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A Model of Yield Variability and Price Effect

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1. Introduction

Suppose a crop can be produced by either of two sets of cultural practices. One promises a higher yield if growing conditions turn out to be favorable; yield under the second will be higher if unfavorable conditions are encountered. An expected utility maximizing, risk averse producer is allocating his acreage between the two regimes. We are particularly interested in how the allotted acreages respond to changes in the price of the crop.

Questions of this kind sometimes arise in developing economies. If the less variable technology is in common use, but the more variable technology is associated with higher average yield, then partial adoption of the more variable system increases supply over time and exerts downward pressure on prices. Will lower prices create additional incentives to adopt the more variable system or will lower prices reduce incentives to change? This was the primary question which led to the present investigation. One example of alternative technologies is the choice between chemical and organic fertilizers. With ample rainfall chemical fertilizer should be more productive. If there is little rain, organic will turn out better.

In this paper price of output is regarded as known when the production decision is made. This corresponds to forward contracting

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or to a controlled price announced in advance. Models involving a random price will be explored later.

The following notation is used:

W represents yield per acre from the more variable technology.

V represents yield per acre from the more stable technology.

W and V are random variables on an underlying probability space $(\Omega, \mathcal{F}, P)$ where $\omega \in \Omega$ is a possible sequence of growing conditions.

y is price of output

ψ is the decision maker's utility of gain function

b is total acreage on the farm in question with a acres cultivated in the more variable way and (b - a) in the less variable way.

η is the expected utility function and the decision problem is

$$(1.1) \qquad \max_{a\varepsilon[o,\ b]} \eta(a;y) = E\psi[yQ(a)] \quad y > 0 \text{ where}$$

Q(a) is a random variable representing prospective total yield when a acres are cultivated by the more variable method.

$$(1.2) \qquad Q(a) = aW + (b - a)V = a(W - V) + b\,V.$$

The semicolon separating the arguments of η is inserted to recognize that a is the decision variable while y is regarded as a parameter by the decision maker. The following assumptions are used:

(1.3) (a) $\psi' > 0,\ \psi'' < 0,\ \psi''' > 0$

(b) $\lim_{x\to\infty} \psi'(x) = 0$

(c) Let $Z = W - V$. For all $a\varepsilon R$, $y > 0$ the following random variables have finite means W, V, W^2, V^2, $Z\psi'(yaZ)$, $Z^2\psi''(yaZ)$.

(a) and (b) have been commonly assumed by economists, $\psi' > 0$ expresses preference for larger gains rather than smaller, $\psi'' < 0$ represents risk aversion, $\psi''' > 0$ is implied by (but weaker than) decreasing absolute risk aversion, (b) is more difficult to assess since it involves what would happen as potential gains became arbitrarily large and, therefore, outside our normal experience and contemplation. It has, however, seemed reasonable to other analysts (e.g. Leland [6], Bertsekas [2]) as well as the present authors.

(c) also seems reasonable and is assumed for mathematical tractability. It will hold providing tails of the distributions of W, V are not too fat; for example if W, V are bounded.

If $\eta(\hat{a}) > \eta(a)$ for all real numbers a, then $\hat{a}$ will be called an unrestricted maximizer of η. If $0 \leq \tilde{a} \leq b$ and $\eta(\tilde{a}) \geq \eta(a)$ for all $a \in [o, b]$ than $\tilde{a}$ will be called a restricted maximizer of η.

Assumptions (1.3), which are maintained throughout this paper, have the following implications:

(1.4) (i) continuous first and second derivatives of η exist and may be obtained by differentiation under the expectation.

(ii) η is strictly concave in a and has, for each y, a unique unrestricted maximizer.

(iii) $\hat{a} < 0 \Rightarrow \tilde{a} = 0$; $\hat{a} > b \Rightarrow \tilde{a} = b$; $0 \leq \hat{a} \leq b \Rightarrow \hat{a} = \tilde{a}$.

(iv) $(\hat{a} - a) \overset{s}{=} D_a\eta(a;y)$ $a \in R$. "$\overset{s}{=}$" means "agrees in sign with".

A proof of (i) is indicated in the third section of this paper. (ii) is proved in Hildreth [4], pp. 9, 10 (note that $P(W - V > 0) > 0$, $P(W - V < 0) > 0$). (iii), (iv) are simple consequences of the fact that, for given y, η is single peaked and everywhere strictly concave.[1/] Thus the contemplated decision problem (summarized in 1.1) has a unique solution $\tilde{a}$ and the associated unrestricted problem has a unique solution $\hat{a}$. It will be convenient to examine the relation between $\hat{a}$ and y and to write

$$(1.5) \qquad \hat{a} = \alpha(y) \qquad y > 0.$$

Assertation (iii) above connects $\hat{a}$ with $\tilde{a}$.

In the next section a particular specification of a possible relation between W and V is introduced. It is shown that, under the specification, response of more variable acreage to a change in price has sign opposite to the derivative of the decision maker's relative risk aversion function. This is the paper's main result and it adds to the importance of finding empirical evidence on how decision makers', and particularly entrepreneurs', relative risk aversion responds to increases in gain.

One expects that per acre costs of the two regimes will differ in most applications. It is convenient to think of gain as being measured after deducting the costs of cultivating the entire acreage in whichever way is more expensive. One regards these cultivation costs as prepaid when the decision is contemplated. Any savings in per acre costs can then be added to gross receipts under the less expensive procedure to allow for lower costs. This enables one to regard W, V, Q as non-negative random variables and does not change the interpretation of the model.

Before introducing the specification, it is perhaps worth reminding some readers that if risk neutrality were assumed we could write $\psi(x) = x$

1/ If (iii), (iv) are not immediately obvious, the reader can draw a peaked, smooth, strictly concave function and compare a, $\hat{a}$, $\tilde{a}$ for various values of a.

and the decision problem (1.1) would have the following trivial solution.

$$(1.6)\qquad \begin{aligned} &EW > EV \Rightarrow \hat{a} = \infty,\ \tilde{a} = b \\ &EW < EV \Rightarrow \hat{a} = -\infty,\ \tilde{a} = 0 \\ &EW = EV \Rightarrow \hat{a},\ \tilde{a} \text{ indeterminate.} \end{aligned}$$

2. Derivation of Principal Result

The notion that one procedure will be advantageous if favorable circumstances are encountered while another will be better if developments in the decision maker's environment are adverse can be made more precise in many ways. Such notions are relevant in many contexts--risky versus less risky securities, uninsured versus insured ventures, etc. We hope the specification (S) below represents some production problems reasonably well. It is assumed that for some intermediate events, yields under the two regimes are equal ($W = V = k$). If experience is better, acreage under the less variable technology benefits somewhat but acreage under the other technology benefits more. Conversely, under less favorable experience yield on acreage under the more variable technology declines more. This can be stated

$$(S)\qquad W - V \overset{s}{=} V - k.$$

Since η is differentiable and strictly concave, $\hat{a}$ may be obtained for given y by solving $D_a\eta(a;\, y) = 0$. As shown in the final section this yields

$$D_a\eta(a;\, y) = y\, E(W - V)\, \psi'[ya(W - V) - ybV] = 0$$

Not much can be said about $\hat{a}$ without further assumptions. One may note

$$(2.1)\qquad D_a\eta(o;\, y) = yE(W - V)\,\psi'(ybV) < y\int_{W>V}(W - V)\,\psi'(ybk) + y\int_{W<V}(W - V)\,\psi'(ybk)$$

$$= y\,\psi'(ybk)\,E(W - V) \overset{s}{=} E(W - V).$$

Thus, by (1.4 - iii),

$$(2.2)\qquad E(W - V) \leq 0 \Rightarrow \hat{a} < 0 \text{ and } \tilde{a} = 0.$$

No acreage will be devoted to the more variable method unless its average yield is higher, and one clearly could have $\hat{a} < 0$, $\tilde{a} = 0$ even though EW were somewhat larger than EV. More precise statements would depend on making further assumptions about ψ, W, V.

The main interest here is in the response of $\hat{a}$ and $\tilde{a}$ to changes in y. It will be convenient to discuss the relation (1.5) $\hat{a} = \alpha(y)$ and let the reader note possible responses of $\tilde{a}$ using (1.4 - iii).

Since $\eta\varepsilon C^2$ (is twice continuously differentiable), the implicit function theorem may be applied to investigate the sign of $D_y\alpha$, namely

$$(2.3)\qquad D_y\alpha = -(D_{aa}\eta(\hat{a};\, y))^{-1}\, D_{ay}\eta(\hat{a};\, y).$$

Differentiation under the expectation yields

$$(2.4)\qquad D_{aa}\eta(\hat{a};\, y) = y^2\, E(W - V)^2\, \psi''(yQ(\hat{a})) < 0$$

so, using (2.3),

$$(2.5)\qquad D_y\alpha \overset{s}{=} D_{ay}\eta(\hat{a};\, y).$$

Again differentiating under the expectation,

$$(2.6)\qquad D_{ay}\eta(\hat{a};\, y) = E(W - V)\,\psi'(yQ(\hat{a})) + E(W - V)\, yQ(\hat{a})\,\psi''(yQ(\hat{a})).$$

The first term on the right equals $y^{-1}\, D_a(\hat{a};\, y)$ and vanishes by definition of $\hat{a}$. Let

$$(2.7) \qquad r(x) = -\frac{\psi''(x)}{\psi'(x)}$$

$$R(x) = -\frac{x\psi''(x)}{\psi'(x)}$$

be the Pratt-Arrow indices of absolute and relative risk aversion. Then

$$(2.8) \qquad D_{ay}\eta(\hat{a};\ y) = -E(W - V)\ \psi'(yQ(\hat{a}))\ R(yQ(\hat{a}))$$

$$= \int_{W>V}(V - W)\ \ \psi'(yQ(\hat{a}))\ R(yQ(\hat{a})) + \int_{W<V}(V - W)\ \psi'(yQ(\hat{a}))\ R(yQ(\hat{a}))$$

If $\hat{a} < 0$ or $\hat{a} > b$, then small changes in y will not affect $\tilde{a}$ so assume $0 \leq \hat{a} \leq b$. Then under (S), $Q(\hat{a}) > Q(ybk)$ in the first integral and $Q(\hat{a}) < Q(ybk)$ in the second integral. Thus, if R is an increasing function,

$$(2.9) \qquad D_{ay}\eta(\hat{a};\ y) < \int_{W>V}(V - W)\ \psi'(yQ(\hat{a}))\ R(ybk) + \int_{W<V}(V - W)\ \psi'(yQ(\hat{a}))\ R(ybk)$$

$$= R(ybk) \int (V - W)\ \psi'(yQ(\hat{a})) = R(ybk)[-y^{-1}\ D_a\eta(\hat{a};\ y)] = 0$$

If R is a decreasing function, the inequality in (2.9) is reversed. Clearly R constant implies $D_{ay}\eta(\hat{a};\ y) = 0$. Putting these together

$$(2.10) \qquad D_y\alpha(y) \stackrel{\leq}{\geq} D_{ay}\ \eta(\hat{a};\ y) \stackrel{\leq}{\geq} -D_xR(x).$$

Thus the acreage response to price turns on how, for the relevant decision maker, relative risk aversion responds to an increase in gain. If relative risk aversion increases with gain, as initially suggested by Arrow [1], then acreage using the more variable technology declines as price increases. If relative risk aversion is a decreasing function of gain than the indicated acreage moves in the same direction as price.

However, existing evidence on relative risk aversion is not entirely clear (see Stiglitz [7] and [8]). There are also many other contexts, e.g. Diamond and Stiglitz [3], in which optimal choice depends on the behavior of relative risk aversion. The present result adds to our need to obtain firmer empirical evidence on this point.

One fragment of recent evidence is furnished by a study of farmer's utility functions by Hildreth and Knowles [5]. Utility functions were fitted to responses of 13 Minnesota farmers to a series of questions on hypothetical decisions under uncertainty. Of several forms of utility functions tried, substantially the best fits were obtained using

$$\psi(x) = -e^{-\lambda_1 x} - \beta e^{-\lambda_2 x} \qquad \lambda_1, \lambda_2, \beta > 0, \qquad \lambda_1 > \lambda_2. \tag{2.11}$$

For such utility of gain functions, differentiation and rearrangement of terms and factors yields

$$\frac{(\psi')^2 e^{(\lambda_1+\lambda_2)x}}{\lambda_1\lambda_2\beta} R'(x) = (\lambda_1 + \lambda_2) + \frac{\lambda_1^2}{\lambda_2\beta e^{\mu x}} + \frac{\lambda_2^2\beta e^{\mu x}}{\lambda_1} - \mu^2 x \tag{2.12}$$

where $\quad \mu = \lambda_1 - \lambda_2 > 0.$

Since coeff $R'(x) > 0$

$$R'(x) \gtreqless \gamma(x) = (\lambda_1 + \lambda_2) + \frac{\lambda^2}{\lambda_2\beta e^{\mu x}} + \frac{\lambda_2^2\beta e^{\mu x}}{\lambda_1} - \mu^2 x\ . \tag{2.13}$$

The first three terms of $\gamma(x)$ are positive. Clearly $\gamma(x) > 0$ for $x \leq \frac{\lambda_1 + \lambda_2}{\mu^2}$ and $\gamma(x) > 0$ for x sufficiently large. For some parameter sets $(\lambda_1, \lambda_2, \beta)$, $\gamma(x) > 0$ for all x therefore $R'(x) > 0$ for all x.

However, in the study cited, 11 of 13 of the estimated utility functions showed an intermediate interval of x for which relative risk aversion was decreasing ($R' < 0$). Also, if we look at "typical" parameter values obtained in this study by taking approximate medians of the 13 estimated values, these approximate medians are $\bar{\lambda}_1 = .1$, $\bar{\lambda}_2 = .003$, $\bar{\beta} = 50$. For a utility function with these parameters, $R(x)$ is decreasing for $15.2 < x < 43.5$ and increasing for $x < 15.2$, $x > 43.5$ (x was measured in thousands of dollars). This suggests that it may be difficult to establish broad generalizations about $R'(x)$ and that this important property may have to be carefully studied in each empirical application.

3. Differentiability and Continuity

Application of the implicit function theorem in Section 2 required $\eta(a; y)$ to have continuous first and second derivatives obtainable by differentiation under the expectation. Two first and three second derivatives are involved. Arguments in the several cases are sufficiently similar that it does not seem worthwhile to present all of them. Instead we shall illustrate proofs by showing that, under Assumptions (1.3),

(i) $D_a \eta(a; y) = yEZ\, \psi'(yaZ + ybV)$

(ii) $D_{ay} \eta(a; y) = EX\, \psi'(yaZ + ybV) + yaEZ(aZ + bV)\, \psi''(yaZ + ybV)$

(iii) $D_{ay}\eta(a; y)$ is continuous.

(i). Recall $\eta(a; y) = E\psi(yaZ + ybV)$. If $D_a\eta$ exists, then

$$D_a\eta = \lim_{h\to 0} \frac{1}{h}[\eta(a + h; y) - \eta(a,y)] \tag{3.1}$$

$$= \lim_{h\to 0} \frac{1}{h}[E\psi(yaZ + ybV + yhZ) - E\,\psi(yaZ + ybV)]$$

By the mean value theorem

$$(3.2) \qquad \psi(yaZ + ybV + yhZ) = \psi(yaZ + ybV) + yZ\,\psi'(yaZ + ybV + hGyZ)$$

where $0 \leq G \leq 1$.

Substituting into (3.1) yields

$$(3.3) \qquad D_a\eta = y \lim_{h\to 0} EZ\,\psi'(ya\,Z + ybV + hGyZ).$$

Without loss of generality assume $|h| < 1$.

Let $Z^+ = \max\{Z, 0\}$, $Z^- = \max\{-Z, 0\}$. Then, since ψ' is a decreasing function and $V > 0$,

$$(3.4) \qquad |Z\psi'(yaZ + ybV + hGyZ)| = |Z^+\psi'(yaZ^+ + ybV + hGyZ^+)$$
$$-Z^-\psi'(-yaZ^- + ybV - hGyZ^-| < |Z^+\psi'((y-1)a\,Z^+|$$

The right side is integrable by (1.3c) so, by the dominated convergence theorem, the order of lim and E may be interchanged in (3.3) producing

$$(3.5) \qquad D_a\eta = yE[\lim_{h\to 0} Z\psi'(yaZ + ybV + hGyZ)] = yEZ\psi'(yaZ + ybV).$$

<u>(ii)</u>. If $D_{ay}\eta$ exists then

$$(3.6) \qquad D_{ay}\eta = \lim_{h\to 0} \frac{1}{h}\,[(y + h)\;EZ\,\psi'(y + h)aZ + (y + b)bV)$$
$$- yEZ\psi'(yaZ + ybV)]$$

By the mean value theorem

$$(3.7) \qquad \psi'(yaZ + ybV + h(aZ + bV)) = \psi'(yaZ + ybV)$$
$$+ h(aZ + bV)\;\psi''(yaZ + ybV + hG(aZ + bV))$$

where $\quad 0 \leq G \leq 1$

Substituting from (3.7) to (3.6),

$$(3.8)\qquad D_{ay}\eta = \lim_{h\to 0} \frac{1}{h}[hEZ\psi'(yaZ + ybV)$$

$$+ hyEZ(aZ + bV)\ \psi''(yaZ + ybV + hG(aZ + bV))$$

$$+ h^2EZ(aZ + bV)\ \psi''(yaZ + ybV + hG(aZ + bV))]$$

$$= EZ\psi'(yaZ + ybV) + ya \lim_{h\to 0} EZ^2\psi''(yaZ + ybV + hG(aZ + bV))$$

$$+ yb \lim_{h\to 0} EZV\ \psi''(yaZ + ybV + hG(aZ + bV))$$

$$+ a \lim_{h\to 0} hEZ^2\ \psi''(yaZ + ybV + hG(aZ + bV))$$

$$+ b \lim_{h\to 0} hEZV\ \psi''(yaZ + ybV + hG(aZ + bV)).$$

Without loss of generality let $|h| < \delta < y$. By (1.3a) ψ'' is negative and increasing so

$$(3.9)\qquad |Z^2\psi''(yaZ + ybV + hG(aZ + bV))| < |(Z^+)^2\psi''(y - \delta)\ a\ Z^+\ |$$

$$+ |(Z^-)^2\psi''(-(y + \delta)aZ^-)|.$$

The right side of (3.9) is integrable by (1.3c) so, again using the dominated convergence theorem, lim may be taken before E in the second and fourth terms of the final expression for $D_{ay}\eta$ in (3.8). Also[2/]

$$(3.10)\qquad |ZV\psi''(yaZ + ybV + hG(aZ + bV)| < |Z^+V\psi''((y - \delta)aZ^+)|$$

2/ Note that $EZ^2\psi''$ integrable implies $EZV\psi''$ integrable.

so lim and E may be reordered in the third and final terms. Simplifying then gives

$$(3.11)\qquad D_{ay}\eta = EZ\psi'(yaZ + ybV) + yaEZ^2\psi''(yaZ + ybV)$$
$$+ ybEZV\psi''(yaZ + ybV)$$
$$= EZ\psi'(yaZ + ybV) + yEZ\ (aZ + bV)\psi''(yaZ + ybV).$$

<u>(iii)</u> Let $(y_n, a_n) \to (y, a)$. Without loss of generality assume $|y_n - y| < \delta < \min\{y, a, 1\}$ and $|a_n - a| < \delta$.

$$(3.12)\qquad D_{ay}\eta(a_n; y_n) = EZ\psi'(y_n a_n Z + y_n bV)$$
$$+ y_n a_n EZ^2\psi''(y_n a_n Z + y_n bV) + y_n bEZV\psi''(y_n a_n Z + y_n bV).$$

Because of the continuity of products it is sufficient to show continuity of the three expectations. Note

$$(3.13)\qquad |Z\psi'(y_n a_n Z + y_n bV)| < |Z^+\psi'(y_n - \delta)\ (a_n - \delta)\ Z^+)|$$
$$|Z^2\psi''(y_n a_n Z + y_n bV)| < |(Z^+)^2\psi''(y_n - \delta)\ (a_n - \delta)\ Z^+|$$
$$+ |(Z^-)^2\ \psi''(-(y_n + \delta)\ (a_n + \delta)Z)|$$
$$|ZV\psi''(y_n a_n Z + y_n bV)| < |Z^+V\psi''((y_n - \delta)\ (a_n - \delta)\ Z^+|$$
$$+ |Z^-V\psi''(-\ (y_n + \delta)\ (a_n + \delta)Z^-)|.$$

Again, the dominating random variables are integrable by (1.3c) so lim as $(y_n, a_n) \to (y, a)$ of each expectation is the expectation of its limit and the sum of these with appropriate coefficients is $D_{ay}\eta(a; y)$.

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