimate the system of net investment equations, they include time trends as exogenous variables to reflect the effect of technical change in agriculture over time. While Vasavada and Chambers add a linear time trend onto equation 11, Taylor and Monson include time explicitly into the optimal value function, but then use the Hamilton-Jacobi equation for an autonomous problem to derive the investment equations

Technical change in the production function provides one case where an infinite-horizon control problem may not be autonomous For example, Hicks-neutral technical change implies that the firm's production function can be written as $y(t) = f(k,I)A(t)$, where $A(t)$ describes the process of technical change in the production of output y If the firm knows or expects that technical change will occur over time, then the firm must solve a nonautonomous problem to find its optimal investment choices

There are many other examples where an infinite horizon nonautonomous formulation would be appropriate As suggested for problem A, the objective functional for infinite-horizon problems could depend explicitly on time if output is a flow product (milk) from a biological asset In problem C, the capital price, p, would also be a function of time if the firm expected prices to rise over time (for example, $p(t) = p(t_0)\exp[m(t-t_0)]$, where m is the expected rate of price increase) The state equation for an infinite-horizon problem could be a function of time if the depreciation rate α depended on an asset's age or if technical change affected the rate of asset depreciation

Problem D is a simple variation of problem C that incorporates time into the production function to represent expected disembodied technical change

$$J(k_0 t_0, p) = \max_{I \geq 0} \int_{t_0}^{\infty} e^{-rt} [f(k,I,t) - pk] \, dt \qquad (12)$$

$$s\,t \quad dk/dt = I - \alpha k, \quad k(t_0) = k_0$$

Problem D is a nonautonomous control problem because the firm's production function is an explicit function of time Therefore, the Hamilton-Jacobi equation can no longer be written as equation 9 Fortunately, a convenient form of the Hamilton-Jacobi equation for problem D can be derived using the same process as followed for problem C First, define

$$J(t_0 k_0, p) = e^{-rt_0} V(t_0, k_0, p), \quad \text{where}$$

$$V(k_0, t_0, p) = \max_{I \geq 0} \int_{t_0}^{\infty} e^{-r(t-t_0)} [f(k,I,t) - pk] \, dt \qquad (13)$$

$$dk/dt = I - \alpha k, \quad k(t_0) = k_0$$

Because equation 13 is a nonautonomous problem, the optimal value function V is an explicit function of initial time t_0 Using the definitions of J and V from equations 12 and 13, which imply that $-\partial J/\partial t_0 = [rV - \partial V/\partial t_0][\exp(-rt_0)]$ and $\partial J/\partial k_0 = [\partial V/\partial k_0][\exp(-rt_0)]$, the Hamilton-Jacobi equation at t_0 can be written as

$$rV(k_o, t_o, p) = \max_{I \geq 0} \{ f(k_o, I, t_o) - pk_o$$

$$+ \frac{\partial V}{\partial k_o} [I - \alpha k_o] + \frac{\partial V}{\partial t_o} \} \qquad (14)$$

By the envelope theorem, the derivative of equation 14 with respect to p yields, after rearranging, the dynamic analogue of Hotelling's Lemma

$$I^*(k_o, t_o, p) = \left[\frac{\partial^2 V}{\partial k_o \partial p} \right]^{-1} \left[r \frac{\partial V}{\partial p} + k_o - \frac{\partial^2 V}{\partial t_o \partial p} \right] + \alpha k_o \quad (15)$$

Equation 15 shows how to derive the optimal investment choice at t_o for problem D, which is an infinite-horizon nonautonomous control problem There are two important differences between equations 15 and 11 First, the optimal value function V is an explicit function of initial time for nonautonomous problems Second, equation 15 has the additional term $\partial^2 V/\partial t_o \partial p$ Since $\partial V/\partial t_o$ is the marginal value of technical change at the initial time in problem D, the term $\partial^2 V/\partial t_o \partial p$ is the change in the marginal value of technical change due to a change in the rental price of capital

Therefore, if the issue to be studied involves an infinite-horizon nonautonomous problem, the empirical model could be based on a Hamilton-Jacobi equation similar to equation 14, which would allow a time variable to be incorporated into the empirical model in a consistent manner [7] The approach followed above can be applied to any infinite-horizon nonautonomous problem with discounting

Conclusions

In this article, the Hamilton-Jacobi equation was derived for four general classes of dynamic optimization problems The envelope theorem can then be applied to the Hamilton-Jacobi equation to specify systems of optimal choice functions The output supply function can also be derived from a minimization problem dual to the Hamilton-Jacobi equation (see 17) While the properties of the optimal value function for infinite horizon autonomous models such as problem C are well known, further research is needed to identify the usable properties of optimal value functions for the other types of problems To date, far

more empirical studies for each class of problems are needed to determine if intertemporal duality will be as useful to applied researchers as static duality

References

1 Aarrestad, J "Optimal Savings and Exhaustible Resource Extraction in an Open Economy," *Journal of Economic Theory* Vol 19, 1978, pp 163-79

2 Arrow, K J , and M Kurz *Public Investment, The Rate of Return, and Optimal Fiscal Policy* Baltimore Johns Hopkins University Press, 1970

3 Benveniste, L M , and J A Scheinkman "Duality Theory for Dynamic Optimization Models of Economics The Continuous Case," *Journal of Economic Theory* Vol 27, 1982, pp 1-19

4 Berndt, E R , and L R Christensen "The Translog Function and the Substitution of Equipment, Structures, and Labor in U S Manufacturing, 1929-68," *Journal of Econometrics* Vol 1, 1973, pp 81-113

5 Binswanger, H P "A Cost Function Approach to the Measurement of Elasticities of Factor Demand and Elasticities of Substitution," *American Journal of Agricultural Economics* Vol 56, 1974, pp 377-86

6 Brock, W A "The Global Asymptotic Stability of Optimal Control with Applications to Dynamic Economic Theory," in *Applications of Control Theory to Economic Analysis* (J Pitchford and S Turnovsky, eds) New York North-Holland, 1977

7 Bryson, A E , and Y C Ho *Applied Optimal Control* Waltham, MA Blaisdell Publishing, Inc , 1969

8 Burgess, D F "Duality Theory and Pitfalls in the Specification of Technologies," *Journal of Econometrics* Vol 3, 1975, pp 105-21

9 Chambers, R G "Duality, the Output Effect, and Applied Comparative Statics," *American Journal of Agricultural Economics* Vol 64, 1982, pp 152-56

10 Chambers, R G , and R E Lopez "A General, Dynamic, Supply-Response Model," *The North-East Journal of Agricultural and Resource Economics* Vol 13, 1984, pp 142-54

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[7]Whether time trends are the appropriate way to model technical change is beyond the scope of this paper However, the possible empirical problems with trend analysis should be considered (28) A possible alternative would be to include expected technical change as a capital-augmenting stochastic process (see 32)

11 Chapman, D "Computational Techniques for Intertemporal Allocation of Natural Resources," *American Journal of Agricultural Economics* Vol 69, 1987, pp 134-42

12 Chavas, J P, J Kliebenstein, and T D Crenshaw "Modeling Dynamic Agricultural Production Response The Case of Swine Production," *American Journal of Agricultural Economics* Vol 76, 1985, pp 636-46

13 Clark, C W *Mathematical Bioeconomics The Optimal Management of Renewable Resources* New York Wiley, 1976

14 Cooper, R J, and K R McLaren "Atemporal, Temporal, and Intertemporal Duality in Consumer Theory," *International Economic Review* Vol 21, 1980, pp 599-609

15 Dasgupta, P, and G Heal "The Optimal Depletion of Exhaustible Resources," *The Review of Economic Studies* Vol 41, Symposium, 1974, pp 3-29

16 Diewert, E "Applications of Duality Theory" *Frontiers of Quantitative Economics* Vol II (M Intrilligator and D A Kendrick, eds) Amsterdam North Holland, 1974

17 Epstein, L G "Duality Theory and Functional Forms for Dynamic Factor Demands," *Review of Economic Studies* Vol 48, 1981, pp 81-95

18 Gallant, A R "Three-Stage Least-Squares Estimation for a System of Simultaneous, Nonlinear, Implicit Equations," *Journal of Econometrics* Vol 5, 1977, pp 71 88

19 Hotelling, H "The Economics of Exhaustible Resources," *Journal of Political Economy* Vol 39, 1931, pp 137-75

20 Intrilligator, M D *Mathematical Optimization and Economic Theory* Englewood Cliffs Prentice-Hall, 1971

21 Jorgenson, D W, and L J Lau "Duality of Technology and Economic Behavior," *Review of Economic Studies* Vol 41, 1971, pp 181-200

22 Kamien, M I, and N L Schwartz *Dynamic Optimization The Calculus of Variations and Optimal Control in Economics and Management* Amsterdam Elsevier/North-Holland, 1981

23 Lopez, R E "Supply Response and Investment in the Canadian Food Processing Industry," *American Journal of Agricultural Economics* Vol 67, 1985, pp 40-8

24 Lucas, R "Optimal Investment Policy and the Flexible Accelerator," *International Economic Review* Vol 81, 1967, pp 78-85

25 McConnell, K "An Economic Model of Soil Conservation," *American Journal of Agricultural Economics* Vol 65, 1983, pp 83-9

26 McLaren, K R, and R J Cooper "Intertemporal Duality Application to the Theory of the Firm," *Econometrica* Vol 48, 1980, pp 1,755-62

27 Mortenson, D "Generalized Costs of Adjustment and Dynamic Factor Demand Theory of the Firm," *Econometrica* Vol 41, 1981, pp 81-95

28 Nelson, C R, and H Kang "Pitfalls in the Use of Time as an Explanatory Variable in Regression," *Journal of Business & Economic Statistics* Vol 2, 1984, pp 73-82

29 Pindyck, R S, and J J Rotemberg "Dynamic Factor Demands and the Effects of Energy Price Shocks," *American Economic Review* Vol 73, 1983, pp 1,066-79

30 Roe, T, and T Graham-Tomasi "Yield Risk in a Dynamic Model of the Agricultural Household," *Agricultural Household Models* (I J Singh, L Squire, and J Strauss, eds) Baltimore Johns Hopkins University Press, 1986

31 Silberberg, E "The Theory of the Firm in Long Run Equilibrium," *American Economic Review* Vol 64, 1974, pp 734-41

32 Stefanou, S E "Technical Change, Uncertainty, and Investment," *American Journal of Agricultural Economics* Vol 69, 1987, pp 158-65

33 Taylor, T G, and M J Monson "Dynamic Factor Demands for Aggregate Southeastern United States Agriculture," *Southern Journal of Agricultural Economics* Vol 17, 1985, pp 1-9

34 Taylor, R "Stochastic Dynamic Duality Theory and Empirical Applicability," *American Journal of Agricultural Economics* Vol 66, 1985, pp 351-57

35 Treadway, A "The Globally Optimal Flexible Accelerator," *Journal of Economic Theory* Vol 7, 1974, pp 17-39

36 Vasavada, U, and R G Chambers "Investment in U S Agriculture," *American Journal of Agricultural Economics* Vol 68, 1986, pp 950-60